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J. B. FOURIER—ON THE OCCASION OF HIS TWO HUNDREDTH BIRTHDAY

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Fourier's work on the conduction of heat has stimulated the most diverse developments in pure mathematics. The object of the pages which follow is to trace these developments in outline.

Fourier's other contributions to mathematics, such as his work on the theory of equations and linear inequalities, will not be discussed.

1. Convergence of Fourier series. The most original aspect of Fourier's work on trigonometric series, and the one which caused the greatest misgivings among his contemporaries, was his insistence that his expansion applied to *arbitrary* functions. In his *Théorie analytique de la Chaleur*, 1822, he says (Section 417): "In general the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary . . . We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity." This general concept of a function had appeared before Fourier, although more commonly "function" meant "function defined by an analytical expression." Fourier asserted that the two were the same. After giving what we would call today a plausibility argument rather than a proof he says (Section 418): "Thus there is no function $f(x)$, or part of a function, which cannot be expressed by a trigonometric series."

In this generality his statement is certainly not true. The first rigorous proof under wide conditions of the possibility of expanding a function in a Fourier series was given by Dirichlet (1829). In the extended form given it by C. Jordan (1881); his result reads: the Fourier series of a function f which is the difference of two increasing functions (i.e. a function of bounded variation) converges at any point x with sum $\frac{1}{2}[f(x+0)+f(x-0)]$.

Hamilton (1843), in a discussion of the convergence of Fourier series, proved that if f is continuous in an interval $[a, b]$ then

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$$\int_a^b f(x) \sin nx \, dx \to 0 \quad \text{as } n \to \infty.$$

This was extended from continuous to integrable functions by Riemann and Lebesgue and the result is now known as the Riemann-Lebesgue Lemma. Hamilton's contribution is forgotten. It follows from this lemma that the convergence of the Fourier series of a function at a particular point depends only on the behaviour of the function in an arbitrarily small neighbourhood of this point. Another almost immediate consequence is the convergence criterion of Dini (1880): the Fourier series of f converges at the point x with sum s if the integral

$$\int_0^{2\pi} |f(x+t) + f(x-t) - 2s| \, dt/t$$

exists.

Although the convergence tests of Dini and Dirichlet suffice for applications, a number of more refined tests have been given. Rather than describe them, I shall mention some results which show in what way a Fourier series may fail to converge. Du Bois Reymond (1876) gave an example of a continuous function whose Fourier series diverges on an everywhere dense set of points. Kolmogorov (1926) gave an example of a Lebesgue-integrable function whose Fourier series is everywhere divergent. Recently Carleson (1966) solved a long-standing problem by showing that the Fourier series of a function f in the space $L^2[0, 2\pi]$ (see Section 3) converges except on a *null-set* (i.e. a set which can be enclosed in a sequence of intervals whose total length is as small as one pleases). In particular, the Fourier series of a continuous function converges except possibly on a null-set. Finally, Kahane and Katznelson (1966) have shown that for any null set E there is a continuous function whose Fourier series diverges at each point of E.

Fejér (1904) showed that the situation is greatly simplified if instead of considering the convergence of the sequence of partial sums

$$s_N(x) = \sum_{n=-N}^{N} c_n e^{inx},$$

one considers the convergence of the sequence of arithmetic means

$$\bar{s}_N(x) = \frac{1}{N+1} [s_0(x) + s_1(x) + \cdots + s_N(x)].$$

If f is Lebesgue integrable, then $\bar{s}_N(x) \to f(x)$ for all x except possibly those in a null set; and if f is continuous at the point x, then $\bar{s}_N(x) \to f(x)$. Moreover if f is everywhere continuous the convergence is uniform. Fejér obtained in this way a simple proof of the Theorem of Weierstrass (1885) that each continuous function of period 2π can be uniformly approximated by trigonometric polynomials, i.e. by functions of the form

$$\sum_{n=-N}^{N} d_n e^{inx}.$$

A far-reaching generalization of Weierstrass' Theorem, and its analogue for ordinary polynomials, has been given by Stone (1948).

The possibilities of representing 'arbitrary' functions by Fourier series are illustrated by the first published example, due to Weierstrass (1875), of a function which is everywhere continuous and nowhere differentiable:

$$\sum_{n=0}^{\infty} a^n \cos (b^n x),$$

where $0<a<1$, b is an odd positive integer, and $ab>1+(3\pi/2)$.

2. Trigonometric series. To establish the convergence of the Fourier series

$$(1) \qquad f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx,$$

for as wide a class of functions f as possible, one must be able to define the integral of such a function. It was for this reason that Riemann (1854) in his *Habilitationsschrift* "On the representation of a function by a trigonometric series" introduced what we now call the Riemann integral, generalizing the definition given by Cauchy for the integral of a continuous function. We shall return to the Riemann integral shortly. The main part of Riemann's paper was concerned with the representation of functions by general trigonometric series $\sum c_n e^{inx}$ in which the coefficients c_n are not necessarily given by the integral formulae (1). By an ingenious argument based on integrating twice term by term, he obtained necessary and sufficient conditions for the possibility of such a representation.

