Phoebe Floats!

Ezra Brown

Ezra (Bud) Brown (brown@math.vt.edu; Virginia Polytechnic Institute & State University, Blacksburg, VA 24061) grew up in New Orleans and has degrees from Rice and LSU. He arrived at Virginia Tech shortly after Hurricane Camille, and has been there ever since, with time out for sabbatical visits to Washington, DC (where he has spent his summers since 1993) and Munich. He has done research in number theory, discrete mathematics, and expository mathematics. He has received a teaching award and three writing awards from the MAA. The idea for this article came from his long-standing interest in calculus, the history of mathematics, and astronomy. The inspiration, however, came from the students mentioned in this story and from his granddaughter Phoebe Rose.

—To Phoebe Rose: the One True Phoebe.

When history of mathematics classes encounter ancient problems, the discussions can be both lively and unpredictable. This story, about one such instance, begins with a seemingly irrelevant remark about a heavenly body, which leads to a question. Along the way, we'll encounter Archimedes' answer to this question, Newton's method, the behavior of Newton's method under iteration, and repelling periodic points.

The problem

We had already discussed Archimedes' solution of how to divide a sphere into two segments whose volumes were in a given ratio of $a : b$, and the students had read up on the Law of Floating Bodies. We were talking about what they had read when Mark, a student with many interests, said, "By the way, did you know that Phoebe floats?"

This brought a halt to the proceedings and prompted many questions: "What do you mean?", "Who's Phoebe?", and "What's that got to do with Archimedes?"

"Phoebe is a satellite of Saturn," replied Mark, "and they say that it's less dense than water, so if you drop it into the ocean, it floats." (See Figure 1.) This intelligence

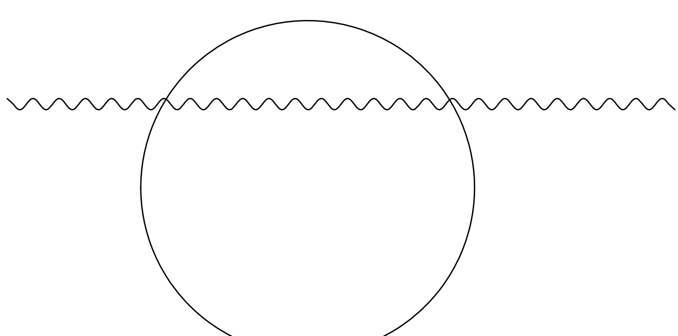

Figure 1. Phoebe afloat

proved to be from a website [1], which also gave Phoebe's diameter as 220 kilometers and its mass as 4.0×10^{18} kilograms. A few seconds later: "That puts the density at about 7/10."

"I think this does have something to do with Archimedes," said Carla. "The book says that if you know the density of a floating ball, you can figure out how far down the bottom of the ball sinks below the surface. So, we can ..."

"... we can see how far Phoebe would sink by knowing her density and comparing the volume below the surface to the volume above the surface," Jason broke in, "and then ..."

"... calculate the volume in terms of the unknown depth, using integrals," added Nick, "and ..."

"... back-solve for the depth," finished Karen triumphantly.

"Great!" I said. "Go home, solve the problem, write it up, and present it to the class next time." So they did.

In this article, we'll see how the students solved the problem. We will also examine a particularly interesting wrinkle to the problem (see [4]), involving the chaotic behavior of Newton's method applied to a cubic polynomial.

Solving the original problem

To standardize the problem, we'll work with a ball of radius 1 and Phoebe's density 7/10, then scale up to find the results for Phoebe. We also assume that the density of water is 1. Here is a statement of the problem.

A ball of radius 1 and density 7/10 is placed in a body of water. How far down does the ball sink below the surface?

Archimedes' Law of Floating Bodies states that a body immersed in a fluid displaces an amount of fluid equal in weight to the weight of the body, provided the body is less dense than the fluid. In plain English, a floating object displaces its weight. An object of density δ and volume V cubic units weighs exactly as much as an amount of water of volume $\delta \cdot V$ cubic units. That means that our ball sinks until exactly δ percent of its volume is underwater.

