THIRTEEN COLORFUL VARIATIONS ON GUTHRIE'S FOUR-COLOR CONJECTURE

Dedicated to the memory of Oystein Ore

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INTRODUCTION

After careful analysis of information regarding the origins of the four-color conjecture, Kenneth O. May **[1]** concludes that:

> "It was not the culmination of a series of individual efforts that flashed across the mind of Francis Guthrie while coloring a map of England . . . his brother communicated the conjecture, but not the attempted proof to De Morgan in October, 1852."

His information also reveals that De Morgan gave it some thought and communicated it to his students and to other mathematicians, giving credit to Guthrie. In 1878 the first printed reference to the conjecture, by Cayley, appeared in the Proceedings of the London Mathematical Society. He wrote asking whether the conjecture had been proved. This launched its colorful career involving a number of equivalent variations, conjectures, and false proofs, which to this day, leave the question of sufficiency wide open in spite of the fact that it is known to hold for a map of no more than 39 countries.

Our purpose here is to present a short, condensed version (with definitions) of most equivalent forms of the conjecture. In each case references are given to the original or related paper. For the sake of brevity, proofs are omitted. The reader will find a rich source of information regarding the problem in Ore's famous book **[1]**, "The Four-Color Problem".

A number of conjectures given here are not in any of the books published so far. Others are found in some but not in others. Even though this array of conjectures may not be complete, it is hoped that the condensed presentation and its order

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would give the interested reader a feeling of the depth and variety in which the problem has been examined by a large number of people.

We have intentionally avoided extending the concepts to important areas of graph theory which do not have direct bearing on the conjectures given here. Otherwise, there would be no end to this paper.

CHAPTER I: THEME

1. Basic definitions and statement of the conjecture.

1.1 DEFINITION: A **graph** is a triple (V, E, Φ) where V is a finite nonempty set called the set of **vertices,** E a (possibly empty) finite set called the set of **edges,** with $E \cap V = \emptyset$, and Φ: $E \to V \& V$ is a function called the **incidence mapping.** Here $V \& V$ is the unordered product of V with itself; i.e., if $(u \& v) \in V \& V$ then $(u \& v) = (v \& u)$. If $\Phi(e) = (v \& w)$, then we say that v and w are **incident** with e. Two vertices connected by an edge (incident with the same edge) are said to be **adjacent.** They are called the **end points** of the edge. Two edges with a vertex in common are also called adjacent.

A graph is **simple** if it has no loops or parallel edges. (An edge is a **loop** if both of its end points coincide; two edges are **parallel** if they have the same end points.)

1.2 DEFINITION: A sequence of n edges $e_1, \cdots, e_n$ in a graph G is called an **edge progression** of length n if there exists an appropriate sequence of $n+1$ (not necessarily distinct) vertices $v_0, v_1, \cdots, v_n$ such that e_i is incident with v_{i-1} and v_i, $i = 1, \cdots, n$. The edge progression is **closed** (open) if $v_0 = v_n$ ($v_0 \neq v_n$). If $e_i \neq e_j$ for all i and j, $i \neq j$, the edge progression is called a **chain progression.** The set of edges is said to form a **chain.** The chain is a **circuit** if $v_0 = v_n$. If the vertices are also distinct, we have a **simple chain progression,** the edges form a simple chain. In this case, if only $v_0 = v_n$ and all other vertices are distinct, the edges are said to form a **simple circuit.** The **length** of (number of edges in) a longest simple circuit is called the **circumference** of G. Frequently one abbreviates a "simple circuit" by a "circuit".

1.3 DEFINITION: The **degree** (or valence) of a vertex is the number of edges incident with that vertex.

1.4 DEFINITION: A graph is: **planar** if it can be embedded (drawn) in a plane (or on a 2-sphere) such that no two edges meet except at a vertex; **connected** if each pair of vertices can be joined by a chain; **complete** if each vertex is connected by an edge to every other vertex; **k-partite** if its vertices can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent; and **complete k-partite** if every pair of vertices in different sets are adjacent. A **connected component** of a graph is a maximal connected subgraph.

Note that a graph is bipartite if and only if every circuit has even length. (Bipartite means 2-partite.)

1.5 Definition: A **map,** or **planar map,** M consists of a planar graph G together with a particular drawing, or embedding, of G in the plane. We call G the **underlying graph** of M and write $G = U(M)$. The map M divides the plane into connected components which we call the **regions,** or **faces,** or **countries,** of the map. Two regions are **adjacent** if their boundaries have at least one common edge, not merely a common vertex. We refer to the edges in the boundary of a region as its **sides.**

Note that a graph may be embedded in the plane to produce several different maps. For example, the graph which consists of a square and two triangles all meeting at one vertex may be embedded in the plane in several ways—one has both triangles on the inside of the square, another has one triangle inside and one triangle outside the square. In the second map there is no four-sided region, while in the first map the region exterior to the square has four sides.

1.6 Definition: A k-**coloring** of a map (sometimes called a **proper** k-**coloring**) is an assignment of k colors to the countries of the map in such a way that no two adjacent countries receive the same color. A map is k-**colorable** if it has a k-coloring.

1.7 Conjecture C_0: *Each planar map is* 4-*colorable.*

K. May points out that the four-color conjecture belongs uniquely to Francis Guthrie and could fairly be called "Guthrie's Conjecture". That four colors are necessary can be seen from the two figures below, the first of which has four regions, each of which is adjacent to the remaining three. However, this type of condition need not hold in order that four colors be necessary as illustrated by the second figure.

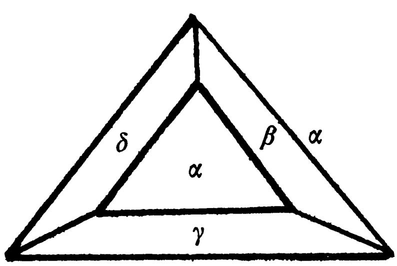


Fig. 1

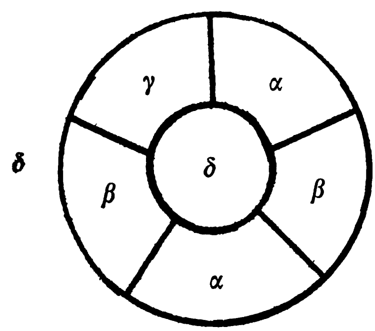


Fig. 2

2. Historical highlights. Because of the many valuable contributions of many people to the four-color problem, we are reluctant to appear to give special mention to some contributors but not to others. Nevertheless, we thought it would be useful to give a brief summary of some of the historical events relating to the conjecture and some of its variations. Occasionally it is difficult to pinpoint the exact date of an idea. The best one can do is give the year of its appearance in print. The names of G. A. Dirac and W. T. Tutte may well be added here for their many contributions to ideas occurring in the context of the four-color problem.

1852. F. Guthrie [May 1] communicated the four-color conjecture to De Morgan.

1878. A. Cayley [1] published an inquiry as to whether the conjecture had been proven.

1879. A. B. Kempe [1] published a "proof" of the conjecture. W. E. Story [1] used Kempe's work to show that the conjecture for arbitrary planar maps can be reduced to cubic maps.

1880. P. G. Tait [1] reduced the conjecture to the colorability of the edges of cubic maps.

1890. P. J. Heawood [1] pointed out the error in Kempe's proof and salvaged enough to prove the sufficiency of 5 colors for planar maps.

1891. J. Petersen [1, p. 219] proved that either the vertices of a planar cubic map can be toured by a Hamiltonian circuit or by a collection of mutually exclusive subcircuits.

1912. O. Veblen [1] transformed the conjecture into equivalent assertions in projective geometry and the solution of simultaneous equations. G. D. Birkhoff [1] introduced a version of chromatic polynomials.

1922. P. Franklin [1] showed that a map with 25 or fewer regions is 4-colorable.

1925. A. Errera [1], referring to Franklin's result that a map requiring five colors must have at least 26 regions, proved that such a map must include at least 13 pentagons.

1926. C. N. Reynolds [1] showed that a map with 27 or fewer regions is 4-colorable.

1931. H. Whitney [1] used the notion of the dual graph and proved that the dual graph to a loopless cubic map always has a Hamiltonian circuit. He also proved the equivalence of the four-color conjecture and the fact that if a planar graph is Hamiltonian, it is 4-colorable.

1932. H. Whitney [4] studied chromatic polynomials.

1936. D. König [1] published the first book on graph theory with notions later used to formulate conjectures equivalent to the four-color problem.

1937. C. E. Winn [1], considering Franklin's paper which was to be published in 1938, in which Franklin proved that a map which requires five colors must have at least 32 regions, showed that it must contain at least 2 regions bounded by more than six edges (see Ball and Coxeter [1, p. 230]).

1938. P. Franklin [2] extended the number to 31 regions (thus if a map were to require 5 colors, it must have at least 32 regions). He also showed that such a map must include at least 15 pentagons.

1940. C. E. Winn [4] extended the number of regions in a 4-colorable map to 35.

1941. R. L. Brooks [1] proved an important theorem giving a bound on the chromatic number of a graph.

1943. H. Hadwiger [1] gave his well-known conjecture of which the four-color problem is a special case.

1952. Dynkin and Uspenskii [1] first published a small book of elementary exercises on the coloring problem.

1959. G. Ringel [1] published the first major book on the coloring of maps and graphs.

1967. O. Ore [1] published the now classic book on the subject containing a number of new ideas.

1969. O. Ore and G. J. Stemple [1] increased the number of regions to 39.

Several other books now include chapters on the theory of graphs and on coloring problems. The leading texts fully given to the subject are the books by C. Berge [1], F. Harary [2], B. Roy [1], and by W. T. Tutte [11]. No library is complete without them. One may also refer to Busacker and Saaty [1], Franklin [3], and Liu [1].

CHAPTER II: VARIATIONS ON THE THEME

1. Duality and coloring. Given a map M there is another graph $D(M)$ which we can derive from it. Replace each region by a vertex, or **capital,** and join two capitals by as many parallel edges as there are edges common to the boundaries of both corresponding regions. Thus an edge which lies on the boundary of only one region in M produces a loop in $D(M)$.

1.1 DEFINITION: The graph described above is called the **dual graph** $D(M)$ of the map M.

Note that the dual graph is the underlying graph of a (dual) map.

1.2 DEFINITION: A **k-coloring** (or **proper k-coloring**) of a graph is an assignment of k colors to the vertices of the graph in such a way that no two adjacent vertices receive the same color. A graph is **k-colorable** if it has a k-coloring.

Thus, a map is k-colorable if and only if its dual graph $D(M)$ is k-colorable.

1.3 PROPOSITION: *Let M be any map. We may subdivide the edges of $U(M)$—i.e., introduce vertices of degree 2—to obtain a new map M' for which $U(M')$ is simple. Hence, to 4-color M, it suffices to 4-color M'.*

Thus, in coloring a map M we may always assume that $U(M)$ is simple. Note, however, that by making $U(M)$ simple we may force $D(M)$ to be non-simple. For example, if $U(M)$ consists of a loop, $D(M)$ is a simple edge. Subdividing $U(M)$ introduces parallel edges in $D(M)$.