Riemann's work on trigonometric series was followed by that of Cantor. Cantor was concerned with the question whether the sum of a trigonometric series uniquely determines its coefficients. He showed first that if a trigonometric series converges to zero at every point of the interval $[0, 2\pi]$, then its coefficients must all be zero. In trying to extend this result he was led to the concept of *derived* set. Let E be a set of real numbers. The derived set E' of E consists of all real numbers x such that any neighbourhood of x contains a point of E distinct from x. One can then form the derived set E'' of E', and so on. Cantor (1872)* proved that any set E, whose nth derived set is empty for some positive integer n, is a *set of uniqueness*, i.e. a trigonometrical series which converges to zero at all points outside E must have all its coefficients zero. From the concept of derived set he was led to the concept of *closed* set (a set which contains its derived set as a subset) and thence to the general study of point set topology.

* Cantor's method of constructing the real numbers from the rationals by means of fundamental sequences appears at the beginning of this paper.

A set with empty nth derived set is either finite or *countable*, i.e. it can be put into 1-1 correspondence with the set of positive integers. Cantor then showed that the set of all algebraic numbers is countable, but the set of all real numbers is not countable. This led him to the general notion of 1-1 correspondence between two sets and the concept of cardinal number. Incidentally it was later shown by W. H. Young (1908) that any countable set is a set of uniqueness. However, not all sets of uniqueness are finite or countable.

3. Integration. We have seen that the discussion of convergence of Fourier series provoked a widening of the concept of "integral." The most satisfactory extension was found by Lebesgue (1902). It will be explained here in the alternative form due to Daniell (1917).

We are all agreed about what value the integral of a (real-valued) step function should have. If f has the constant value c_k of an interval J_k of length l_k $(k=1, \cdots, N)$ and is everywhere else zero then the integral is defined by

$$I(f) = \sum_{k=1}^{N} c_k l_k.$$

The set S of all step-functions has the property that if f and g are in S then so are $|f|$, $f+g$ and cf for any real number c. Moreover,

$$(2_1) \qquad I(f + g) = I(f) + I(g),$$

$$(2_2) \qquad I(cf) = cI(f),$$

$$(2_3) \qquad I(f) \geqq 0 \quad \text{if } f \geqq 0,$$

$$(2_4) \qquad I(f_n) \to 0 \quad \text{if } f_1 \geqq f_2 \geqq \cdots \quad \text{and} \quad f_n \to 0.$$

The problem of integration is to extend the set of integrable functions so that these properties are preserved. Riemann solved this problem in the following way. Suppose there exists an increasing sequence of step functions $s_1 \leqq s_2 \leqq \cdots$ and a decreasing sequence of step functions $t_1 \geqq t_2 \geqq \cdots$ such that $s_n \leqq f \leqq t_n$ for all n and

$$(*) \qquad \lim_{n\to\infty} I(s_n) = \lim_{n\to\infty} I(t_n).$$

Then we say that f is Riemann integrable and we define $I(f)$ to be the common value of the limits $(*)$.

Lebesgue's more general solution proceeds in two stages. Suppose first that we have an increasing sequence of step functions $s_1 \leqq s_2 \leqq \cdots$ whose integrals are bounded, $I(s_n) \leqq c$ for all n. Then $f(x) = \lim_{n\to\infty} s_n(x)$ exists for all x and $\lim_{n\to\infty} I(s_n)$ exists. We define $I(f) = \lim_{n\to\infty} I(s_n)$. The function f has values in the extended real number system, i.e. it can assume infinite values, although the boundedness of the integrals of the approximating step functions does restrict the set of points at which their limit is infinite (it must be a null set). It is not difficult to show that this definition does not depend on the approximating sequence of step functions, i.e. if $t_1 \leqq t_2 \leqq \cdots$ is another increasing sequence of step functions such that $I(t_n) \leqq d$ for all n and if $f(x) = \lim_{n\to\infty} t_n(x)$ for all x then $\lim_{n\to\infty} I(s_n) = \lim_{n\to\infty} I(t_n)$.

Let us call the functions f for which the integral is now defined "over" functions. Also let us call f an "under" function if $-f$ is an over function, and set $I(f) = -I(-f)$. There is no inconsistency in this if f is both "under" and "over." We now complete our definition by saying that a function f is Lebesgue integrable if there exists an increasing sequence of under functions $s_1 \leqq s_2 \leqq \cdots$ and a decreasing sequence of over functions $t_1 \geqq t_2 \geqq \cdots$ such that $s_n \leqq f \leqq t_n$ for all n and $\lim_{n\to\infty} I(s_n) = \lim_{n\to\infty} I(t_n)$. Moreover we define $I(f)$ to be the common limit.