We now translate the problem into mathematics. We represent the ball by its cross-section through the poles, namely a circle of radius 1 with its center at the origin. The waterline is a horizontal line, and the depth underwater is a length we'll call r. In the case of density 1/2, half the ball is submerged and the depth equals 1, the radius. Since our density is greater than 1/2, it's clear that more than half of Phoebe is submerged, and so r is between 1 and 2.

We place the ball so that its south pole is at $(0, -1)$ and so that the waterline intersects the y-axis at the point $(0, -1 + r)$; see Figure 2. The volume of a ball of radius 1 is equal to $4\pi/3$, so our mission is to find that value of r so that the segment of the ball between $y = -1$ and $y = -1 + r$ has volume $(4\pi/3)(7/10)$, or 70% of the volume of the ball.

We may now set up an integral to find the volume of the segment. Since the equation of the unit circle is $x^2 + y^2 = 1$, a slice of the segment through y_k of thickness Δy_k has radius $\sqrt{1 - y_k^2}$. The volume V_k of the slice is then $\pi(1 - y_k^2)\,\Delta y_k$. Hence, the volume V_S of the segment is approximated by $\sum_{k=1}^{n} \pi(1 - x_k^2)\,\Delta x_k$, and so

$$\frac{7}{10} \cdot \frac{4\pi}{3} = V_S = \int_{-1}^{-1+r} \pi(1 - y^2)\,dy$$

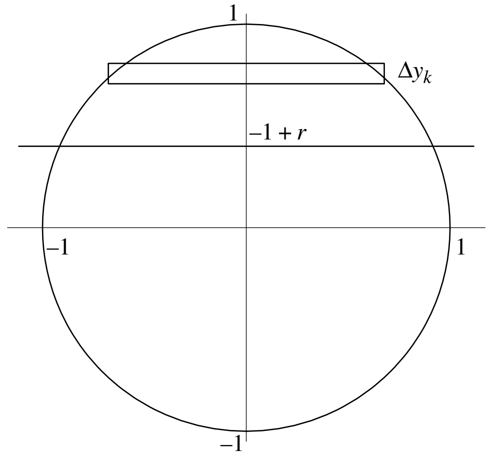

Figure 2. Phoebe with calculus

$$= \pi \left[y - \frac{y^3}{3} \right]_{-1}^{-1+r},$$

and a little algebra shows that

$$\frac{28\pi}{30} = \pi \left(r^2 - \frac{r^3}{3} \right).$$

Thus, r satisfies $r^3 - 3r^2 + \frac{14}{5} = 0$. More generally, if the ball has density δ then r satisfies $r^3 - 3r^2 + 4\delta = 0$.

Nick, who kept up with the latest events in the scientific world, asked, "Is this how Archimedes solved the problem? I once thought that he didn't have integrals, but aren't the people who are studying his *Method* saying that he came awfully close to integrals?" I pounced on this teachable moment: "Good question, why don't some of you read about what's in the *Method* and let the rest of us know."

What the students learned was that Archimedes' solution to this problem was extremely clever. First, he observed that a spherical *sector* (think of an ice-cream cone with vertex at the center of the sphere) is composed of a spherical segment sitting on top of a right circular cone. Then, he used Eudoxus's Method of Exhaustion to prove that the volume of a spherical sector is equal to one-third of the product of its radius and the area of its spherical curved surface. Since he knew how to find the volume of a right circular cone, all he had to do was subtract volumes, and what remained was the volume of his spherical segment. In particular, he found the formula for the volume of a sphere by pointing out that the entire sphere is also an ice-cream cone. See [**5**, pp. 62–66] for Archimedes' original proof, and [**7**, Chapter 10] for an illuminating discussion.

In their own solutions, the students did the integration and arrived at the last equation without much difficulty. They were surprised by the appearance of a cubic polynomial, but most of them found the roots of $r^3 - 3r^2 + 14/5$ using a calculator or a computer algebra system. However, Matt, our chief skeptic, had a good question: "Wait a minute, $r^3 - 3r^2 + 4\delta$ is a cubic polynomial. Isn't there a cubic formula?" "Yes, there is," I said. "It was found in the sixteenth century, and the tale of its discovery is full of intrigue, treachery, murder, and—but that's a whole nother story." (See [**2**, pp. 291–309] for further details.)