If G is a graph, we write $S(G)$ for the simple graph obtained from G by deleting loops and replacing parallel edges by a single edge. Obviously, we have the following result:

1.4 LEMMA. *G is k-colorable if and only if $S(G)$ is k-colorable.*

1.5 CONJECTURE C_1: *Every planar graph is* 4-*colorable.*

REMARK: Some misunderstanding can result from not making the distinction between Conjectures C_0 and C_1. Sometimes authors speak of graphs in both cases and refer to coloring regions or vertices as the case may be. Perhaps it is best when using Conjecture C_0 to refer to a map and when using Conjecture C_1 to refer to a graph, the first suggesting a coloring of regions and the second a coloring of vertices. Thus, in the sequel when speaking of equivalent conjectures, whenever we speak of graphs, the equivalence is to Conjecture C_1. The equivalence of Conjecture C_1 and Conjecture C_0 follows from the definition of a dual graph. Characterization of planar graphs in terms of an *abstract* duality was established by H. Whitney **[1]**. In particular he showed that if M is planar, so is $D(M)$.

As a consequence of the easier half of the theorem of Kuratowski **[1]**, proving that the complete graph on five vertices is nonplanar, one can conclude that there are no planar maps in which five countries are pairwise adjacent.

Heawood's proof that any planar map can be 5-colored is inductive and surprisingly simple, and it exemplifies the many ingenious approaches which have been taken in pursuit of the four-color problem. However, rather than prove the sufficiency of five colors, we prefer to use the method of Heawood's proof to show that a planar map containing a region with no more than four sides must be 4-colorable, provided that we first assume it is *irreducible* — i.e., minimally non-4-colorable. Thus, we shall see that every region in an irreducible map has 5 or more sides.

Note that in particular any map with at most 12 regions has some region with no more than four sides. To see this, suppose that the map has n vertices, m edges, and r regions. Then Euler's formula (satisfied by planar maps) gives

$$n - m + r = 2. \tag{1}$$

Assuming without loss of generality that M has no vertices of degree one or two, we always have $3n \leqq 2m$ and if we assume that every region of a 4-colorable map is bounded by at least five edges, then $5r \leqq 2m$. Substitution in (1) gives $m \geqq 30$. Substituting $3n \leqq 2m$ alone in (1) gives $m \leqq 3r - 6$ which for $r < 12$ gives $m < 30$. Hence, a map of less than 12 regions has at least one region bounded by less than five edges.

To color the vertices of the dual graph $D(M)$ of such a map M with four colors, let v be the vertex adjacent to (1) four other vertices, v_1, v_2, v_3, v_4, or (2) three other vertices (the proof of this case is trivial).

By minimality of M, we may assume that on suppressing v and its four incident edges, the vertices of the resulting graph have been colored with four colors, which we denote by c_1, c_2, c_3, c_4. Let this assignment result in giving v_i the color $c_i, i = 1, \cdots, 4$. See Fig. 3.

Now if there is a chain from v_1 to v_3 whose vertices are alternately colored with c_1 and c_3 starting at v_1 and ending at v_3, then there cannot be a chain whose vertices

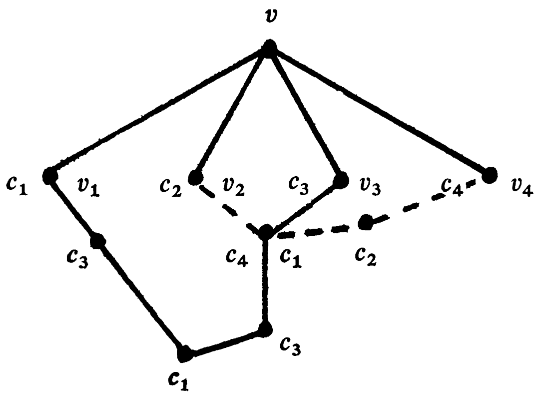


FIG. 3

are alternately colored with c_2 and c_4 starting at v_2 and ending at v_4. Otherwise the two chains must cross (see diagram) at a vertex whose color would conflict in the two chains. Thus, the second chain of alternating colors may have the colors of its vertices reversed. In that case, v_2 could be assigned the color c_4, and the remaining color c_2 would then be assigned to v.

If the first chain starting at v_1 does not terminate at v_3, then the color of its vertices may be reversed, assigning c_3 to v_1, leaving c_1 to be assigned to v. This completes the argument.

In every planar map there is at least one region bounded by five or fewer edges. Otherwise we have $3n \leqq 2m$, $6r \leqq 2m$, and substitution in Euler's formula gives $2m/3 - m + 2m/6 \geqq 2$, a contradiction.

A slight adaptation of the foregoing approach, again applied inductively to a vertex of the dual graph which has five or less neighbors, can be used to prove the following theorem (Heawood [**1**]).

1.6 THEOREM. *Any planar graph is* 5*-colorable.*

Of course, the problem is to show that any planar graph is 4-colorable.

Sketch of Heawood's Argument (Fig. 4). Heawood's counterexample [**1**] is directed at Kempe's chain coloring reversals. He is not concerned with whether one can by a judicious choice recolor some of the vertices. The above example with 25 vertices is known to be 4-colorable by existing theory.

Using the inductive argument on the number of vertices n, assume that every planar graph on $(n - 1)$ vertices is 4-colorable. Consider a graph on n vertices and remove a vertex v (which has five neighbors) and its connecting edges and 4-color the resulting graph on $n - 1$ vertices. Suppose the coloring is as shown. Reinstate v and attempt to color the resulting graph.

There is a b–g chain from 2 to 4. There is also a b–y chain from 2 to 5.

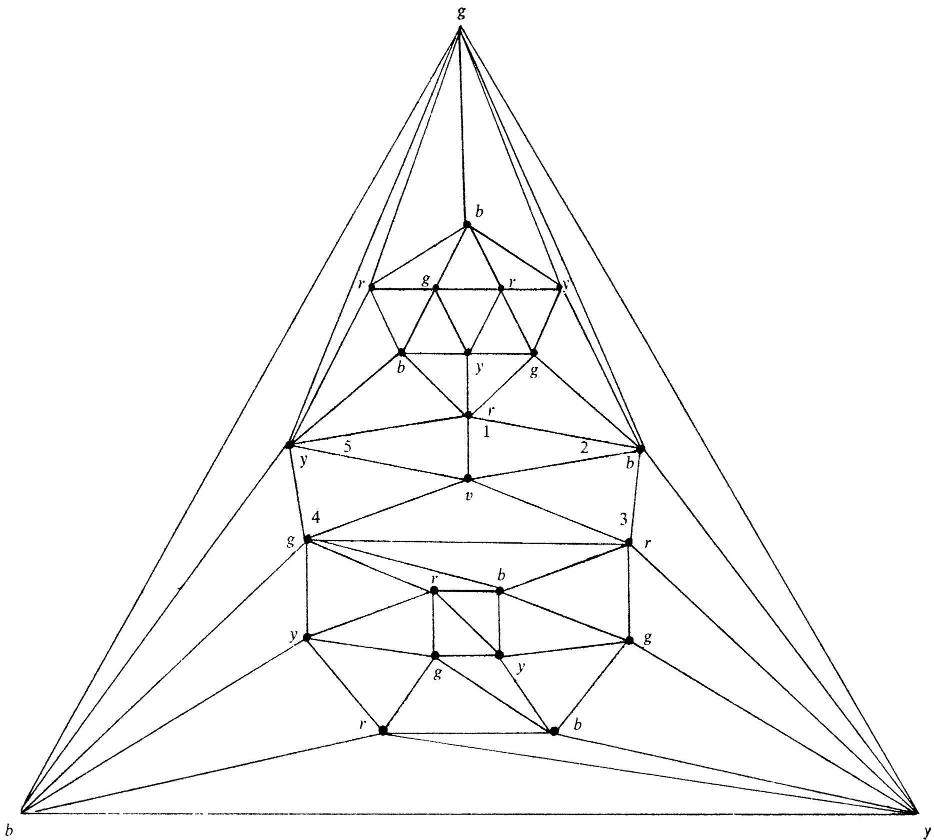


FIG. 4 — Heawood's counterexample to Kempe's proof

Reversal of colors on either chain will not free a color for v. This leaves r in two places. Now there is no r–g chain from 1 to 4. Therefore, one can reverse r to g in the r–g chain starting at 1. But the other r at 3 must also be turned to g or to y to obtain a spare color for v. This is not possible because 4 which has color g is adjacent to 3 which will become colored with g. On the other hand, if we reverse colors on the r–y chain starting at 3, the two vertices of the outer triangle which are connected by an edge would both be assigned r by the r–g and r–y reversals, starting at 1 and at 3 respectively, contradicting proper coloring. Thus, one cannot replace r by g at both 1 and 3 nor by g at 1 and by y at 3. Note that at 1, r cannot be turned to y because it is adjacent to a y at 5. Heawood [**1**] wrote "Unfortunately, it is conceivable that though either transposition would remove an r both may not remove both r's." (It is clear that reversal of colors on the y–r chain starting at 5 followed by a reversal on the r–g chain starting at 1 frees the color y for v, but this does not justify Kempe's argument.) See also Saaty [**1**].

2. Cubic Maps.

2.1 DEFINITION: A graph is **cubic (normal, regular, regular of degree three, trivalent)** if all of its vertices are of degree 3. A map is **cubic (normal, regular, trivalent)** if $U(M)$ is cubic.

2.2 DEFINITION: A graph is **bridgeless** (or doubly edge-connected) if there is no edge whose removal disconnects the vertices (i.e., after any edge is removed, it is still possible to connect any two vertices by a chain). An edge e is called a **bridge** (or **isthmus**) if the set of vertices can be partitioned into two sets T and U such that e is the only edge with one end point in T and the other end point in U.

Obviously, a graph is bridgeless if and only if it has no bridges. A map M is bridgeless if $U(M)$ is bridgeless.

REMARK: In a cubic graph, a loop is counted twice. A cubic graph with a loop must have a bridge.

In coloring maps, we can really assume that the maps are bridgeless as the following argument will show.

2.3 LEMMA. *Let e be an edge of a graph G. Then e is a bridge if and only if e lies on no circuit.*

2.4 LEMMA: *Let e be an edge of a map M (i.e., e is an edge of $U(M)$). Then e is a bridge if and only if e lies on the boundary of exactly one region.*

2.5 THEOREM. *Let M be any map. Then there exists a map M' such that* (i) *M' is bridgeless,* (ii) *M' can be k-colored if and only if M can be k-colored.*

Proofs: Lemmas 2.3 and 2.4 are trivial. We obtain M' by simply shrinking each bridge to a point. By Lemma 2.4, M' satisfies the conclusions of the theorem.

2.6 CONJECTURE C_2: *Every bridgeless cubic planar map is 4-colorable.*

As we indicated in Section 2 of Chapter 1, reduction of the four-color problem to cubic maps is due to Story. A proof of the equivalence of Conjectures C_0 and C_2 is given in Harary's book [**2**, p. 132]. To go from any map to a cubic map, each vertex is blown into a polygon with as many vertices as there are edges incident with the vertex. Out of each of these vertices of the polygon emanates one of the edges. Thus, each vertex is of degree three, and the resulting map is cubic. After coloring the cubic map, the added polygons are contracted back to the vertex to obtain a coloring for the original map.

2.7 DEFINITION: A region is called **odd** (even) if it is bounded by an odd (even) number of edges. A circuit is called odd (even) if its length is odd (even).

REMARK: In Problem E 1756, this MONTHLY, 72 (1965) p. 76, it is shown that in a 4-colored cubic map, the number of odd regions colored by any two colors is even.

2.8 DEFINITION: A map, all of whose vertices have even degree, is said to be **triangle-colored** when its regions can be colored in two colors such that all regions colored with one of the colors are triangles.

2.9 CONJECTURE C_3: *The vertices of a planar triangle-colored map without multiple edges and all of whose vertices have degree four can be* 3-*colored.*

This conjecture is equivalent to Conjecture C_1 (Ore [**1**, p. 126]).