The integral thus defined has the properties $(2_1)-(2_4)$. Also it is easy to show that we get no further by repeating the process. If $f_1 \leqq f_2 \leqq \cdots$ is an increasing sequence of integrable functions such that $I(f_n) \leqq c$, then $f = \lim_{n\to\infty} f_n$ is already integrable and $I(f) = \lim_{n\to\infty} I(f_n)$.

Complex valued functions can be included by saying that f is integrable if its real and imaginary parts are integrable, and setting

$$I(f) = I(\Re f) + iI(\Im f).$$

It is customary to denote by $L^p(a, b)$, where $p \geqq 1$, the set of all functions f such that $|f|^p$ is Lebesgue integrable over the interval (a, b) and such that for any positive integer n there is a step function s_n with $I(|f - s_n|^p) < 1/n$.

4. Eigenfunction expansions. Fourier considered the conduction of heat in homogeneous bars. In seeking to extend his work to *inhomogeneous* bars, Sturm and Liouville (1836–37) were led to consider eigenfunction expansions defined by general second order linear differential equations. If we try to solve the inhomogeneous heat equation by the method of separation of variables, we obtain an ordinary differential equation

$$(k(x)y')' + [\lambda g(x) - l(x)]y = 0,$$

with boundary conditions

$$y'(a) - hy(a) = 0, \qquad y'(b) + Hy(b) = 0.$$

Here $k(x)$ and $g(x)$ are positive continuous functions representing the conductivity and specific heat, while the continuous nonnegative function $l(x)$ and the nonnegative constants h, H depend on the emissivity at the surface and ends of the bar respectively.

The values of λ for which there is a nontrivial solution y are called the *eigenvalues* of the boundary value problem and the corresponding solutions the *eigenfunctions*. Sturm and Liouville showed that there is an infinite sequence of positive eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with $\lambda_n \to \infty$. Moreover each eigenvalue λ_n is simple, i.e. the corresponding eigenfunction y_n is uniquely determined up to a constant factor, and eigenfunctions corresponding to different eigenvalues are *orthogonal* in the sense that

$$\int_a^b y_n(x)y_m(x)g(x)dx = 0 \quad \text{if } n \neq m.$$

They also obtained many results on the zeros of the eigenfunctions y_n.

With an "arbitrary" function f we associate the eigenfunction expansion

$$f(x) \sim \sum_{n=1}^{\infty} c_n y_n(x),$$

where

$$c_n = \int_a^b f(x)y_n(x)g(x)dx \Big/ \int_a^b y_n^2(x)g(x)dx.$$

This generalises the ordinary Fourier series, to which it reduces for $k(x) = \text{const.}$, $g(x) = \text{const.}$, $l(x) = 0$ and $h = H = 0$. Probably the simplest way of proving the convergence of the eigenfunction expansion is to introduce a Green's function. This replaces the boundary value problem by an equivalent integral equation, to which Hilbert's (1904) theory of integral equations with symmetric kernel can be applied. Indeed this was the first application which Hilbert made of his theory and it may be assumed that this was one of his motives for its construction.

De la Vallée Poussin (1893) proved that for each Riemann integrable function f with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{3}$$

This is usually known as Parseval's equation. It could with equal historical justification be attributed to Pythagoras, since it is an infinite-dimensional generalisation of the fact that in any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides. Parseval's equation was extended to functions f in $L^2[0, 2\pi]$ by Fatou (1906) and to Sturm-Liouville eigenfunction expansions by Steklov (1901). F. Riesz (1907) and Fischer (1907) independently found a converse to Parseval's equation: for any sequence $\{c_n\}$ of complex numbers for which the series $\sum |c_n|^2$ is convergent there exists a function f in L^2 with Fourier series $\sum c_n e^{inx}$ such that (3) holds. Fischer showed that this was a corollary of a much more general result. If $\{f_n\}$ is a sequence of functions in L^2 such that

$$\lim_{m,n\to\infty} \int_a^b |f_m(x) - f_n(x)|^2 dx = 0,$$

then there exists a function f in L^2 such that

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 dx = 0.$$

This is an analogue of Cauchy's general convergence principle in which the norm of a function f in L^2 is defined by

$$\|f\| = \left[\int_a^b |f(x)|^2 dx \right]^{1/2}.$$

It is just such closure properties which make the Lebesgue integral more convenient than the Riemann integral.

One method of proving Parseval's equation for Sturm-Liouville eigenfunction expansions, due to G. D. Birkhoff (1917), may be mentioned here, not because it is simpler than the method of reduction to an integral equation, but because there has been a revival of interest in it recently. Liouville showed that the eigenvalues and eigenfunctions of his problem behave asymptotically for $n \to \infty$ like those of an ordinary Fourier series. On the other hand it can be shown that if $\{y_n\}$, $\{z_n\}$ are two orthogonal sequences in L^2 with $\|y_n\| = \|z_n\| = 1$ for all n, and if the sequences are close in the sense that the series $\sum \|y_n - z_n\|^2$ is convergent, then Parseval's equation holds for one sequence if it holds for the other. In this way the validity of Parseval's equation for general Sturm-Liouville expansions follows from its validity for the ordinary Fourier expansion.