But Matt would not be put off. "Don't start with your digressions ... just give us the word on the cubic formula." So I did.

The Cardano–Tartaglia formula, as it's called, is an algebraic expression for the three roots of an arbitrary cubic polynomial. The formula gives the three roots of $r^3 - 3r^2 + 14/5$ as

$$r_j = 1 + 2\cos\Big(\big(2j\pi + \arccos(-2/5)\big)/3\Big), \quad \text{for } j = 0, 1, 2. \tag{1}$$

At this point, Karen pointed out that it is not at all obvious which of the three values of r_j from equation (1) gives us "the answer." "And another thing," she added. "What would we have done without the cubic formula? If we had to find the roots of a seventh-degree polynomial, we'd need a computer, so how do computers find roots of functions?"

This was an excellent question, so we pushed on. One of the root-finding strategies a computer algebra system (or CAS) employs is popularly known as Newton's method, so let's talk about Newton's method.

Cubic equations, CAS, and Newton's method

From the graph of $y = r^3 - 3r^2 + 14/5$, we see that there is a root in each of the intervals $(-1, 0)$, $(1, 2)$, and $(2, 3)$. Our friendly CAS comes to the rescue and discloses that, to six decimal places, the three roots are -0.852523, 1.273485, and 2.579038, and this seems to agree with the graph in Figure 3. These three roots correspond to the solutions in equation (1) for $j = 1$, 2, and 0, respectively. Problem solved—or is it?

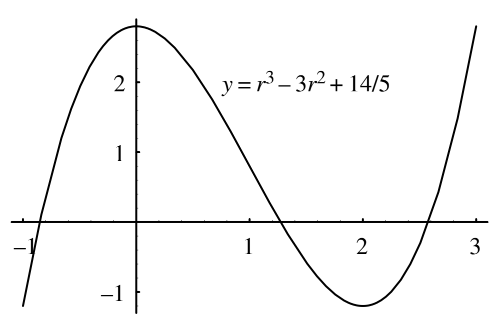

Figure 3. The roots of $r^3 - 3r^2 + 14/5$

You see, the physical problem has only one solution: if you throw a body, er, a *moon of Saturn*, into water, it sinks to only one level. Which of these roots is "the answer"? To figure that out, we go back to our original formulation. The ball had radius 1 and the depth r was between 1 and 2. Thus, the answer to our problem is r_2, or 1.273485 to six places. Phoebe's radius is about 110 km, so she sinks to a depth of about 140.083 km. This means that when viewed while in the water, Phoebe looks like a spherical segment about 80 km high (about 49 miles) at its highest point. Problem solved; we're done. Or are we?

Well, not quite. How, for example, did the CAS find those three roots? Surely we don't believe the answers merely because they were found by a **computer**?

In fact, we are on solid ground here. The typical CAS package uses a number of strategies for finding roots, but mainly Newton's Method.

Newton began with some convenient approximation x_0 to a root of a given differentiable function f. He then noticed that under favorable conditions, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is closer than x_0 to the root in question.

So he defined a sequence of numbers $\{x_n\}$ by putting $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Now under those favorable conditions, which occur for example in Figure 4, the sequence $\{x_n\}$ converges to a root of f. (See [**2**, pp. 58–66] for further details.)

