THE ELEMENTARY CASES OF LANDAU'S PROBLEM OF INEQUALITIES BETWEEN DERIVATIVES

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INTRODUCTION

In 1913 Landau initiated in [5] a new kind of extremum problem: The sharp inequalities between the supremum-norms of derivatives. He wrote two further papers, [6] and [7], on this subject (see also [3, 139–142]). Here we are only concerned with his first paper [5]. A lively activity on this subject culminated in 1939 with Kolmogorov's remarkable paper [4], where Landau's $\mathbb{R}$-problem was solved for all values of n (Landau had solved it for $n = 2$ only). In 1941 Bang [2] gave a second proof of Kolmogorov's theorem using the theory of almost periodic functions. Recently, the author gave a third proof in [13]. This third proof is in essence an elaboration of Landau's original direct approach and may be regarded as an application of spline theory. The analogue of Kolmogorov's theorem for the halfline $\mathbb{R}_+$ has recently been established in [11].

The present paper discusses for both $\mathbb{R}$ and $\mathbb{R}_+$ those cases of Landau's problem that require no knowledge beyond the elements of the Differential and Integral Calculus of functions of one variable. The novel contribution of this paper, besides the proofs, is the discussion of the extremizing functions in Theorems 4, 5, 6, and 7, for the $\mathbb{R}$-problem, and Theorems 9 and 11 for the $\mathbb{R}_+$-problem.

The author believes that the subject can be used to supplement the contents of a calculus course, of an introductory course in numerical analysis, or for lectures in undergraduate, or beginning graduate, seminars. In doing this there is a good deal of flexibility. The main object of discussion are the Euler splines $\mathscr{E}_n(x)$, and the essential section of Part I is §2. The §§1 and 3 only furnish further background and may be omitted. If I were to make a selection, I would choose Theorems 1, 2, 4, Corollaries 1, 2, and Theorem 5. This choice was implemented on when in the

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framework of the Visiting Lectureship Program of the MAA the author gave three one-hour lectures on this subject at Wichita State University on December 6 and 7, 1971. He wishes to thank Professor Keith Moore, Albion College, and Professor William M. Perel, Wichita State University, for arranging these lectures. This experience encouraged the author to write this paper.

I. THE EULER SPLINES

1. Cardinal spline interpolation. Let n be a natural number and let $\mathscr{S}_n = \{S(x)\}$ be the class of functions $S(x)$ having the following two properties

(i) $S(x) \in C^{n-1}(\mathbb{R})$.

(ii) The restriction of $S(x)$ to every interval $(\nu, \nu + 1)$ between consecutive integers in a polynomial of degree $\leqq n$.

Such functions $S(x)$ are called *cardinal spline function of degree n*. Evidently $\pi_n \subset \mathscr{S}_n$, where π_n denotes the class of polynomials of degree not exceeding n. We may even consider $\mathscr{S}_0$, the class of step-functions with discontinuities at the integers. Indefinite integration of the elements of $\mathscr{S}_0$ gives the elements of $\mathscr{S}_1$, also called cardinal *linear splines* (the term "spline" can be used either as an adjective or as a noun). Integrating the elements of $\mathscr{S}_1$ we obtain those of $\mathscr{S}_2$, also called cardinal *quadratic* splines a.s.f. The term "cardinal" is to remind us that we pass from one polynomial component of $S(x)$ to the next at the integers. These transition points are called the *knots* of the spline.

It is also useful to introduce the class

$$\mathscr{S}_n^* = \{S(x); S(x + \tfrac{1}{2}) \in \mathscr{S}_n\}. \tag{1.1}$$

The elements of $\mathscr{S}_n^*$ are again defined by the properties (i) and (ii), provided that we replace in (ii) the interval $(\nu, \nu + 1)$ by $(\nu - \frac{1}{2}, \nu + \frac{1}{2})$. The knots of $S(x)$ are now half-way between the integers, and $S(x)$ may be called a *midpoint spline.*

With elements of the class $\mathscr{S}_n$, or perhaps $\mathscr{S}_n^*$, we may attempt to solve the following

CARDINAL INTERPOLATION PROBLEM. *Given the sequence of numbers*

$$(y_\nu) = (\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots) \tag{1.2}$$

we are to find $S(x)$ such that

$$S(\nu) = y_\nu \text{ for all integers } \nu. \tag{1.3}$$

We restrict our discussion to the case when (y_ν) is a *bounded* sequence. This means that for an appropriate K

$$|y_\nu| < K \text{ for all } \nu. \tag{1.4}$$

A main result is the following

THEOREM OF CARDINAL SPLINE INTERPOLATION. *We assume that* (1.4) *holds.*

1. *If n is odd, then there exists a unique $S(x) \in \mathscr{S}_n$ such that $S(x)$ is bounded for all real x and satisfies the interpolation conditions* (1.3).