5. The Fourier integral. Fourier series had to some extent been anticipated in the work of Clairaut, Euler, and Lagrange. The Fourier integral was Fourier's own. He obtained it from his series by a limiting process in the manner which is still given in textbooks. It is most simply stated as an inversion formula:

$$(4) \qquad \hat{f}(y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) e^{-iyx} dx, \qquad f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \hat{f}(y) e^{iyx} dy$$

and is valid under conditions analogous to those for the convergence of Fourier series. Only the analogue of the Parseval equation, due to Plancherel (1910), will be mentioned here. It states that if f is in $L^2 = L^2(-\infty, \infty)$, the sequence

$$\hat{f}_n(y) = \frac{1}{(2\pi)^{1/2}} \int_{-n}^{n} f(x) e^{-iyx} dx$$

converges *in* L^2 to a function $\hat{f}$ such that

$$f_n(x) = \frac{1}{(2\pi)^{1/2}} \int_{-n}^{n} \hat{f}(y) e^{iyx} dy$$

converges *in* L^2 to f and $\|\hat{f}\| = \|f\|$.

The Fourier integral is associated with the differential equation $y'' + \lambda y = 0$ over the interval $(-\infty, \infty)$. The extension to general second order linear differential equations over an infinite interval was first made by H. Weyl (1910). The situation is complicated by the fact that the spectrum, instead of being discrete (viz. the sequence of eigenvalues λ_n) as in the ordinary Sturm-Liouville case, or continuous (viz. the whole line $-\infty < \lambda < \infty$) as in the case of the

Fourier integral, may be a combination of the two. The extension to differential equations of arbitrary order, which presents little difficulty for ordinary boundary value problems, was first achieved for singular boundary value problems by Kodaira (1950) and M. G. Krein (1950). The most elementary way of obtaining their results is to follow Fourier and apply a limiting process to the results for a finite interval.

We consider next the algebraic properties of the Fourier integral. Let $L=L^1(-\infty, \infty)$ denote the set of all complex-valued functions which are (Lebesgue) integrable over the interval $(-\infty, \infty)$. For any function f in L the Fourier transform $\hat{f}$ is defined and is a continuous function. The transformation $f\to\hat{f}$ is *linear*, i.e. the transform of $f+g$ is $\hat{f}+\hat{g}$ and the transform of cf is $c\hat{f}$, for any complex number c. Again, if f and g are in L their *convolution* product $f * g$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

is also in L and the transform of $f * g$ is the ordinary product $\hat{f}\hat{g}$. This important property seems to have been first observed by Čebyšev (1890/1) in the context of probability theory. Finally the transform of $f(x+a)$ is $e^{iya}\hat{f}(y)$ and, if the derivative $f'(x)$ is in L, its transform is $iy\hat{f}(y)$. It is the last property, which replaces differentiation by a simple algebraic process, that makes Fourier transforms especially useful in the solution of differential equations.

New applications of the Fourier integral were found by Wiener (1932), whose general 'Tauberian' theorem embraced a vast number of analytical results which previously had been obtained by different and quite special arguments. We state first the analogue of his theorem for series, which Wiener used as a stepping stone towards the corresponding result for integrals: Let $\hat{f}$ be a continuous complex valued function of period 2π with an absolutely convergent Fourier expansion:

$$\hat{f}(x) = \sum_{n=-\infty}^{\infty} f(n)e^{inx}, \qquad \sum_{n=-\infty}^{\infty} |f(n)| < \infty. \tag{5}$$

If $\hat{f}$ never vanishes then its reciprocal $1/\hat{f}$ also has an absolutely convergent Fourier expansion. A much clearer proof of this result has been given by Gelfand (1941) by means of the theory of Banach algebras, then in its infancy. It is a fine example of the application of algebraic ideas to problems in analysis.

Let $L(Z)$ denote the set of all functions f defined on the integers for which the series $\sum_{n=-\infty}^{\infty} |f(n)|$ is convergent. $L(Z)$ becomes an *algebra* if we define the sum $f+g$ and product $f * g$ of two functions by

$$(f+g)(n) = f(n) + g(n), \qquad (f * g)(n) = \sum_{m=-\infty}^{\infty} f(n-m)g(m).$$

The function 1 which takes the values 1 for $n=0$ and 0 for $n\neq 0$ is an identity for multiplication. An *ideal* in this algebra is a subset I such that if f and g are in I and h is in $L(Z)$ then $f+g$ and $f*h$ are in I. For example, the set of all functions $f*h$. where h runs through $L(Z)$, is an ideal, the ideal *generated* by f. An ideal is *maximal* if it is not the whole algebra $L(Z)$ and is not a subset of any other ideal. Any ideal, apart from $L(Z)$ itself, is contained in a maximal ideal.