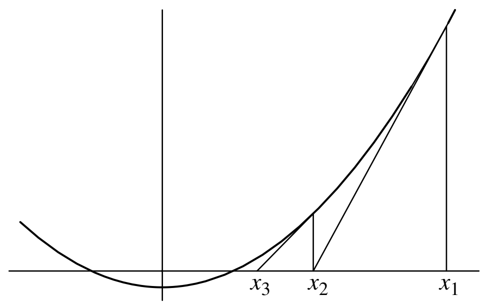

Figure 4. Newton's Method

To find how much of Phoebe rises up out of the ocean, we want to use Newton's method to approximate the roots of $f(x) = x^3 - 3x^2 + 14/5$. Since $f'(x) = 3x^2 - 6x$, our "Newton function" for this problem is

$$N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 3x^2 + 14/5}{3x^2 - 6x} = \frac{2x^3 - 3x^2 - 14/5}{3x^2 - 6x}.$$

To begin, we choose some convenient value for x_0; the only restriction is that it can't be either 0 or 2 (why so?). We then evaluate $N(x_0)$ and call that x_1; we evaluate $N(x_1)$ and call *that* x_2, and so forth. But if a function has more than one root, then different starting values for x_0 might converge to different roots, as we see in Table 1 (values are calculated to 10 places):

Table 1. $N(x_i)$ for $x_0 = -1$, 1 and 3

i	x_i	x_i	x_i
0	−3	1	5
1	−1.8637735810	1.2899370483	3.8251153078
2	−1.2145647824	1.2988070666	3.1116597461
3	−0.9287789619	1.2988323699	2.7309306013
4	−0.8651469860	1.2988323701	2.5868016405
5	−0.8620310100	1.2988323701	2.5637711504
6	−0.8620236739	1.2988323701	2.5631916674
7	−0.8620236739	1.2988323701	2.5631913037
8	−0.8620236739	1.2988323701	2.5631913037

It is clear that what we are doing is iterating the function N. If all goes well, the iterates will converge to a root. It doesn't always go well, however; but for $f(x) = x^3 - 3x^2 + 14/5$ it appears that N behaves itself and everything works out just fine.

At this point, I threw the class a curve.

Sensitive dependence on initial conditions

"Here are three starting values for Newton's method," I told them, "that differ only by 1 or 2 in the fifth decimal place: 1.86693, 1.86695, and 1.86696. Any guesses as to what will happen when you run the algorithm on them?" Most of the class seemed to think that such a small difference would not affect the outcome.

Carla, however, was suspicious. "What's so special about those numbers? You must have something tricky up your sleeve." Carla was right, for it turns out that Newton's method can be quite sensitive to the starting value. Just look at the following successive values of N:

Table 2. $N(x_i)$ for $x_0 = 1.86693$, 1.86695 and 1.86696

i	x_i	x_i	x_i
0	1.86693	1.86695	1.86696
1	0.32495	0.324788	0.324627
2	1.86669	1.8673	1.86792
3	0.327563	0.320908	0.314178
4	1.85679	1.88237	1.90951
5	0.426005	0.135938	−0.359362
6	1.58572	3.74961	−1.28961
7	1.20201	3.07184	−0.94907
8	1.27228	2.71974	−0.858907
9	1.27348	2.59596	−0.852554
10	1.27349	2.57933	−0.852524
11	1.27349	2.57904	−0.852524
12	1.27349	2.57904	−0.852524

The students were surprised at this turn of events, and immediately wanted to know what this was all about. "You rigged the starting numbers," said Jason. "The successive approximations look like they bounce back and forth for a while between somewhere around 1.867 and somewhere around 0.325, and then they escape." "And their escape routes all head to different roots of $f(x)$," added Mark.

Apparently, the Newton function is extremely sensitive to changes in the starting value whenever that starting value is close to 1.867 or 0.325—that is, N has *sensitive dependence on initial conditions*.

In his *Method of Fluxions*, written around 1671, Newton describes the method for approximating solutions of equations; he does not discuss situations such as the behavior we see in Table 2. Historically, the first person to identify sensitive dependence on initial conditions was Henri Poincaré. In his 1892 study [**6**] of the three-body problem, Poincaré points out that in some physical situations, small differences in the initial conditions may produce great differences in the outcomes.

But let us return to our Newton function. Is it sensitive dependence on initial conditions that is causing this odd behavior?