2.10 DEFINITION: We call a map M **triangular** if its dual $D(M)$ is a cubic graph. We shall discuss triangular maps later.

3. Edge Coloring.

3.1 DEFINITION: A (proper) coloring of the edges of a cubic map (called a **Tait-coloring or edge-coloring**) is a 3-coloring of the edges such that all three edges incident with the same vertex have different colors.

3.2 CONJECTURE C_4: *The edges of a bridgeless cubic planar map are* 3-*colorable.*

The equivalence of Conjectures C_2 and C_4 is due to P. Tait [**1**]. Proofs are found in Ball and Coxeter [**1**, p. 226], Ore [**1**, p. 121], and Liu [**1**, p. 253] (in dual form—see Conjecture C_5). A cubic map with a bridge has no Tait coloring. According to a previous remark, if the map has a loop, it has no Tait coloring.

3.3 CONJECTURE C_5: *The edges of a triangular map can be colored with three colors so that the edges bounding every triangle are colored distinctly.*

Let us actually see how to construct Tait-colorings from region-colorings and region-colorings from Tait-colorings.

Suppose that we are given a bridgeless cubic map M whose regions have been 4-colored using colors $0, 1, 2, 3$. We may then Tait-color the edges according to the following scheme:

Color edge:	if edge lies on boundaries of	regions colored:
α		0 and 1, or 2 and 3
β		0 and 2, or 1 and 3
γ		1 and 2, or 0 and 3

It is easy to check that this scheme actually works.

Conversely, suppose we are given a Tait-coloring of the edges of M using the colors α, β, γ. Those edges labelled α and β form disjoint simple circuits (of even length) which we call α-β circuits.

Now every region R of M is contained in the interiors of either an odd or an even number of α-β circuits. Let us pre-color R with $1'$ if R is contained in an odd number of α-β circuits and $0'$ if R is contained in an even number of α-β circuits. Similarly, we have α-γ circuits and every region R of M is contained in either an even or odd number of α-γ circuits. In the former case, we pre-color R with $0''$ and in the latter case with $2''$. Now color the regions of M according to the following scheme:

Color region:	if region has already been	pre-colored:
0		$0'$ and $0''$
1		$1'$ and $0''$
2		$0'$ and $2''$
3		$1'$ and $2''$

Thus, each region is pre-colored twice and two regions are colored the same if and only if both of their pre-colorings are the same.

This yields a proper coloring of the regions. For if two regions R_1 and R_2 have a common edge e, then e may be colored either α, β, or γ. If e is colored β, then e lies on exactly one α-β circuit C which contains either R_1 or R_2, but not both, in its interior. Hence, R_1 and R_2 are pre-colored with $1'$ and $0'$ or $0'$ and $1'$, respectively. Thus, they cannot be colored the same. The same argument holds when e is colored γ. If e is colored α, then e lies on both an α-β and an α-γ circuit so the argument above shows that both pre-colorings of R_1 and R_2 are different, and we may again conclude that R_1 and R_2 are colored differently.

3.4. DEFINITION: The **line** or **interchange** graph $L(G)$ of a given graph G (without multiple edges) is obtained by associating a vertex with each edge of the graph and connecting two vertices by an edge if and only if the corresponding edges of the given graph are adjacent.

3.5 CONJECTURE C_6: *The vertices of the line graph of a bridgeless cubic planar map can be colored with* 3 *colors.*

The equivalence of Conjectures C_4 and C_6 is trivial.

For more information on line graphs, see Ore [**1,** p. 124]. Ore quotes the following two results of Sedláček [**1**]:

3.6 THEOREM. *A planar graph G has a planar line graph L(G) if and only if no vertex in G has degree exceeding* 4, *and when a vertex has degree* 4, *then its removal must disconnect the graph.*

3.7 THEOREM. *If G is nonplanar, then L(G) is nonplanar.*

4. Hamiltonian circuits.

4.1 DEFINITION: A graph is said to be **Hamiltonian** if it has a simple circuit called a **Hamiltonian circuit** which passes through each vertex exactly once.

It is clear that if M is a cubic map and $U(M)$ has a Hamiltonian circuit C, then the edges of the map M can be 3-colored. (Recall that in the cubic graph $U(M)$, there must be an even number of vertices because $3n = 2m$ where m is the number of edges. Thus two colors are alternately assigned to the edges of C, and the third color is assigned to the remaining edges.) This implies that M is 4-colorable.

4.2 CONJECTURE C_7: *Every Hamiltonian planar graph is* 4-*colorable.*

Proof of the equivalence of Conjectures C_1 and C_7 is due to Whitney [**1**]. It is clear that if a planar graph is 4-colorable, then also every Hamiltonian planar graph is 4-colorable. The proof of the converse is not obvious. It depends on the result of Whitney [**1**] that every maximal planar graph (see 6.4) has a Hamiltonian circuit.

4.3 CONJECTURE C_8: *It is possible to* 4-*color the vertices of a planar graph consisting of a regular polygon of n sides with non-crossing diagonals dividing the interior of the polygon into triangles and with non-crossing edges dividing the exterior of the polygon into triangles.*

Whitney [**1**] proves the equivalence of Conjecture C_8 and Conjecture C_0. Conjecture C_8 is essentially Conjecture C_7. For a discussion of the following conjecture, see Ball and Coxeter [**1,** p. 226], Petersen's 1891 paper, page 219, and Ore [**1,** p. 121].

4.4 Conjecture C_9: *In a bridgeless cubic map it is possible either to tour all the vertices by a Hamiltonian circuit or to make a group of mutually exclusive subcircuits (subtours) of the vertices in several even-length simple circuits.*

The equivalence of this conjecture with Conjecture C_4 is essentially due to Tait who preceded Petersen and is easy to establish. We give a sketch here. We assume that the edges have been 3-colored. We start at any vertex and follow a chain whose edges alternate with two colors. Such a chain must return to its starting point to form a simple circuit. The reason is that since the degree of each vertex is 3, and the three edges meeting at any vertex have all three colors, returning to an intermediate vertex would mean that the tour would have used the third color contrary to assumption. Because of connectedness, the tour must return to the starting vertex and hence it must have even length. If all the vertices are included in this tour, we have a Hamiltonian circuit of even length. Otherwise, the process is repeated on the remaining vertices to form another simple circuit (subtour) disjoint from the first and so on.

If, on the other hand, we have the disjoint subtours of even length, we color their edges alternately with two colors and assign the third color to edges not on any subtour. In this manner we can 3-color the edges.

4.5 DEFINITION: Let G be a graph and G' a subgraph. We call G' a **section graph** of G if two vertices are adjacent in G' whenever they are adjacent in G.

Thus, a section graph of G is determined by its set of vertices. Let G be a graph which has been 4-colored (say red, blue, yellow, and green).

4.6 DEFINITION: A **Kempe chain** in G is a connected component of a section graph determined by all of the vertices in two of the colors.

4.7 DEFINITION: Let M be a map which has been 4-colored. Then a collection of regions in M forms a **Kempe chain** in M if its dual is a Kempe chain in (DM).

4.8 DEFINITION: A family of disjoint simple closed curves of even length including every vertex in M is called a **Tait cycle.**

Suppose we have a red-blue Kempe chain K in a cubic map M. Let R_1 be a region of K. If R_2 is a region not in K and R_1 and R_2 are adjacent, then R_2 must be colored either yellow or green. Thus every edge on the boundary of K separates a red or blue face from a yellow or green face, and hence, by our construction scheme, we can Tait-color the edges of M using only two colors for the boundary edges of K. This implies that the boundary of K consists of a family of even-length simple closed curves. Moreover, since every vertex in M is on the boundary of three differently colored faces, every vertex belongs to one, and only one, of the simple closed curves in the boundary of a Kempe chain. Thus a 4-colored cubic map has a Tait cycle. Note that in fact the coloring has three Tait cycles, one for each separation of the four colors into pairs.

One can reformulate Conjecture C_9 in terms of Tait cycles. We use this nomenclature later on in the paper.

4.9 DEFINITION: A graph is said to be p-**connected** if each pair of vertices v and w is connected by at least p chains which have no vertices in common other than v and w.

A graph G is p-connected if and only if G is not disconnected or made trivial by the removal of $p - 1$ or fewer vertices.

There are special types of graphs which are known to be Hamiltonian; e. g., complete graphs with $n \geqq 3$ vertices. As another example, Tutte [3] has proved that a 4-connected planar graph with at least two edges has a Hamiltonian circuit. Whitney [1] has shown that if M is a cubic map then $D(M)$ has a Hamiltonian circuit.

That not every planar graph is Hamiltonian is illustrated in Fig. 5 which shows a graph with 20 vertices and 12 pentagonal faces. It is easy to show that this graph is 4-colorable.

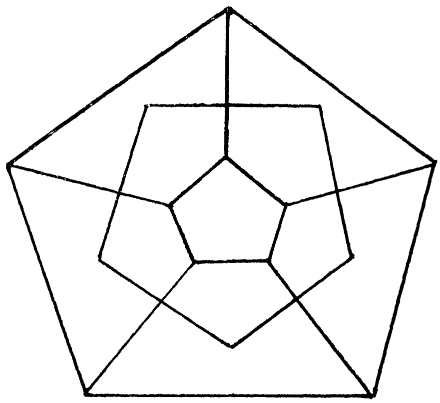
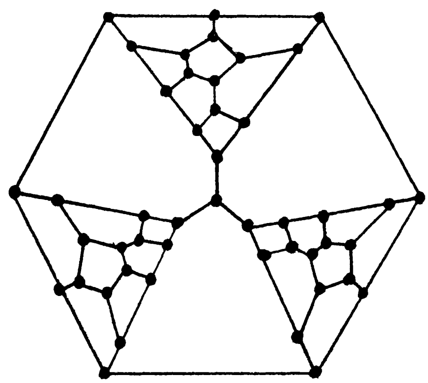


FIG. 5



Tutte's counterexample
FIG. 6

Dirac [3] has shown that each graph on n vertices, the degree of each vertex of which is at least $n/2$, has a Hamiltonian circuit. L. Posa [1] proved that a graph on $n \geqq 3$ vertices has a Hamiltonian circuit if for each integer i with $1 \leqq i < n/2$, the number of vertices of degree not exceeding i is less than i. See the book by B. Roy [1] for additional results.

Tait [3] once conjectured that every 3-connected planar graph is Hamiltonian but Tutte [3] gave a counterexample (Fig. 6) with 46 vertices. Had Tait's conjecture been true, the truth of Conjecture C_0 would have followed. For as we shall see in the last chapter, to prove Conjecture C_0, it suffices to show that every cubic map M with $U(M)$ 3-connected can be 4-colored. But Tait's conjecture would imply that every such map had a Hamiltonian circuit and hence was 4-colorable.

Tait himself did not supply an adequate proof as to how the four-color conjecture would be true if his conjecture were true. He thought his conjecture was true from all the evidence he had. Chuard [1] went on to "complete" the story in 1932. Doubts as to the validity of Chuard's claim were expressed by Pannwitz [1].

In any event, Tutte's example has made the entire debate academic as a means of settling the four-color conjecture.

5. Flow ratio.

5.1 DEFINITION: A graph is called **directed** or **oriented** if each edge is assigned a direction (indicated by an arrow) from one of its end vertices toward the other.

5.2 Definition: The flow ratio of a simple circuit is the ratio m_1/m_2, where m_1 and m_2 are the numbers of edges of the circuit directed clockwise and counter-clockwise around the circuit with $m_1 \geqq m_2$. If $m_1 < m_2$ then the roles of m_1 and m_2 are interchanged (the flow ratio may be $+\infty$).