For any function f in $L(Z)$, let $\hat{f}$ denote the continuous function of period 2π defined by (5). It is readily seen that $(f*g)^{\wedge}=\hat{f}\hat{g}$ and hence that the set of all functions g in $L(Z)$ whose transforms $\hat{g}$ vanish at a particular point y is a maximal ideal. Gelfand showed that, conversely, each maximal ideal in $L(Z)$ is obtained in this way. Thus if f has the property that its transform $\hat{f}$ never vanishes, it is contained in no maximal ideal. Therefore the ideal generated by f is the whole of $L(Z)$. Thus f has an inverse f^{-1} in $L(Z)$ and the transform of f^{-1} is the reciprocal of $\hat{f}$.

Wiener's Tauberian Theorem says that if f is a function in $L=L'(-\infty, \infty)$, then each function in L can be approximated arbitrarily closely in L by finite linear combinations $\sum_{k=1}^{N} c_k f(x-x_k)$ of translates of f if and only if the Fourier transform $\hat{f}$ never vanishes. Since the convolution product $f*g$ is a limit of linear combinations of translates of f, the set of limits of such linear combinations is the same as the closed ideal in L generated by f. The argument is now similar to that in the series case, although there is no multiplicative identity.

Wiener's Theorem on absolutely convergent Fourier series was extended by Levy (1933). The analogue of this extension for integrals was stated by Paley and Wiener (1934) and may again be proved by the method of maximal ideals: If $\hat{f}(y)$ is the Fourier transform of a function $f(x)$ in L and if $\phi(u)$ is analytic over the range of values of $\hat{f}(y)$ for $-\infty \leqq y \leqq \infty$ (i.e. 0 is included), then $\phi[\hat{f}(y)]$ is also the Fourier transform of a function in L.

6. Almost periodic functions and positive definite functions. H. Bohr (1924–26) defined a continuous complex valued function f on $(-\infty, \infty)$ to be *almost periodic* if for each $\epsilon>0$ there is a corresponding $T=T(\epsilon)>0$ such that every interval of length T contains at least one point a with the property

$$|f(x+a)-f(x)| < \epsilon \qquad \text{for } -\infty < x < \infty.$$

Bochner (1927) showed that this was equivalent to requiring each sequence $\{a_n\}$ of real numbers to contain a subsequence $\{a_n'\}$ for which the sequence of translates $f(x+a_n')$ converges uniformly on $(-\infty, \infty)$. Bohr's main object was the construction of a theory of Fourier series for almost periodic functions. He showed that the limit

$$c(\lambda) = \lim_{X\to\infty} \frac{1}{2X}\int_{-X}^{X} f(x)e^{-i\lambda x}dx$$

exists for each real number λ and is different from zero for at most countably many values of λ. For the corresponding Fourier series

$$f(x) \sim \sum_{\lambda} c(\lambda)e^{i\lambda x}$$

the Parseval equation holds:

$$\lim_{X\to\infty} \frac{1}{2X}\int_{-X}^{X} |f(x)|^2 dx = \sum_{\lambda} |c(\lambda)|^2.$$

Moreover any almost periodic function can be approximated uniformly on $(-\infty, \infty)$ by generalised trigonometric polynomials $\sum_{k=1}^{N} d_k e^{i\lambda_k x}$, and conversely any function which can be uniformly approximated by generalised trigonometric polynomials is almost periodic.

Fourier integrals are an invaluable tool in the theory of probability. A random variable is described by its *distribution function*, a bounded nondecreasing function $\mu(y)$ with $\mu(y+0)=\mu(y)$ such that $\mu(-\infty)=0$ and $\mu(\infty)=1$. Its *characteristic function* is the Fourier-Stieltjes transform

$$f(x) = \int_{-\infty}^{\infty} e^{ixy} d\mu(y). \tag{6}$$

(Note: In the definition of the integral of a step function in Section 3 l_k no longer represents the length of the interval $J_k=(a_k, b_k)$ but the quantity $\mu(b_k)-\mu(a_k)$.) The convolution theorem states that to the sum of two independent random variables corresponds the product of their characteristic functions. In this way the Fourier-Stieltjes transform becomes the most powerful method for establishing the convergence of a sequence of random variables.

Bochner (1932) found an interesting intrinsic characterization of characteristic functions. A complex valued function f defined on $(-\infty, \infty)$ is said to be *positive definite* if for any finite set of real numbers $x_1, \cdots, x_n$ and any finite set of complex numbers $c_1, \cdots, c_n$,

$$\sum_{j,k=1}^{n} f(x_j - x_k)c_j\bar{c}_k \geqq 0.$$

Bochner showed that a function f could be represented in the form (6) for some bounded nondecreasing function μ, if and only if it was continuous and positive definite.