2. *If n is even, then there exists a unique $S(x) \in \mathscr{S}_n^*$ such that $S(x)$ is bounded and satisfies* (1.3).

The first part of this theorem was first established by Subbotin [**14**]. For the complete theorem under more general conditions (the condition (1.4) is replaced by the requirement that y_ν should grow at most like some power of $|\nu|$ as $\nu \to +\infty$ or $\nu \to -\infty$) see [**10**].

The theorem is trivial if $n = 1$, but is no longer so if $n > 1$. Indeed, a linear spline $S_1(x)$ satisfying (1.3) is immediately obtained by successive linear interpoaltion between consecutive ordinates y_ν and $y_{\nu+1}$. The condition (1.4) is not needed in this case and $S_1(x)$ is evidently unique for any sequence (y_ν).

Remarkable cardinal splines are obtained from the above theorem for particular simple sequences (1.2). Here are two examples.

A. *The fundamental splines.* For the special sequence

$$y_0 = 1, \; y_\nu = 0 \text{ if } \nu \neq 0. \tag{1.5}$$

The theorem furnishes a unique bounded solution that we denote by $L_n(x)$. Thus

$$L_n(0) = 1, \; L_n(\nu) = 0 \text{ if } \nu \neq 0. \tag{1.6}$$

Of course

$$L_n(x) \in \begin{cases} \mathscr{S}_n & \text{if } n \text{ is odd,} \\ \mathscr{S}_n^* & \text{if } n \text{ is even.} \end{cases} \tag{1.7}$$

The following is also true: The unique bounded solution $S(x)$ of the interpolation problem (1.3) may be represented by the formula

$$S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_n(x-\nu), \tag{1.8}$$

where the series converges uniformly in every finite interval. This is a cardinal spline analogue of Lagrange's interpolation formula (see [**10**]).

B. *The Euler splines.* Very likely the most interesting examples of cardinal spline functions arise if we apply the above theorem to the sequence

$$y_\nu = (-1)^\nu \text{ for all } \nu. \tag{1.9}$$

For each n we denote the solution by $\mathscr{E}_n(x)$ and call it the *Euler spline of degree n.* Thus

$$\mathscr{E}_n(\nu) = (-1)^\nu \text{ for all } \nu, \text{ and } \mathscr{E}_n(x) \in \begin{cases} \mathscr{S}_n & \text{if } n \text{ is odd,} \\ \mathscr{S}_n^* & \text{if } n \text{ is even.} \end{cases} \tag{1.10}$$

These properties, together with the requirement that $\mathscr{E}_n(x)$ is bounded, defines this function uniquely on the basis of the cardinal interpolation theorem. We may also apply (1.8) and define $\mathscr{E}_n(x)$ by

$$\mathscr{E}_n(x) = \sum_{-\infty}^{\infty} (-1)^\nu L_n(x-\nu).$$

Our entire discussion so far was to show how the Euler splines fit into the theory of cardinal spline interpolation. However, this approach to $\mathscr{E}_n(x)$ does not help us much, because we have not established here the general interpolation theorem, nor have we learnt anything concerning $L_n(x)$ beyond its existence and uniqueness. Fortunately, there is a direct constructive approach to the Euler spline $\mathscr{E}_n(x)$ to which we now proceed.

2. A direct construction of the Euler splines. Let $f(x)$ be defined on $\mathbb{R}$ and integrable in every finite interval.