5.3 Conjecture C_{10}: *The edges of a planar graph can be oriented in such a way that the flow ratio of each cycle is at most* 3.

A proof of the equivalence of Conjectures C_1 and C_{10} is due to Minty **[1]**. Actually Minty proves the equivalence of k-colorability to the fact that the flow ratio of each circuit does not exceed $k - 1$.

5.4 Conjecture C_{11}: *The edges of a planar graph can be so directed that for any circuit C with $m(C)$ edges and any direction associated with the circuit (clockwise or counter-clockwise), the number of edges of C oriented opposite to the given direction and denoted by $m_1(C)$ satisfies*

$$m_1(C) \geqq \tfrac{1}{4}m(C).$$

This is obviously equivalent to the previous result (see Ore [**1,** p. 104]).

6. Partition of vertices; chromatic number. When the vertices of a planar graph are 4-colored, they are divided into four disjoint sets such that the vertices in each set are assigned the same color and no two vertices of the same color are joined by an edge. Clearly a graph can be 4-colored if and only if it is 4-partite. Each pair of these four sets, together with their interconnecting edges, forms a bipartite graph.

6.1 Definition: A planar graph is said to have **bipartite dichotomy** if there is a disjoint decomposition of its vertices into two sets such that each set defines a bipartite graph.

We sometimes call a bridge a **separating edge.**

6.2 Conjecture C_{12}: *The dual of a planar map without separating edges has a bipartite dichotomy.*

6.3 Conjecture C_{13}: *Any planar graph without loops has a bipartite dichotomy.*

See Ore [**1,** page 105] for the equivalence of these conjectures to Conjecture C_1.

6.4 Definition: A graph G is called **maximal planar** if it is planar and has no loops and no multiple edges and it is not possible to add a new edge to G without violating one of these restrictions.

Remark. The following statements are equivalent:

(i) G is maximal planar;

(ii) For every map M with $G = U(M)$, M is triangular;

(iii) There exists a triangular map M with $G = U(M)$.

It is known that every uniquely 4-colorable planar graph is maximal planar (Harary [**2**, page 140]).

6.5 Conjecture C_{14}: *Every maximal planar map has a bipartite dichotomy.*

The equivalence of Conjectures C_1 and C_{14} is proved in Ore [**1**, page 122].

6.6 Definition: The **chromatic number** $\chi(G)$ of a graph G is the minimum number of disjoint subsets into which its vertices can be partitioned such that no two vertices in the same subset are adjacent.

6.7 Conjecture C_{15}: *The dual graph G of a planar map satisfies $\chi(G) \leqq 4$.*

Remark. Ershov and Kozhukhin [**1**] have shown that a connected graph G with n vertices and m edges satisfies the following bounds on its chromatic number (using $[x]$ and $\{x\}$ to denote the integral and fractional parts of x, respectively):

$$-\left[-\frac{n}{[(n^2-2m)/n]}\left(1-\frac{\{(n^2-2m)/n\}}{1+[(n^2-2m)/n]}\right)\right] \leqq \chi(G) \leqq \left[\frac{3+\sqrt{9+8(m-n)}}{2}\right].$$

If the vertices of a graph G are numbered $i = 1, \cdots, n$ according to the decreasing order of their degree d_i, and if k is the last number of a vertex which satisfies

$$k \leqq d_k + 1, \text{ then } \chi(G) \leqq k.$$

It follows from this that $\chi(G)$ is at most equal to the highest degree of any vertex plus unity. Welsh and Powell [**1**] give an algorithm for coloring the vertices of a graph with a number of colors equal to the bound k.

6.8 Definition: A graph G is called **critical**, or **vertex-critical,** (Dirac [**2**]) if after the removal of any vertex v and its connecting edges we have

$$\chi(G-v) < \chi(G).$$

G is k-critical if $\chi(G) = k$ (in which case, for every v, $\chi(G-v) = k-1$). A graph is **edge-critical** if similar relations hold on removing an edge.

It is known (Ore [**1**, p. 164]) that the removal of a complete subgraph cannot separate a critical graph. Dirac [**3**] has shown that if a graph G is k-critical with $k \geqq 3$, then either G has a Hamiltonian circuit or the circumference of G is $2k-2$. He has also proved that every k-chromatic graph contains a critical k-chromatic subgraph.

6.9 Definition: The **chromatic index** $q(G)$ of a graph G is the smallest number of colors necessary to color its edges so that no two adjacent edges have the same color.

Thus $q(G) = \chi[L(G)]$ when G is simple.

6.10 Definition: A **p-graph** is a graph with multiple edges between its vertices such that no two vertices are jointly incident with more than p edges.

Vizing [1] and Shannon [1] have shown that if d_m is the maximum degree of any vertex in a graph, then we have:

$$d_m \leqq q(G) \leqq \text{Min}\left(p, \left[\frac{d_m + 1}{2}\right]\right) + d_m.$$

It follows that if G has no multiple edges, $q(G)$ is either d_m or $d_m + 1$.

6.11 CONJECTURE C_{16}: *Let G be a planar bridgeless cubic graph. Then* $q(G) = 3$.

This conjecture is just a restatement of Conjecture C_4.

7. Partitions of edges; factorable graphs.

7.1 DEFINITION: A graph (or map) is k-**factorable** if its edges can be partitioned into edge disjoint subsets in such a way that in each subset any vertex meets exactly k edges of that subset. See König [**1**, pp. 155–195].

7.2 CONJECTURE C_{17}: *Every cubic bridgeless planar map is* 1-*factorable.*

This conjecture, first formulated by Tait in 1884, is obviously equivalent to Conjecture C_4. See also Harary [**2**, p. 135].

7.3 CONJECTURE C_{18}: *The dual of every connected planar map is the sum of three edge-disjoint subgraphs such that each vertex has either an even number of edges incident with it from each of the three subgraphs or it has an odd number from each of them.*

The equivalence of Conjectures C_1 and C_{18} is given in Ore [**1**, p. 103]. Alternatively, one can give a direct proof that Conjectures C_{18} and C_4 are equivalent.

8. Vertex characters.

8.1 CONJECTURE C_{19}: *It is possible to associate a coefficient* $k(v)$ *equal to* $+1$ *or* -1 *with each vertex in a bridgeless cubic map in such a way that* $\Sigma k(v) = 0 \pmod 3$, *where the summation is taken over the vertices occurring in the boundary of any region.*

Heawood [**2**] proved the equivalence of this conjecture with Conjecture C_4. A reformulation of this conjecture would be to take the above congruences and require a solution, for all of them taken together, none of whose members is congruent to zero modulo 3. Thus, if A is the $(0, 1)$ region-vertex incidence matrix, the above is equivalent to the existence of a vector X such that $AX = 0 \pmod 3$, where none of the components of X is zero.

To see how this conjecture implies Conjecture C_4, label the edges of the map a, b, or c, such that the three edges incident with each vertex are labelled differently and the ordering of the edges $a \to b \to c$ is a clockwise rotation if $k(v) = +1$ and counter-clockwise if $k(v) = -1$. This labelling is consistent if and only if the vertex character assignment is proper; i.e., for each region $\Sigma k(v) = 0 \pmod 3$.

Using a computer code, Yamabe and Pope developed an assignment method for cubic maps of up to 36 vertices and illustrated their method by an example in their brief paper [1].

8.2 CONJECTURE C_{20}: *It is always possible repeatedly to cut off corners (replace a vertex by a triangle) from a convex polyhedron so that eventually a polyhedron is obtained whose faces have a number of edges which is divisible by* 3.

This conjecture due to Hadwiger [2] is a modification of the previous conjecture of Heawood. Cutting off corners yields vertices of degree 3, and hence the truth of the last conjecture implies Heawood's conjecture (Conjecture C_{19}). The proof in the reverse direction is more elaborate.

Conjecture C_{20} may have been suggested by a result of Heawood [2] in which he proved that if the regions of a map could each be subdivided (by the simple operation of adding a new edge to connect some pairs of adjacent edges thereby forming triangles) into new regions such that all the regions are bordered by edges whose number is congruent to zero mod 3, then the map is 4-colorable.

Heawood first shows constructively that such a map is 4-colorable. Then he shows that any 4-coloring of the constructed map is also a 4-coloring of the initial map by removing the edges.

9. Modular equations and Galois fields. Let $GF(k)$ denote the Galois field of order k. Thus, k is a prime power and $GF(k)$ is the unique (finite) field with k elements. Obviously, one may view a k-coloring of the vertices (or edges or regions) of a graph (or map) as an assignment of an element of $GF(k)$ to every vertex (or edge or region) of the graph (or map).

We shall consider in this section the cases $k = 2, 3, 4$. When $k = 4$, note that two elements in $GF(k)$ are equal if and only if their sum is zero. Thus, if we assign to every edge e in a bridgeless map which has been 4-colored, the sum of the colors of the two regions adjacent to e, this sum will never be zero. We may give this a matrix formulation as follows: List the edges $e_1, \cdots, e_m$ and regions $r_1, \cdots, r_n$ of a bridgeless map M. Let B be the matrix defined by putting $B_{ij} = 1$ if e_i is in the boundary of r_j and putting $B_{ij} = 0$ otherwise. Thus, each row of B contains two unit elements. B is sometimes called the **edge-region incidence matrix** of M, or simply an **incidence matrix.**

Suppose M is 4-colored. Then define a column vector $Z = (z_1, \cdots, z_n)$, where z_j is the color of the jth region, and each z_j belongs to $GF(4)$. The matrix product BZ is a column vector $P = (p_1, \cdots, p_m)$, and each p_i is the sum of two distinct elements in $GF(4)$ since e_i is on the boundary of two distinctly colored regions. Hence, each p_i is non-zero.

Now we can state the following conjecture due to O. Veblen [1]:

9.1 CONJECTURE C_{21}: *Let B be any edge-region incidence matrix. Then there is a* *column vector $Z = (z_1, \cdots, z_n)$ with entries z_j in $GF(4)$ such that the matrix product BZ has no zero entries.*

The discussion above shows that Conjecture C_{21} is equivalent to Conjecture C_0 since the existence of the column vector Z provides us with a 4-coloring of the map.

We can also form an edge-vertex incidence matrix for a graph G and make a conjecture as before. Obviously, this procedure is equivalent to the above by duality.

We may now restate Conjecture C_{19} using the Galois field $GF(3)$. We can also define a region-vertex incidence matrix for a map M and then make the following conjecture:

9.2 CONJECTURE C_{22}: *Let B be the region-vertex incidence matrix of a map M. Then there is a column vector $Z = (z_1, \cdots, z_n)$ with each z_j in $GF(3)$ such that BZ is identically zero but no z_j is equal to zero.*

In an interesting generalization of these ideas, Tutte **[6]** has developed a framework for merging the two questions of 4-colorability and Tait-colorability of a planar map. Some of the work is motivated by a conjecture due to Tutte that any bridgeless cubic map with no Tait-coloring can be reduced to a Petersen graph (illustrated later) by deleting some edges and contracting others to single vertices. (The converse of this conjecture is known to be false—see Watkins **[1]**). It leads to the classification of 2-blocks where the term k-block refers to a set of points of a projective geometry $PG(q, 2)$ over the Galois field $GF(2)$ whose dimension is $\geqq k$. A k-block is **tangential** if it cannot be converted to a similar k-block by a particular process of projection. It is not known if any tangential 2-blocks (sets of points in $PG(q, 2)$ that meet every $(q-2)$ space) other than the following three exist:

— The Fano block (the plane which has exactly 7 points),

— The Desargues block (a 3-dimensional 2-block consisting of 10 points lying in three's on 10 lines in a Desargues configuration), and

— The Petersen block (this is the only 5-dimensional 2-block) which is an embedding closely related to the Petersen graph, and its existence is associated with the non-existence of a Tait coloring of the Petersen graph. In a private communication, W. T. Tutte has informed me that Mr. Biswa T. Datta of Ohio State University proved in his Ph.D. thesis that there are no 6-dimensional tangential 2-blocks.