To help describe what is going on we introduce some notation from *dynamical systems theory*, which describes how functions behave under iteration. Let F be a function, and let F^k denote the kth iterate of F. We also let F^0 denote the identity function, and, to avoid confusion, write $(F(x))^k$ for the kth power of $F(x)$. If a is a point and k is a positive integer for which $F^k(a) = a$, then a is called a *point of period k* of F. The least such value of k is called the *period* of a, and for such k, the set $\{a, F(a), F^2(a), \ldots, F^{k-1}(a)\}$ is called a *k-cycle* of F. A *fixed point* of F is a point a such that $F(a) = a$.

As before, let r_1, r_2, and r_3 be the roots of $f(x) = x^3 - 3x^2 + 14/5$. It turns out that:

(1) $N(r_i) = r_i$, so that the roots of f are the fixed points of N.

(2) There exist two points, namely $s_1 = 0.32488\ldots$ and $s_2 = 1.86693\ldots$ for which $N(s_1) = s_2$ and $N(s_2) = s_1$ (see Figure 5). Thus, s_1 and s_2 are points of period 2, and $\{s_1, s_2\}$ is a 2-cycle of N. Note that s_1 and s_2 are fixed points of N^2.

(3) Points near r_i move closer to r_i under iterates of N. Thus, we call the roots of f *attracting fixed points* of N. Points near s_1 (or s_2) move away from s_1 (or s_2) under iterates of N. Thus, we call $\{s_1, s_2\}$ a *repelling 2-cycle* of N.

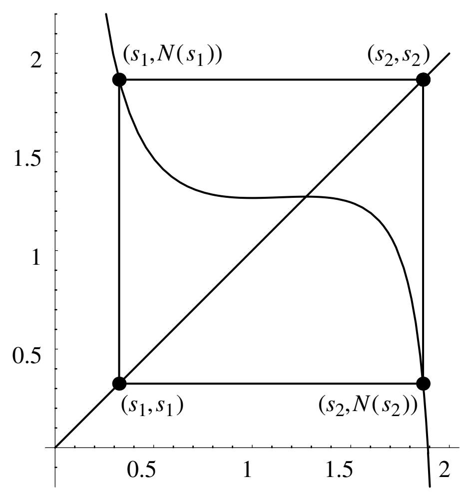

Figure 5. A Newtonian 2-Cycle

Here is the explanation of the behavior of $N(x)$ in Tables 1 and 2. The starting points in Table 1 are not close to any repelling cycle of N. The starting points in Table 2, however, *are* close to the repelling 2-cycle $\{s_1, s_2\}$, and that accounts for the Newton function's sensitive dependence on initial conditions.

The following theorem tells most of the story. For a proof, see [**3**, §1.4].

Theorem 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function.*

(a) *Assume s is a fixed point of g. If $|g'(s)| < 1$, then s is an attracting fixed point of g, while if $|g'(s)| > 1$, then s is a repelling fixed point of g.*

(b) *Assume s is a point of period n of g. If $|(g^n)'(s)| < 1$, then s is on an attracting n-cycle of g, and if $|(g^n)'(s)| > 1$, then s is on a repelling n-cycle of g.*

Matt looked up from his calculator and announced that $\frac{d}{dx}(N^2(s_i))$ is about $41.1774\ldots$ for both period-2 points s_1 and s_2, "and that is clearly greater than 1, so by Theorem 1(b), $\{s_1, s_2\}$ is a repelling 2-cycle of $N(x)$." He added, "So this explains why the Newton function's behavior depends sensitively on those starting values."

“Absolutely right,” I said, “and furthermore . . .”

“Wait a minute,” broke in Leigh, who usually had the last word. “Aren’t there a number of things wrong with this whole Phoebe experiment?” Before I could answer, the bell rang. “Next time, we’ll talk about the thirteen Archimedean semiregular polyhedra . . . or is it fourteen?” Immediately, the class wanted to know what a semiregular polyhedron is, and whether it was 13 or 14. “Ah, yes, 13 or 14? Well, that’s another story. And by the way,” I called out as the class scattered, “did you know that Saturn also floats?”