7. Fourier analysis on groups. Let G be a *locally compact topological group*, i.e. a group on which a topology is defined such that the group operations (multiplication and inversion) are continuous and such that each point has a compact neighbourhood. Haar (1933) showed how to define for all real-valued continuous functions on G which vanish outside compact sets, an integral I, not identically zero, with the properties $(2_1)-(2_4)$ and with the additional left invariance property

$$I(f_y) = I(f), \quad \text{where} \quad f_y(x) = f(yx).$$

Moreover this integral is uniquely determined apart from a positive constant factor. The domain of definition of the integral can then be extended by the process used in Section 3. We denote by $L=L(G)$ the set of all complex-valued functions which are integrable on G.

For any two functions f, g in L the functions $f+g$ and $f * g$ defined by

$$(f+g)(x) = f(x) + g(x), \qquad (f * g)(x) = I[f(xy)g(y^{-1})]$$

are again in L. With these definitions of addition and multiplication L forms an algebra, the *group algebra* of G. If we define a norm by setting

$$\|f\| = I(|f|),$$

then L is actually a Banach algebra.

Suppose now that G, and hence also L, is commutative. Then a *character* of G is defined to be a continuous mapping γ of G into the complex numbers of absolute value 1 such that

$$\gamma(xy) = \gamma(x)\gamma(y) \qquad \text{for all } x, y \text{ in } G.$$

If we define the product of two characters γ_1, γ_2 by

$$(\gamma_1\gamma_2)(x) = \gamma_1(x)\gamma_2(x)$$

then the set Γ of all characters becomes a commutative group, the *dual* group of G. We give Γ a topology by defining the basic open sets to be the sets $H(C, \epsilon, \gamma_0)$ of all γ in Γ such that

$$|\gamma(x) - \gamma_0(x)| < \epsilon \qquad \text{if } x \text{ is in } C,$$

for some compact set C in G, some $\epsilon > 0$, and some γ_0 in Γ. With this topology Γ is also a locally compact topological group.

The Fourier transform of a function f in $L(G)$ is the continuous function $\hat{f}$ on Γ defined by

$$f(\gamma) = I_G[f(x)\overline{\gamma(x)}].$$

Then the Fourier transform of $f * g$ is $\hat{f}\hat{g}$. A function f on G is said to be positive definite if for any finite set $x_1, \cdots, x_n$ of elements of G and any finite set $c_1, \cdots, c_n$ of complex numbers

$$\sum_{j,k=1}^{n} f(x_j x_k^{-1}) c_j \bar{c}_k \geqq 0.$$

The Fourier Inversion Theorem holds in the following form: if f is a continuous positive definite function in $L(G)$ then $\hat{f}$ is a continuous function in $L(\Gamma)$ and

$$f(x) = I_\Gamma[\hat{f}(\gamma)\gamma(x)],$$

provided the invariant integral on Γ is suitably normalised.

A. Weil (1938) showed that Plancherel's Theorem and Bochner's Theorem on positive definite functions can be extended to this general situation, as can Wiener's Tauberian Theorem. The duality Theorem of Pontryagin (1939) says that conversely G is the dual of Γ.

Ordinary Fourier series and integrals both appear as special cases. In the first case G is the additive group of all integers and its dual Γ is the multiplicative group of complex numbers of absolute value 1. In the second case G is the additive group of all real numbers and is its own dual. Seeing the two as special cases of the same phenomenon adds to our understanding of them.

The theory of almost periodic functions has been extended by von Neumann (1934) to arbitrary groups.

8. Singular integral equations. Numerous problems in mathematical physics lead to integral equations of the form

$$f(t) - \int_0^\infty k(t-s)f(s)ds = g(t) \qquad (0 \leqq t < \infty), \tag{7}$$

where f is the unknown function and k and g are given. The first explicit solutions of the corresponding homogeneous equation ($g=0$) were obtained by Wiener and Hopf (1931) for kernels k which are exponentially small at infinity. Their method depended on taking Fourier transforms and representing a function analytic in the strip $|Iz| < c$ as the product of two functions, one analytic in the half-plane $Iz > -c$ and the other analytic in the half-plane $Iz < c$. Rapoport (1948) made less stringent restrictions on the kernel k by reducing the integral equation (7) to Hilbert's problem on the boundary values of analytic functions. Then M. G. Krein (1958) treated the equation (7) under the sole conditions that k is in $L = L^1(-\infty, \infty)$ and that its Fourier transform

$$K(\lambda) = \int_{-\infty}^\infty k(t)e^{i\lambda t}dt \neq 1 \qquad \text{for } -\infty < \lambda < \infty.$$

(Note: The sign of the exponent in the integrand has been chosen to agree with Krein.) The basis of his method is the Theorem of Wiener and Levy mentioned at the end of Section 5.