That many excellent mathematicians have constructed erroneous proofs of the four color conjecture is perhaps a measure of the difficulty and subtlety of the problem. For example, in a recent paper, J. M. Thomas **[1]** attempts to prove the four color conjecture. His argument is based on Veblen's modular equation approach. However, we can point out the fault with his paper in simpler terms. Essentially, his line of argument is the slitting operation which he describes as follows:

> Let side s bond faces K, L which are unequal and do not join. Slit side s lengthwise so that its two pieces border a channel making K, L into a single face in map M' with $n-1$ faces. Let K', L' be the sums of the unknowns

at the vertices of K, L with those u, v at s omitted. A root of map system X in which u, v are numbered $+$, $-$ becomes a root of the system $X' + (K' + 1) + (L' - 1)$, where X' is the map system for M'. Conversely, such a special root of X' augmented by the values $+1$, -1 for u, v becomes a root of the map system X.

The difficulty occurs in the inductive step when he claims that he can extend a root for the slit back into a root for the original map. This means that the two regions along the slit would have to be differently colored in the slit map. If this were true, the four color conjecture would follow trivially. Unfortunately, this part of the paper appears to be as difficult as any of the other formulations.

10. Hadwiger's Conjecture.

10.1 DEFINITION: An **edge contraction** of a graph G is obtained by removing two adjacent vertices u and v and adding a new vertex w, adjacent to those vertices to which u or v was adjacent. A graph G is **contractible** to a graph H if H can be obtained from G by a sequence of edge contractions. We shall also call H a **contraction** of G. Note that G is contractible to H if and only if there is a connected homomorphism (see Ore [**2**, p. 85]) from G onto H.

10.2 HADWIGER'S CONJECTURE: *Every connected k-chromatic graph is contractible to a complete graph on k vertices.*

10.3 CONJECTURE C_{23}: *Hadwiger's conjecture is true for* $k = 5$.

The equivalence of this conjecture and Conjecture C_1 is due to K. Wagner [**4**]; a simpler proof of the equivalence has been given by R. Halin [**2**]. The truth of this conjecture for $k < 5$ was established by G. A. Dirac [**2**].

An equivalent statement of the above conjecture using the notion of conformal graphs, (Ore [**1**, p. 26]) is due to Halin [**1**].

One may use the notion of contraction to formulate a criterion for planarity which is dual to the well-known result of Kuratowski. The following theorem was discovered independently by Harary and Tutte [**1**] and by Wagner [**3**]. It was also probably known to Ringel, since he realized that any contraction of a planar graph is planar. Let K_5 denote the complete graph on 5 vertices and $K_{3,3}$ the complete bipartite graph on two sets each with three vertices.

10.4 THEOREM. *A graph is planar if and only if it has no subgraph contractible to* K_5 *or* $K_{3,3}$.

11. Amalgamation.

11.1 DEFINITION: A graph G is a **conjunction** of two disjoint graphs G_1 and G_2 if it is obtained by taking an edge $e_1 = \{a_1, b_1\}$ in G_1 and an edge $e_2 = \{a_2, b_2\}$ in G_2, identifying (or coalescing) a_1 with a_2, deleting the edges e_1 and e_2, and introducing a new edge $e_3 = \{b_1, b_2\}$.

11.2 Definition: Suppose that we are given two sets A_1 and A_2 of vertices of a simple graph G such that no edge is incident with vertices of both sets. Let μ be a 1-1 correspondence between the elements of the two sets. A **μ-coalition** of G is the graph obtained from G by identifying corresponding vertices in A_1 and A_2. Vertices which are connected by two edges as a result of the identification are connected by a single edge in the μ-coalition by eliminating one of the edges.

11.3 Remark: Conjunctions and μ-coalitions do not decrease chromatic numbers.

11.4 Definition: Let G be a conjunction of G_1 and G_2 as in 11.1. Consider a 1-1 correspondence μ between sets A_1 and A_2 where $a_1 \in A_1$, $a_2 \in A_2$, $\mu(a_1) = a_2$, and $\mu(b_1) \neq b_2$. The graph obtained by applying this μ-coalition to G is called a **merger.**

11.5 Remark: A conjunction is a merger in which $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$.

11.6 Definition: A graph G is called an **amalgamation** of the disjoint graphs $G_1, \cdots, G_p$ if it is derived by repeated mergers of the G_i. A **k-amalgamation** is an amalgamation of graphs G_i, $i = 1, \cdots, p$, each of which is a complete graph on k vertices.

11.7 Conjecture C_{24}: *No 5-amalgamation is planar.*

The equivalence to Conjecture C_1 is given in Ore [**1**, p. 180] utilizing ideas from Hajös [**1**].

12. Other algebraic and number-theoretic approaches. The first two approaches give statements equivalent to the four color problem but for specific maps. They are useful in applying computer methods, to test whether a given map of a reasonable size (within the bounds of computer capability and of time) is 4-colorable or not. The third and fourth approaches are number-theoretic.

Diophantine Inequalities. Let the regions of a planar map be labelled $r = 1, 2, \cdots, n$. Let the variable t_r be integer-valued $0 \leqq t_r \leqq 3$. Thus, t_r assigns one of the four colors, labelled 0, 1, 2, 3 to the region whose number is r. If two regions r and s have a boundary in common, then $t_r - t_s \neq 0$. Such a relation is written down for every pair of adjacent regions. The relation for one pair may be reduced to two inequalities as follows:

$$\textit{either } t_r - t_s \geqq 1 \quad \textit{or} \quad t_s - t_r \geqq 1.$$

This pair of inequalities may now be written as

$$t_r - t_s \geqq 1 - 4\delta_{rs} \text{ and } t_s - t_r \geqq -3 + 4\delta_{rs},$$

where $\delta_{rs} = 0$ or 1. We obtain a system of such inequalities by allowing r and s to vary from 1 to n. The problem then is to determine whether it is possible to choose the integers $0 \leqq t_r \leqq 3$, $r = 1, \cdots, n$, and the binary variables δ_{rs}, $r, s = 1, \cdots n$, such that the system of inequalities has a solution. If not, then our assumption that t_r take on only four values is untenable.

We have now proved that the following conjecture is equivalent to Conjecture C_0:

12.1 CONJECTURE C_{25}: *For every planar map the corresponding system of diophantine inequalities formulated here has a solution.*

According to G. Dantzig, this formulation was informally communicated to him by Ralph Gomory of Integer Programming fame.

Optimization. Another formulation is due to Dantzig himself [**1**, p. 549]. Referring back to Conjecture C_9, consider each subtour of a cubic map, and starting at any vertex, assign a direction to an edge. Then assign the opposite direction to the edge of the circuit adjacent to it and continue around the (even-length) circuit in this manner so that for each vertex the two edges incident with it (now called arcs) are directed away from it or directed towards it.

Label the vertices $1, 2, 3, \cdots, n$. For any pair of adjacent vertices i and j, we write $x_{ij} = 1$ if there is an arc directed from i to j. Otherwise we write $x_{ij} = 0$. Thus we always have

$$0 \leqq x_{ij} \leqq 1.$$

We also write

$$\sum_j x_{ij} = 2\delta_i \text{ where } \delta_i = 1 \text{ or } 0,$$

expressing the fact that there must be two arcs on some subtour leading away from vertex i if $\delta_i = 0$ and none if $\delta_i = 1$. The problem now is to find δ_i and x_{ij} which satisfy these three conditions. The three conditions constitute a bounded Transportation Problem, and so one may attempt to apply the techniques of integer programming to this formulation.

Arrangements. Consider the sum $a_1 + a_2 + a_3 + \cdots + a_n$. If we add brackets to this sum as one usually does to evaluate a sum, one never adds the brackets in such a way that the numbers are added more than two at a time. The result is called an **arranged sum.** For example, $a_1 + a_2 + a_3 + a_4$ can be written as an arranged sum

(1) $$((a_1 + a_2) + (a_3 + a_4))$$

or

(2) $$(((a_1 + a_2) + a_3) + a_4), \qquad \text{etc.}$$

We can define a **partial sum** to be the sum within any pair of brackets; e.g., in (2) the partial sums are

$$(a_1 + a_2), (a_1 + a_2 + a_3), (a_1 + a_2 + a_3 + a_4).$$

In (1) the partial sums are

$$(a_1 + a_2), (a_3 + a_4), (a_1 + a_2 + a_3 + a_4).$$

12.2 CONJECTURE C_{26}: *If a sum of n numbers is expressed in any two ways as an arranged sum, then one can choose integer values for the a_i's in such a way that no partial sum of either arranged sum is divisible by* 4.

For example: for (1) and (2) $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 2$. The equivalence of conjectures C_0 and C_{26} is due to H. Whitney [7].

Sequences.

12.3 DEFINITION: A **cartesian sequence** is a finite sequence $c(0), c(1), \cdots$ of four colors such that

(i) $c(r) \neq c(r + 1)$, $r = 0, 1, 2, \cdots$,

i.e., the same color never appears in two consecutive positions.

(ii) $c(2r) \neq c(2r + 2)$, $r = 0, 1, 2, \cdots$, is also cartesian.

12.4 CONJECTURE C_{27}: *Given any integer n and an arbitrary increasing sequence of integers $0 \leqq i_0 < i_1 < \cdots < i_m \leqq n$, $m \leqq n$, there exists a cartesian sequence $c(s)$, $s = 0, 1, 2, \cdots, n$, such that the subsequence $d(s) = c(i_s)$ is also cartesian, $s = 0,1, \cdots, m$.*

The equivalence of Conjectures C_0 and C_{27} is discussed by B. and R. Descartes in [**1**].

13. Chromatic polynomials.

13.1 DEFINITION: Let $P_r(\lambda)$ be the number of ways to color an r-country map in at most λ colors. Then $P_r(\lambda)$ is called the **chromatic polynomial** of the map. It is clear that a chromatic polynomial may correspond to many maps with r countries and that a classification of r-country maps is essential in order to give $P_r(\lambda)$ more precise meaning; i.e., the number of ways to color two r-country maps can be different.

13.2 CONJECTURE C_{28}: *For any r-country planar map, $\lambda = 4$ is not a root of $P_r(\lambda) = 0$.*

Conjectures C_0 and C_{28} are clearly equivalent. Chromatic polynomials are due to G. D. Birkhoff [**1**] and to H. Whitney [**4**]. A chromatic polynomial is a counting method of testing the 4-colorability of a map.

In 1946 Birkhoff and Lewis [**1**] considered cubic maps (for these $P_r(0) = P_r(1) = P_r(2) = 0$) and gave the following conjecture:

13.3 CONJECTURE C_{29}:

$$(\lambda - 3)^r \ll \frac{P_{r+3}(\lambda)}{\lambda(\lambda - 1)(\lambda - 2)} \ll (\lambda - 2)^2 \quad \textit{for } \lambda \geqq 4.$$

They were only able to show this for $0 \leqq r \leqq 8$. The double inequality has the following meaning: If $f(\lambda)$ and $g(\lambda)$ are polynomials, then $f(\lambda) \ll g(\lambda)$ if and only if the coefficients of $f(\lambda)$ are non-negative and not greater than the corresponding coefficients of $g(\lambda)$. Such a relation with an additional condition such as $\lambda \geqq 4$ means that the relation holds with λ replaced by $\lambda - 4$. Note that Conjecture C_{29} implies Conjecture C_{28}. Thus, Conjecture C_{29} is a strong form of Conjecture C_0.