Questions

What happens with Newton functions for other polynomials? It depends. Play around with Newton’s method for different functions and see what happens. If you have a CAS with a nice graphics package, you can construct a picture of which starting points converge to which roots. For cubic polynomials, you might use three different colors for the roots. You may even try complex numbers for the starting points, and get a very interesting picture as the end result. For example, you might prove the following somewhat surprising theorem: Let $f(x) = x^3 - a$, let $N(x)$ be its Newton function and let $\{s_1, s_2\}$ be a 2-cycle of $N(x)$. Then for all values of a and for all 2-cycles, $\frac{d}{dx}(N^2(s_i)) = 6$.

How did you find those starting values in Table 2 in the first place? For $f(x) = x^3 - 3x^2 + 14/5$, the function $N^2(x)$ can be written as a quotient $p(x)/q(x)$, where p and q are polynomials with integer coefficients and degrees 9 and 8, respectively. A CAS has very little trouble finding accurate numerical approximations to the roots of $N^2(x) = x$. There are the three fixed points r_1, r_2, and r_3, the 2-cycle $\{s_1, s_2\}$, and two nonreal 2-cycles, each of which is a pair of conjugate complex numbers.

Are there functions with attracting 2-cycles, and what about 3-cycles? The function $g(x) = x^2 - 31/25$ has two fixed points, namely $(5 - \sqrt{149})/10$ and $(5 + \sqrt{149})/10$, and the 2-cycle $\{-6/5, 1/5\}$. You can show that the 2-cycle is attracting, and that the fixed points are both repelling. As for 3-cycles, the function $h(x) = 4x(1 - x)$ has a repelling 3-cycle that I’ll let you find. Here is a hint: what is $h(\sin^2\theta)$?

What makes Phoebe float? And while we’re on the subject, how do they know the densities and sizes of distant planets, satellites, stars and galaxies? The website [**8**] has a wealth of accessible information that gives the answers to these and many other questions about astronomy.

What’s wrong with “this whole Phoebe experiment?” Haven’t you made a great many simplifying assumptions? Guilty as charged. The simplifying assumptions fly in the face of the following realities: sea water does not have density 1, the density of water in a real ocean is not uniform, the earth is not flat, and gravity is not uniform. Worse than that, there is one big mistake that invalidates the entire set-up. It *is* difficult to float a moon whose diameter is about 137 miles in an ocean whose greatest depth is not quite seven miles, isn’t it?

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Proof Without Words: A Partial Fraction Decomposition

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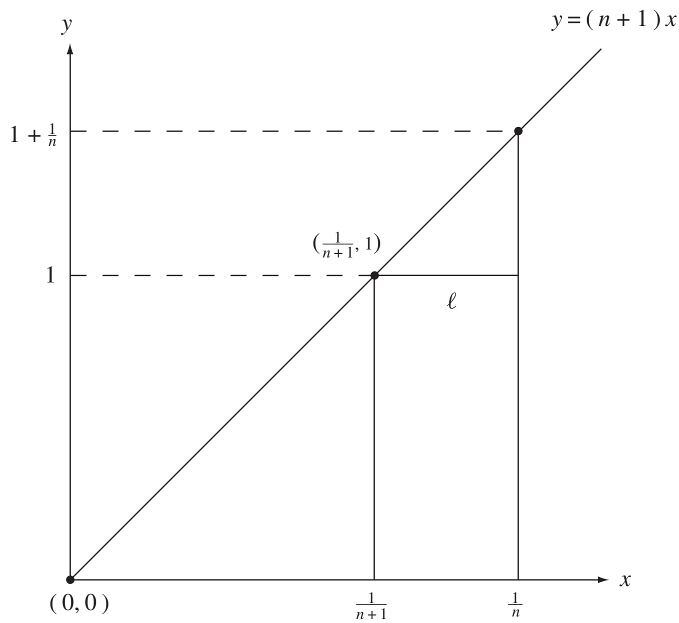

$$\ell = \frac{1}{n} - \frac{1}{n+1} \quad \text{and} \quad \frac{\frac{1}{n+1}}{1} = \frac{\ell}{\frac{1}{n}}$$

$$\Downarrow$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n}\left(\frac{1}{n+1}\right)$$