Let

$$\nu = -\frac{1}{2\pi}\Delta \arg[1 - K(\lambda)],$$

where $\Delta \arg \phi(\lambda)$ denotes the net increase in $\arg \phi(\lambda)$ as λ increases from $-\infty$ to ∞. ν is an integer, since $K(\lambda) \to 0$ as $\lambda \to \pm\infty$ by the Riemann-Lebesgue Lemma.

Krein shows that the integral equation (7) has a unique solution f in L for every g in L if and only if $\nu = 0$. If $\nu > 0$ then (7) is always soluble but the solution is not unique, since the corresponding homogeneous equation

$$f(t) - \int_0^\infty k(t-s)f(s)ds = 0$$

has exactly ν linearly independent solutions.

If $\nu < 0$ then (7) either has no solution in L or a unique solution. The latter case occurs if and only if

$$\int_0^\infty g(t)h_j(t)dt = 0 \qquad (j = 1, \cdots, |\nu|),$$

where $h_1, \cdots, h_{|\nu|}$ is a basis for the solutions of the adjoint homogeneous equation

$$h(t) - \int_0^\infty k(s - t)h(s)ds = 0.$$

Thus if $\nu \neq 0$, the homogeneous equation and its adjoint do not have the same number of linearly independent solutions, contrary to what occurs in the ordinary Fredholm theory of integral equations. These results have analogues for systems of linear equations of the form

$$\sum_{m=0}^{\infty} k_{n-m} f_m = g_n \qquad (n = 0, 1, 2, \cdots),$$

and extensions to the case where f and g in (7) are vector functions and k is a matrix function.

9. Generalized functions. The theory of distributions of L. Schwartz (1950–51) and the various "generalized functions" of Gelfand and Šilov (1958) are closely connected with the Fourier transform. Indeed this is the main feature which distinguishes Schwartz's theory from its precursors. We describe here the elementary approach used by Temple (1955).

An infinitely differentiable function on $(-\infty, \infty)$ is said to be *rapidly decreasing* if it, and its derivatives of all orders, tend to zero faster than any negative power of $|x|$ as $x \to \pm\infty$. For example, e^{-x^2} is rapidly decreasing. We denote the set of all rapidly decreasing functions by S. It is a linear space which contains $f(ax+b)$ for real $a \neq 0$ and b if it contains $f(x)$.

A sequence $\{f_n\}$ of functions in S is said to be *convergent* if for any function g in S the numerical sequence

$$(f_n, g) = \int_{-\infty}^{\infty} f_n(x)g(x)dx$$

converges. We call two convergent sequences *equivalent* if the corresponding limits are the same for every g in S. We then define a *generalized function* F to be an equivalence class of convergent sequences and we set

$$(F, g) = \lim_{n \to \infty} (f_n, g).$$

We can regard any rapidly decreasing function f as a generalized function by identifying it with the equivalence class containing the sequence $\{f_n\}$ in which $f_n = f$ for all n. The sequence $\{(n/\pi)^{1/2} e^{-nx^2}\}$ is easily seen to be convergent. The corresponding generalized function will be denoted by δ. It has the property

$(\delta, g) = g(0)$. Dirac's popular delta-function thus acquires a precise meaning.

The sum of two generalized functions, linear transformations of the independent variable, and the product of a generalized function by a constant or, more generally, a slowly increasing function are naturally defined. Here an infinitely differentiable function is said to be *slowly increasing* if it and all its derivatives are bounded by some power of $|x|$ as $x \to \pm\infty$. For example, $e^{i\lambda x}$ is slowly increasing for any real λ.

For any two functions f, g in S we have

$$(f', g) = -(f, g').$$

This enables us to define the derivative DF of a generalized function F: if the equivalence class F contains the convergent sequence $\{f_n\}$ then DF is the equivalence class containing the convergent sequence $\{f_n'\}$. Also the Fourier transform maps S onto itself. (It is this property and closure under differentiation that determine the choice of S.) Finally, for any two functions f, g in S we have

$$(f, g) = (\hat{f}, \hat{g}).$$

This enables us to define the Fourier transform $\hat{F}$ of a generalized function F: if the equivalence class F contains the convergent sequence $\{f_n\}$, then $\hat{F}$ is the equivalence class containing the convergent sequence $\{\hat{f}_n\}$. These definitions are easily shown to be consistent, i.e. they do not depend on the choice of sequence $\{f_n\}$ within an equivalence class. By the inversion theorem the Fourier transform of $\hat{F}(x)$ is $F(-x)$. Moreover the transform of δ is the constant $(2\pi)^{-1/2}$ and the transform of DF is the product of $\hat{F}$ by the slowly increasing function ix.