Rota **[1]** has proved that the coefficients of every chromatic polynomial alternate in sign. Read **[2]** conjectured that in their absolute values, these coefficients strictly increase and then strictly decrease.

We now give some interesting results due to W. T. Tutte **[8]** and **[10]** on chromatic polynomials. Let M be a triangular map with k vertices. Then the chromatic polynomial of M, $P(M, \lambda)$, with respect to vertex-coloring satisfies the relation

$$|P(M, 1 + \tau)| \leqq \tau^{5-k},$$

where $\tau = (1 + \sqrt{5})/2 = 1.618$, the "golden ratio" which is one of the solutions of the quadratic equation

$$x^2 = x + 1.$$

Tutte gives this result as a theoretical explanation of the empirical observation that $P(M, \lambda)$ appears to have a zero near $\lambda = 1 + \tau$. Note that there are no λ-colorings for the case where an edge forms a loop. For any loopless triangular map T, Tutte **[10]** shows that $P(T, \tau + 2) > 0$. Since $\tau + 2 = 3.618$, this result tells something of the behavior of $P(T, \lambda)$ near $\lambda = 4$. It is known that $P(T, \lambda)$ is not positive throughout the interval $\tau + 2 < \lambda < 4$.

If the map consists of triangles except for one region which is an m-gon with $2 \leqq m \leqq 5$, then

$$|P(M, 1 + \tau)| \leqq \tau^3 + m - k.$$

Recently, Tutte **[12]** has shown that if M is a triangular map with n vertices, then

$$P(M, \tau + 2) = (\tau + 2)\tau^{3n-10}P^2(M, \tau + 1).$$

CHAPTER 3. REDUCIBILITY

1. Irreducible graphs and maps.

1.1 DEFINITION: We call a 5-chromatic planar map (graph) **irreducible** if any other planar map (graph) with fewer regions (vertices) has a chromatic number less than 5.

Thus an irreducible planar map or graph is minimal 5-chromatic.

Suppose that an irreducible map or graph exists. We shall be able to show that it **must** have certain properties which we shall call **forced**—for example, an irreducible map is forced to have simply connected regions. On the other hand, we shall show that an irreducible map may be assumed without loss of generality, to have certain **optional** properties; i.e., if an irreducible map exists, then we may construct an irreducible map possessing the optional property. For example, if an irreducible map exists, then we may construct an irreducible cubic map from it.

1.2 CONJECTURE C_{30}: *There are no irreducible graphs.*

Clearly, if Conjecture C_1 is false, then 5-chromatic planar graphs exist and, hence, so does a 5-chromatic planar graph with a minimal number of vertices. Conversely if an irreducible graph exists, then it is a 5-chromatic planar graph so Conjecture C_1 is false.

We have two main reasons for studying irreducible maps (aside from trying to show that they don't exist). First of all, in order to show that every map is 4-colorable, it suffices to show that every irreducible map is 4-colorable and hence we may assume that the map we are trying to 4-color has any forced or optional property. Secondly, we study irreducible maps in hopes of raising the Birkhoff number whose definition follows:

1.3 DEFINITION: We define the **Birkhoff number** N to be the minimum number of regions (vertices) in an irreducible map (graph).

By the usual convention, $N = \infty$ if there is no irreducible map. Any map with fewer than N regions is 4-colorable.

Very little is known about the Birkhoff number. Franklin **[1]** proved that $N \geqq 26$, and Reynolds **[1]** improved the result slightly, showing $N \geqq 28$. Franklin **[2]** improved on the improvement, obtaining $N \geqq 32$. Finally, Winn **[4]** proved that $N \geqq 36$. After a hiatus of nearly thirty years, Ore and Stemple **[1]** succeeded in raising the lower bound for N once again by proving the following theorem:

1.4 THEOREM: $N \geqq 40$.

Being irreducible is a (very!) strong requirement, and we shall be able to deduce many properties of irreducible graphs. Since loops and parallel edges do not affect colorability of a graph, we may always assume that an irreducible graph is simple. Suppose that G is a simple irreducible graph. We can embed G in a maximal planar graph $\bar{G}$ with the same number of vertices as G. $\bar{G}$ is 5-chromatic and hence irreducible. Thus, we have shown that if any irreducible planar graph exists, then there is an irreducible simple maximal planar graph. Whitney's result **[1]** guarantees that any simple maximal planar graph has a Hamiltonian circuit, and as we have seen, any map with a Hamiltonian circuit can be 4-colored. Thus, we obtain the following paradoxical result (cf. Ore **[1**, p. 193]):

1.5 THEOREM. *It is optional to assume that any map obtained by embedding an irreducible graph can be face-colored in* 4 *colors.*

Of course, this does not imply that we can *vertex*-color the graph using 4-colors. We shall see later that any triangular map except the tetrahedron is 3-colorable.

By considering maps and dualizing, we can show that the above optional conditions for irreducible graphs yield the following optional conditions for irreducible maps:

1.6 THEOREM. *The following characteristics are optional for irreducible maps*:

(a) *Bridgeless,*

(b) *Two regions meet along at most one edge,*

(c) *Cubic.*

On the other hand, certain characteristics are forced for irreducible maps. Any map divides the plane into open connected components, and the regions of the map are just the closures of these components.

1.7 THEOREM. *Let M be an irreducible map. Then any region in M is simply-connected.*

Proof: Suppose some region R is not simply-connected. Then the region divides the plane into an inside and an outside. The region R and the regions interior to it form a map M_1; the region R and regions exterior to it form a map M_2, and no internal region shares a common boundary edge with an external region. Now, since both M_1 and M_2 have fewer regions than M, we can color both M_1 and M_2 using 4 colors. By rearranging the coloring of M_1, we can insure that R receives the same color in each of the colorings of M_1 and M_2. This allows us to put the two colorings together to obtain a 4-coloring of M.

The same argument would allow us to prove that the union of any two regions in M is simply-connected. Thus, 1.6(b) is *forced*. In other words, an optional property may be forced. Actually, if Conjecture C_0 is true, *any* property is forced.

This theorem is equivalent to the fact that an irreducible planar graph has no point of articulation and thus is 2-connected (i.e., it is a block). In fact, any maximal planar, simple, irreducible graph G must be 3-connected. For if we embed G in the sphere, we obtain a triangulation so, by a theorem of Steinitz (see Steinitz and Rademacher **[1]**, or Grünbaum **[2**, p. 235]) G is 3-connected. (Steinitz's theorem states that the vertices and edges of a 3-dimensional convex polyhedron constitute a planar 3-connected graph and conversely.)

Now we can use duality to prove the following theorem:

1.8 THEOREM. *Let M be an irreducible map satisfying the optional conditions* 1.6 (a), (b), (c). *Then $U(M)$ is 3-connected.*

Proof. Think of M as a map on the sphere. Then M is the dual of its own dual.

But the dual of M is a triangulation of the sphere and the dual of any triangulation is a convex polyhedron (see E. C. Zeeman [**1**]). Hence, M is a convex polyhedron and so, by Steinitz' theorem, $U(M)$ is 3-connected.

1.9 COROLLARY. *Let M be an irreducible map. Then it is optional that $U(M)$ be 3-connected.*

This result seems particularly interesting in view of Whitney's theorem [**8**] which says that a 3-connected planar graph embeds uniquely in the plane. Thus, Corollary 1.9 says that M is completely determined by $U(M)$. But, by the dual of Theorem 1.5, $U(M)$ can be vertex-colored in 4-colors!

2. Critical graphs and irreducibility.

2.1 DEFINITION: Let G be a graph. Then G is **contraction-critical** if any edge contraction reduces the chromatic number of G.

Obviously, any irreducible graph G is vertex-critical and contraction-critical since removing a vertex or contracting an edge both lower the total number of vertices and hence either operation decreases the chromatic number. Thus, we may examine properties of vertex-critical or contraction-critical graphs to derive information about irreducible graphs.

2.2 DEFINITION: A graph G is **k-edge connected** if removing fewer than k edges does not disconnect the graph.

Ore ([**1**, p. 165]) proves the following theorem:

2.3 THEOREM. *Any 5-chromatic vertex-critical graph is 4-edge connected.*

Analogous information about contraction-critical graphs is due to Dirac:

2.4 THEOREM. (Ore [**1**, p. 169]). *Let G be a contraction-critical graph with $\chi(G) \geqq 5$. Then G is 5-connected.*

Thus, every irreducible planar graph is 5-connected.

We can use the last result to rederive Theorem 1.5. For suppose G is irreducible planar and hence 5-connected. Tutte's theorem [**3**] (we only need 4-connected) implies that G has a Hamiltonian circuit, and we complete the argument as before. Theorem 2.4 implies that the degree of every vertex in an irreducible planar graph is at least 5. Of course, our earlier modification of Heawood's argument also proves this fact.

2.5 DEFINITION: Let G be a graph. We call a set T of vertices of G a **minimal disconnecting set** if $G - T$ is disconnected or trivial, but no proper subset of T has this property.

The preceding theorem shows that a minimal disconnecting set T must contain at least 5 vertices if $\chi(G) \geqq 5$. If T is a minimal disconnecting set in G, the section graph determined by T, $G(T)$, is called the **separating** graph. What properties must a separating graph have? The following theorem (Ore [**1**, p. 192]) provides a partial answer.

2.6 THEOREM. *Let G be a maximal planar graph with minimal disconnecting set T. Then $G(T)$ is a simple circuit.*

2.7 THEOREM. *Let G be a contraction-critical 5-chromatic planar graph. Then G cannot be separated by a simple circuit C of length five except when one of the connected components of $G - C$ is a single vertex which is adjacent (in G) to every vertex of C.*

Let us translate this result into a statement about maps.

2.8. DEFINITION: A sequence $R_1, R_2, \cdots, R_p$ of regions in a map with R_i adjacent to R_{i+1}, $1 \leqq i \leqq p-1$, R_p adjacent to R_1, and no other pairs R_i and R_j adjacent is lcalled a **ring** of length p, or p-**ring.**

Obviously, a ring of length p in a map M corresponds to a simple circuit of length p in $D(M)$ which separates the graph. The dual to the conclusion of the theorem holds if and only if either the inside or outside of the ring consists of a single region. Thus, we have shown that Theorem 2.7 implies the following result of Birkhoff [**2**]:

2.9 THEOREM. *If M is an irreducible map, then M may* **not** *contain a ring of five regions unless they surround a pentagon.*

3. Reducible configurations.

Theorem 2.9 of the last section suggests a definition:

3.1 DEFINITION: Let G be a graph. Then we call G a **reducible configuration** if G cannot occur as a subgraph of an irreducible graph. We define reducible configurations in maps using duality.

Thus, the previously mentioned result says that a ring of five regions not surrounding a pentagon is a reducible configuration.

We already have other types of reducible configurations; for example, any region with at most four sides. This allows us to derive a lower bound on the Birkhoff number.

If M is a cubic map and r_i denotes the number of regions bounded by i sides in the map, we have from Euler's formula

$$2m = \sum_i ir_i .$$

Putting these equations together yields the following well-known lemma (see, for example, Franklin [**3**, p. 154]):

3.2 LEMMA. *Let M be a cubic map. Then*

$$\sum_i (6-i)r_i = 12.$$

If a map is irreducible, $r_i = 0$ for $i < 5$ and hence the only positive term in the sum is $(6 - 5)\, r_5 = r_5$, the number of pentagons. We conclude immediately that any irreducible cubic map must have at least 12 pentagons.

If a map has exactly 12 pentagons, then it is a dodecahedron and can be 4-colored. Thus an irreducible map must have at least 13 regions. This proves that the Birkhoff number is at least 13.

To improve on this lower bound for the Birkhoff number, one must obtain more reducible configurations. Even then, however, increasing the lower bound can be very difficult because of the many combinatorial possibilities to be considered at every step.

Before listing other reducible configurations, we shall need some jargon. Our results will be in terms of vertices and degrees but of course can be dualized for regions and number of faces.

3.3 DEFINITION: We call a vertex v of degree k a **k-vertex** and write $d(v) = k$. Any vertex of degree 6 or less is called **minor**; vertices of degree 7 or more are called **major.** Let v_0 be a fixed vertex. A **neighbor** is a vertex adjacent to v_0. If a neighbor is a k-vertex, we call it a **k-neighbor.** Three vertices are in **triad** when they form the three corners of a triangle. Two neighbors of v_0 are **successive** when they form a triad with v_0. A vertex is **reducible** if it belongs to a reducible graph. A sequence $v_1, \cdots, v_r$ of neighbours of v_0 is called **successive** or **consecutive** if v_{i-1} and v_i are successive for $i = 1, \cdots, r$.

The following was one of the first reduction theorems:

3.4 THEOREM (Birkhoff [2]). *A 5-vertex is reducible when it has three consecutive 5-neighbors.*

Franklin [1] proved an analogous theorem about 6-vertices.

3.5 THEOREM. *A 6-vertex is reducible if it has three consecutive 5-neighbors.*

These results yield a corollary (Franklin [1]):

3.6 COROLLARY. *A 5-vertex v_0 is reducible when it has three 5-neighbors and a 6-neighbor.*

Proof: By Theorem 3.4, v_0 must have three consecutive neighbors v_1, v_2, v_3, where v_2 is a 6-vertex and v_1 and v_3 are 5-vertices or else v_0 is reducible. But now v_2 has three consecutive 5-neighbors v_1, v_0, v_3 so it is reducible by Theorem 3.5.

Franklin [2] also proved the following result:

3.7 THEOREM. *A 5-vertex with two 5-neighbors and three 6-neighbors is reducible.*

Winn [1] proved still another reduction theorem:

3.8 THEOREM. *A 5-vertex is reducible if it has one 5-neighbor and four 6-neighbors.*

Choinacki [1] and Winn [1] obtained another reduction result for 5-vertices.

3.9 THEOREM. *A* 5-*vertex all of whose neighbors are* 6-*vertices is reducible.*

Putting together the preceding results, we obtain the following corollary due to Winn [**1**]:

3.10 COROLLARY. *A* 5-*vertex is reducible when all of its neighbors are minor vertices.*

Thus, in an irreducible graph every 5-vertex is adjacent to a major vertex.

Bernhart ([**1**] and [**2**]) proved the following reduction theorem for a 6-vertex:

3.11 THEOREM. *A* 6-*vertex is reducible if it has three successive neighbors with degrees* 5, 6, *and* 5, *respectively.*

Winn [**1**] went on from there to obtain an analogue to Corollary 3.10 for 6-vertices.

3.12 THEOREM. *A* 6-*vertex is reducible when all of its neighbors are minor.*

Errera [**1**] obtained some general results about the number of consecutive 5-neighbors of an n-vertex in an irreducible graph.

3.13 THEOREM. *An n-vertex in an irreducible graph can have at most $n - 3$ consecutive* 5-*neighbors for n even and at most $n - 2$ for n odd.*

For $n = 7$, his result was improved by Winn [**2**].

3.14 THEOREM. *A* 7-*vertex with more than four consecutive* 5-*neighbors is reducible.*

Thus, a 7-vertex with six or more 5-neighbors is reducible; that is, in an irreducible graph, there are at most five 5-vertices adjacent to any 7-vertex.

Several new reducible configurations were discovered by Ore and Stemple [**1**]. For example, we have the following result:

3.15 THEOREM. *Let v_0 be a* 5-*vertex with neighbors v_1, v_2, v_3, v_4, v_5. If the corresponding list of degrees is* (6, 5, 5, 6, 7) *and v_4 and v_5 are in triad with a* 5-*vertex $w \neq v_0$, then the configuration is reducible.*

We have not attempted here to list all, or even nearly all, reducible configurations, but rather to give the flavor of the sorts of manipulations involved in obtaining them.

For a listing of most reducible configurations, see the paper of Ore and Stemple [**1**]. One may also consult Ore [**1**, Chapter 12] and Franklin [**3**, p. 156].

CHAPTER 4. RESULTS

1. Some sufficiency theorems. Any of the following conditions is sufficient to insure that a planar map be 4-colorable:

1.1 CONDITION: *Some region is bounded by at most* 4 *edges* (see Chapter 2, Section 1).

1.2 Condition: *Each region is bounded by at most five edges* (Aarts and de Groot [1]).

1.3 Condition: *There are at most* 21 *vertices of degree* 3 (Finck and Sachs [1]).

1.4 Condition: *There is at most one region of more than six sides and the map is irreducible* (Winn [1]).

1.5 Condition: *The countries with more than four neighbors can be divided into two classes such that one class has at most one country and no two countries in the other class are neighbors* (Dirac [8]).

1.6 Condition: *The number of edges in the boundary of each region is a multiple of* 3, *and the map is bridgeless cubic* (Winn [1]).

Very few constructions have been given which show how to color some general class of maps. The following scheme shows us how to 3-color the edges of a particular kind of map.

Let M be a cubic bridgeless map. Suppose that the number of edges in the boundary of every region is a multiple of 3. Ringel [1, p. 19] has given a constructive scheme for 3-coloring the edges of M.

Call the three colors 1, 2, and 3, and give them the usual cyclic ordering so that 2 follows 1, 3 follows 2, and 1 follows 3. If e, f, and g are the three edges of M incident with some vertex, give them the cyclic ordering induced by the clockwise orientation of the plane; that is, f follows e if, moving clockwise from e, we first encounter f.

1.7 Coloring Scheme: Begin with some edge e of M and color it arbitrarily, say with 1. Now consider the four edges adjacent to e, two at each endpoint. In the cyclic orderings at each endpoint, these four edges either follow or precede e. Give them the corresponding color. (Thus, if f follows e, color f with 2.) Continue the process until all edges have been colored.

This procedure is unambiguous—in other words, only one color is assigned to each edge. Hence, no two adjacent edges receive the same color.

This provides us with a constructive proof of the sufficiency of Condition 1.6 since, given a 3-coloring of the edges of a cubic bridgeless map, we can then construct a 4-coloring of the regions of the map.

1.8 Conjecture C_{31}: *If a critical* 5-*chromatic graph contains a complete graph on three vertices, then the graph can be contracted to a complete graph on five vertices.*

The truth of this conjecture implies the truth of Conjecture C_1 (Dirac [5]). Conjecture C_1 implies Conjecture C_{23} of which this Conjecture is a special case.

1.9 Theorem. *If* $k\ (>2)$ *is the maximum degree of any vertex in a graph without loops and without complete subgraphs on* $k+1$ *vertices, then the graph is* k-*colorable.*

This is the famous result of Brooks [1] which contains the dual of Condition 1.2

as a corollary. The following results indicate that k-chromatic graphs may be somewhat pathological.

1.10 THEOREM. *For any $k > 1$ there exists a k-chromatic graph which has no circuit (region) of less than 6 edges* (B. Descartes **[1]**).

1.11 THEOREM. *If $d \geqq k \geqq 2$, then there exist regular connected k-chromatic graphs of degree d and of an arbitrarily large number of vertices* (Dirac **[4]**).

For $k \geqq 4$ Dirac constructs a k-chromatic graph which does not contain a complete k-graph as a subgraph and in which the degree of every vertex except one is $k - 1$.

2. Coloring problems on surfaces other than the plane. In view of the fact that the four-color problem is unsolved, it is perhaps surprising that the analogous problems on other orientable surfaces have been solved completely!

2.1 DEFINITION: A surface is said to have **genus** p if it is a homeomorph of a sphere with p handles.

2.2 THEOREM. *For any positive integer p, the chromatic number of a graph embedded in the (orientable) surface of genus p is at most χ_p where*

$$\chi_p = \left[\frac{7 + \sqrt{1 + 48p}}{2}\right].$$

This is Heawood's Map-Coloring Theorem—see Busacker and Saaty **[1**, p. 94] for the proof. Note that if this theorem held for $p = 0$, we would have a proof of Conjecture C_1. Unfortunately, the only known proof of Theorem 2.2 depends on having $p > 0$.

Recently, Ringel and Youngs **[2]** have shown that if $p \geqq 1$, then there always exists a graph which can be embedded in the surface of genus p whose chromatic number is exactly equal to χ_p (see also Youngs **[1]** and Berge **[1**, p. 218]).

We might also mention here that Ringel **[2]** has given an interesting six-color problem on the sphere in which he asks for a coloring of both regions and vertices using 6 colors so that no 2 adjacent vertices or regions are colored the same and so that no vertex receives the same color as the regions on whose boundaries it lies.

3. One, two, and three and more colorability. Clearly a graph is 1-colorable if and only if it consists of isolated vertices (i.e., it is totally disconnected).

3.1 THEOREM. *A map is properly colorable with two colors if and only if every vertex is of even degree.*

This follows from the fact that a graph is bipartite if and only if it has no circuits of odd length (König **[1**, p. 151]).

3.2 THEOREM. *A cubic map is properly colorable with three colors if and only if each region is bounded by an even number of edges* (Franklin [**3**, p. 198]).

Dually, a maximal planar graph is 3-colorable (i.e., 3-partite) if and only if every vertex has even degree. Unfortunately, no general useful characterization of 3-partite graphs or 3-partite planar graphs is known at present.

3.3 THEOREM. *The edges of a cubic map can be properly colored with four colors* (Golovina and Yaglom [**1**, p. 43]).

This is also a corollary of the Shannon-Vizing bound on the chromatic index.

Grünbaum [**1**] has shown that every planar map with less than 4 triangles is 3-colorable. As a consequence of the theorem of Brooks, triangular maps (other than the tetrahedron) are 3-colorable.

3.4 THEOREM. *If a triangular map can be properly colored with two colors, then its vertices can be properly colored with three colors.*

See Dynkin and Uspenskii [**1**].

3.5 THEOREM. *The edges of a cubic map can be colored with two colors α and β so that each vertex is incident with one edge colored with α and two edges colored with β.*

This theorem is due to Petersen [**1**]. It can be restated in the form: Every bridgeless cubic map is the sum of a 1-factor and a 2-factor. Petersen gave an example to show that a similar result with three 1-factors cannot be obtained. (See Fig. 7.)

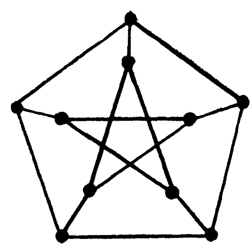


FIG. 7

Marathe [**1**] has shown that Petersen's theorem is a corollary of the following result:

3.6 THEOREM. *Any triangular map with an even number of triangles can be colored with two colors α and β so that each triangle is bounded by one edge colored α and two edges colored β.*

4. The sufficiency of six colors. We already know that 5 colors suffice to color any planar map, but we shall give a short direct proof here that 6 colors suffice since the argument demonstrates, once again, the ubiquity of Euler's formula in these coloring problems and since it gives us a method for reducing the number of regions in a cubic map.