Finally, a sequence $\{F_n\}$ of generalized functions is said to *converge* to the generalized function F if

$$(F, g) = \lim_{n\to\infty} (F_n, g)$$

for any function g in S. If $F_n \to F$ then also $DF_n \to DF$ and $\hat{F}_n \to \hat{F}$.

The theory of trigonometric series is particularly simple within the domain of generalized functions. A trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges to a generalized function F if and only if its coefficients c_n increase no faster than some power of $|n|$ as $n \to \pm\infty$. Moreover F is invariant under translation by 2π, and any generalized function which is invariant under translation by 2π can be uniquely represented as the sum of a convergent trigonometric series.

Generalized functions have found applications in several branches of mathematics, notably in the study of linear partial differential equations with constant coefficients, where they are now indispensable.

10. Miscellany. The summation formula of Poisson (1823) connects the values of a function f on a subgroup of the real line with the values of its Fourier transform $\hat{f}$ on another subgroup:

$$(2\pi)^{1/2} \sum_{n=-\infty}^{\infty} f(2n\pi) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

It holds for functions f in L which are of bounded variation and are normalised so that $f(x) = [f(x+0)+f(x-0)]/2$. Consequences of this formula include Jacobi's imaginary transformation of the theta functions, the reciprocity law for Gaussian sums, and Riemann's functional equation for the zeta-function.

Fejér and F. Riesz (1916) showed that any trigonometric polynomial

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

such that $f(x) \geqq 0$ for all real x can be expressed in the form $f(x) = |g(e^{ix})|^2$, where

$$g(w) = \sum_{n=0}^{N} a_n w^n.$$

Moreover g is uniquely determined if we require further that $g(0) > 0$ and $g(w) \neq 0$ for $|w| < 1$. This was extended by Szegö (1921): Let $f \neq 0$ be a nonnegative function in $L[0, 2\pi]$. Then there exists a function g in $L^2[0, 2\pi]$ such that $f = |g|^2$ and

$$\int_0^{2\pi} g(x)e^{inx}dx = 0 \qquad \text{for } n = 1, 2, \cdots$$

if and only if $\log f$ is in $L[0, 2\pi]$. Moreover there exists a unique g for which also

$$\frac{1}{2\pi}\int_0^{2\pi} g(x)dx = \exp\left[\frac{1}{4\pi}\int_0^{2\pi} \log f(x)dx\right].$$

Szegö's result has found applications to the prediction theory of stationary stochastic processes.

Paley and Wiener (1934) considered Fourier transforms in the complex domain. Only two of their results will be mentioned here:

A function $F(z)$ can be represented in the form

$$F(z) = \int_0^\infty f(t)e^{-zt}dt,$$

where f is in $L^2(0, \infty)$, if and only if it is analytic in the half-plane $Rz > 0$ and

$$\int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \text{constant} \qquad \text{for } 0 < x < \infty.$$

A function $F(z)$ can be represented in the form

$$F(z) = \int_{-a}^{a} f(t)e^{itz}dt,$$

where f is in $L^2(-a, a)$, if and only if it is an entire function (i.e. the sum of an everywhere convergent power series), it is in L^2 on the real axis and

$$|F(z)| \leqq Ce^{a|z|},$$

for some positive constant C. It can be shown that any such function is uniquely determined by its values at a suitable sequence of equally spaced points on the real line, in fact

$$F(z) = \sum_{n=-\infty}^{\infty} F(n\pi/a) \sin(az - n\pi)/(az - n\pi).$$

11. Conclusion. Enough has been said to show how profoundly Fourier's work has influenced the development of mathematics, directly and indirectly.

An expanded version of a lecture given at the Australian National University, Canberra, on April 1, 1968. In addition Professor J. C. Jaeger spoke on the significance of Fourier's work for applied mathematics. Some of the material here also formed part of the third Behrend Memorial Lecture, given at the University of Melbourne on August 2, 1968.

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FUNCTIONAL ANALYSIS PROOFS OF SOME THEOREMS IN FUNCTION THEORY

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We present here functional analysis proofs of three theorems in function theory; the first two theorems are classical and the third is well known. The first theorem is Runge's Theorem on approximation by rational functions, which readily implies the Cauchy Integral Theorem. The second is the familiar theorem that there exists an analytic function that interpolates arbitrary values on any discrete subset of a given open set in the complex plane. This result readily implies the Mittag-Leffler Theorem, which in turn easily implies the Weierstrass Theorem about the existence of analytic functions with arbitrarily prescribed discrete zero set. The third result is that every closed ideal in the ring of functions analytic on a region is principal. The proofs are new, although their substance seems to be known to some workers in the field. For example, a closely related proof of Runge's Theorem appears in [2, pp. 47–48], which is relatively inaccessible.

Our proofs are based on the duality between the space $H(G)$ of all functions holomorphic on the region G in the complex plane, in the topology of uniform convergence on compact subsets of G, and the space $H_0(G')$ of germs of functions