Consider Euler's formula $n - m + r = 2$ and substitute $n = 2m/3$ (for a cubic map). This gives $6(r - 2) = 2m$. Since $6r > 6(r - 2) = 2m$ we prove that 6 colors are sufficient to color any cubic map. This is clear when $r < 6$. If $r \geqq 6$, then there must be (as we already know) at least one region bounded by 5 or less edges. Applying induction, we may assume that all maps are 6-colorable for $r - 1$ regions. If we remove a less than six sided region of the map and extend the edges of its neighbors in such a way that each vertex is of degree three and the entire removed region is covered by its five neighbors as in the diagram below, we can 6-color the map and then reinstate the removed region, coloring it with the sixth color not appearing in any of its five neighbors. (See Fig. 8.)

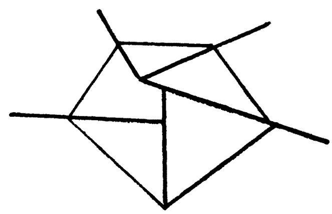


FIG. 8

5. The uniqueness of colorings. The uniqueness of the colorability of a graph has also been investigated. A complete presentation is given in the book by Harary [**2**, p. 137]. Note that in a unique coloring, each vertex must be adjacent to vertices whose totality is colored with all the remaining colors (at least once). We have the following results for uniqueness of coloring with k colors:

5.1 THEOREM. *In the partition of the vertices into subsets induced by the coloring, the vertices of every pair of subsets with their connecting edges form a connected subgraph* (Cartwright and Harary [**1**]).

5.2 THEOREM. *The graph is* $(k - 1)$*-connected. The corresponding subgraph for* m *subsets,* $2 \leqq m \leqq k$ *is* $(m - 1)$*-connected.*

5.3 THEOREM. *For each* $k \geqq 3$ *there is a uniquely* k*-colorable graph with no subgraph isomorphic to the complete graph on* k *vertices* (Harary, Hedetniemi, and Robinson [**1**]).

It is also known (Chartrand and Geller [**1**]) that no planar graph is uniquely 5-colorable; every uniquely 4-colorable planar graph is maximal planar; and that a planar 3-colorable graph in which each vertex belongs to the last triangle of a linear sequence of triangles each sharing an edge with its immediate neighbors is uniquely 3-colorable. A uniquely 3-colorable planar graph on $n \geqq 4$ vertices contains at least two triangles.

In general, the coloring of a map or a graph is not unique. There are a number of papers studying the number of colored graphs. We give a sample of the known ones in addition to the discussion of chromatic polynomials already given.

Let $F_n(k)$ denote the total number of k-colored graphs on n labelled vertices and let $M_n(k)$ denote the number of graphs on n vertices that are colored in at most k colors; also let $f_n(k)$ denote the number of connected k-colored graphs on n vertices. Read [1] gives:

$$\sum_{n=1}^{\infty} 2^{-\frac{1}{2}n^2} F_n(k) \frac{x^n}{n!} = \left\{ \sum_{s=1}^{\infty} 2^{-\frac{1}{2}s^2} \frac{x^s}{s!} \right\}^k,$$

$$\sum_{n=0}^{\infty} 2^{-\frac{1}{2}n^2} M_n(k) \frac{x^n}{n!} = \left\{ \sum_{s=0}^{\infty} 2^{-\frac{1}{2}s^2} \frac{x^s}{s!} \right\}^k,$$

$$1 + \sum_{n=1}^{\infty} 2^{-\frac{1}{2}n^2} F_n(k) \frac{x^n}{n!} = \left\{ \sum_{n=1}^{\infty} f_n(k) \frac{x^n}{n!} \right\}.$$

Wright [1] has proved some asymptotic formulas for $F_n(k)$, $M_n(k)$, $f_n(k)$. Carlitz [1] has analyzed some arithmetic properties of these numbers. An interesting and rather simple one to quote is:

$$M_n(k) \equiv k \pmod{2^n} \qquad (n > 2)$$

from which it follows that $M_n(k)$ is odd if and only if k is odd.

5.4 DEFINITION: A map is **rooted** when a vertex, an edge and a face that are mutually incident are specified as **root-vertex, root-edge** and **root-face**, respectively.

Consider a bridgeless cubic rooted map with $2n$ vertices. Two colorings are not considered as distinct if they differ only by a permutation of the four colors. Suppose that the root-face is red, the other face incident with the root-edge is blue, the third face incident with the root-vertex green, and the fourth color, yellow.

5.5 DEFINITION: The Tait cycle separating blue and green from red and yellow is called the **basic Tait cycle** of the coloring (it passes through the root-edge).

5.6 DEFINITION: The **rank** of the coloring is equal to the number of components of the basic Tait cycle minus one.

W. T. Tutte [7, 9] has shown that the average number of 4-colorings for such maps with 2 n-vertices is asymptotically equal to the following expressions:

$$8(3\pi n)^{-\frac{1}{2}}(32/27)^n \qquad \text{for rank } 0,$$

$$8(3\pi n)^{-\frac{1}{2}}(4/\pi - 1)n^{\frac{1}{2}}(32/27)^n \text{ for rank } 1.$$

One can also introduce the notion of semi -uniquely 4-colorable graphs.

5.7 DEFINITION: Suppose $\chi(G) = 4$. Let v and w be vertices of G. Then we say that v and w are **brothers** if any 4-coloring of G assigns the same colors to v and w. We say that G is **semi-uniquely 4-colorable** if it has a pair of vertices which are brothers.

D. L. Greenwell [1] has proved that the following conjecture is equivalent to Conjecture C_1:

5.8 CONJECTURE C_{32}: *Let G be a semi-uniquely 4-colorable planar graph and let v and w be a pair of brothers in G. Then the graph G′ obtained from G by joining v and w with an edge is not planar.*

6. Some recent developments. It would be totally beyond the scope of this paper to discuss the problem of coloring infinite planar graphs. We might mention here, however, some recent work of R. Halin [3] on coloring numbers which has applications to finite graphs. The **coloring number**, col (G), of a (possibly infinite) graph G was first introduced by Erdös and Hajnal [1] and is defined as the smallest cardinal k for which there exists a well-ordering of the vertices of G such that every vertex v of G is adjacent to less than k vertices preceding it in the ordering. Clearly, $\chi(G) \leqq \text{col}(G)$. Halin shows that if col (G) is sufficiently large, then G must contain subdivisions of any complete graph on fewer than col (G) vertices.

We should also like to draw the reader's attention to some other recent papers. S. Hedetniemi [1] defines a **disconnected-coloring** (or **D-coloring**) of a graph $G = (V, E)$ as a partition $V = V_1 \cup \cdots \cup V_n$ of V such that, for every i, the section graph of G induced by the subset V_i is disconnected. The **D-chromatic** number $\chi_d(G)$ is the smallest number of subsets in any D-coloring of G. The D-chromatic number shares many properties with the chromatic number but differs in others. For example, Hedetniemi gives the following theorem:

6.1 THEOREM. *If G is planar, then* $\chi_d(G) \leqq 4$.

Other recent results have dealt with edge coloring. M. Rosenfeld [1] proved the following theorem:

6.2 THEOREM. *Let G be a cubic graph with n vertices. Then G is homomorphic to a Tait-colorable cubic graph G′ with* $(6n + 5)/5$ *vertices.*

In a recent paper, M. R. Williams [1] suggests an improvement of a heuristic coloring procedure developed by Peck and Williams [1]. The latter procedure takes a graph and proceeds as follows to determine which vertices should be colored with the kth color (cf. Welsh and Powell [1]).

(i) Find the uncolored vertex v of highest degree.
(ii) Check to see if v is adjacent to any vertex already colored with the kth color.
(iii) If not, then color v with color k.
(iv) If yes, then remove v from consideration for color k and return to step (i).

This heuristic procedure uses the vector d whose ith component is the degree of the ith vertex. Williams modifies the above procedure by replacing d with a vector d^m defined recursively by setting $d = d^1$ and $d^{m+1} = Ad^m$, where A is the adjacency or vertex-vertex matrix of G. The vectors d^m converge to the dominant eigenvector of A as $m \to \infty$. Williams observes that convergence generally occurs after $m = \sqrt[3]{n}$ iterations where n is the number of vertices in the graph.

Williams used his modified heuristic to color one graph of over 700 vertices using 28 colors. The graph was later found to contain a complete subgraph on 26 vertices so Williams' estimate was certainly not too high!

Striking out into other new directions, J. W. T. Youngs [2] indicates how his joint work with Ringel (Ringel and Youngs [2]), in which they settled the Heawood Conjecture, can be used to provide "slick" proofs that various conjectures, e.g., Conjecture C_4, are equivalent with the four color conjecture. Hopefully, these methods (current graphs, graphs with rotation, Kirchhoff's Law) will eventually provide us with some new information in this area although they have not yet done so.

Finally, we should like to mention some recent work of ours with P. Kainen [1] in which we have considered the problem of relative colorings. Suppose we consider some planar graph G with a section subgraph, G', that has already been colored. A **relative coloring** of (G, G'), with respect to the given coloring of G', is a coloring of G which agrees with the given coloring on the vertices of G'.

Note that if G' is 4-colored, we may need as many as 4 new colors to color G relative to the coloring of G'. Let us write $\chi(G, G')$ for the maximum number of new colors needed in any relative coloring of (G, G'). We call this the **relative chromatic number** of (G, G').

We prove that the following conjecture is equivalent to the four color conjecture.

6.3 CONJECTURE C_{33}: *For any pair* (G, G') *with* G *planar and* G' *a (possibly empty) subgraph of* G, *we have* $\chi(G, G') \leqq 4$.

If we require G' to be connected, then we know of no examples where $\chi(G, G') > 3$. This leads us to make the following conjecture which implies Conjecture C_1.

6.4 CONJECTURE C_{34}: *For any pair* (G, G') *with* G *planar and* G' *a connected subgraph of* G, *we have* $\chi(G, G') \leqq 3$.

We do not know whether this conjecture is implied by the four-color conjecture.

Conclusion. To conclude, it may be of interest to give a quotation from a paper by a great living geometer, H. S. M. Coxeter [1]:

> If I may be so bold as to make a conjecture, I would guess that a map requiring five colors may be possible, but that the simplest such map has so many faces (maybe hundreds or thousands) that nobody, confronted with it, would have the patience to make all the necessary tests that would be required to exclude the possibility of coloring it with four colors. Many people believe, on the other hand, that the four-color theorem may be true; in fact, editors of journals often have the unhappy experience of receiving manuscripts in which it is "proved." Such manuscripts are either obviously incompetent or else so lengthy that the referee has a tedious job finding the flaw. The problem

has been considered by so many able mathematicians that anyone who can prove that a particular map really needs five, will become world-famous overnight.

There is still great and lively interest in the problem: Shimamoto of the Brookhaven National Laboratory Computer Center, is presenting a paper on a proof of the four-color problem. One of the steps in the proof depends on a complicated computer program which is still being worked on at this time.

My heartfelt thanks to my colleague and friend, Paul Kainen, for careful reading and suggestions which enriched the manuscript. I would also like to thank Michael Albertson and David Burman for help in obtaining information and Marilyn Dalick for her great patience in typing many versions of the manuscript over the past two years.

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