DEFINITIONS. 1. *We say that $f(x)$ is even about the point $x = a$, provided that it satisfies $f(x) = f(2a-x)$ for all x. Likewise $f(x)$ is odd about $x = a$ if $f(x) = -f(2a-x)$.*

2. *We say that $f(x)$ has the property P_0, or $f(x) \in P_0$, provided that $f(x)$ is even about $x = 0$, and odd about $x = 1/2$.*

3. *We say that $f(x)$ has the property P_1, or $f(x) \in P_1$, provided that $f(x)$ is odd about $x = 0$, and even about $x = 1/2$.*

LEMMA 1. *If $f(x) \in P_0$, or $f(x) \in P_1$, then $f(x)$ is a periodic function of period 2, hence $f(x-2) = f(x)$.*

Proof. If $f(x) \in P_0$, then

$$f(x) = -f(1-x) = -f(x-1) = f(2-x) = f(x-2).$$

If $f(x) \in P_1$, then

$$f(x) = f(1-x) = -f(x-1) = -f(2-x) = f(x-2). \qquad \square$$

We may omit the proof of the easily established

LEMMA 2. *If $f(x)$ is even (odd) about $x = a$ then $\int_a^x f(t)dt$ is odd (even) about $x = a$.*

LEMMA 3. 1. *If $f(x) \in P_0$ and $g_0(x) = \int_0^x f(t)dt$, then $g_0(x) \in P_1$.*

2. *If $f(x) \in P_1$ and $g_1(x) = \int_{1/2}^x f(t)dt$, then $g_1(x) \in P_0$.*

Proof: 1. Let $f(x) \in P_0$. By Lemma 2 $g_0(x)$ is *odd* about $x = 0$. Let us show that it is *even* about $x = 1/2$. By Lemma 2 applied with $a = 1/2$ we have

$$g_0(x) = \int_0^x f(t)dt = \int_0^{1/2} f(t)dt + \int_{1/2}^x f(t)dt = \int_0^{1/2} f(t)dt + \int_{1/2}^{1-x} f(t)dt$$
$$= \int_0^{1-x} f(t)dt = g_0(1-x).$$

2. Let $f(x) \in P_1$. By Lemma 2 $g_1(x)$ is *odd* about $x = 1/2$. Let us show that it is *even* about $x = 0$. Again, by Lemma 2

$$g_1(x) = \int_{1/2}^x f(t)dt = \int_{1/2}^0 f(t)dt + \int_0^x f(t)dt = \int_{1/2}^0 f(t)dt + \int_0^{-x} (t)dt$$
$$= \int_{1/2}^{-x} f(t)dt = g(-x). \quad \square$$

We start with the function $f_0(x)$ defined by

$$f_0(x) = (-1)^\nu \text{ if } \nu \leqq x < \nu + 1, \tag{2.1}$$

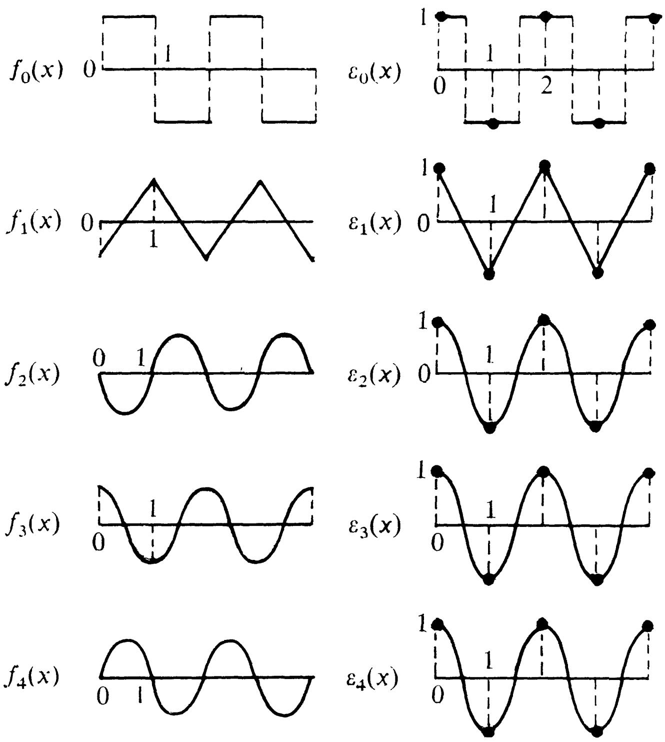

Fig. 1

whose graph is the "square-wave" of Figure 1. From it we derive the functions (see Figure 1)

$$f_1(x) = \int_{1/2}^{x} f_0(t)dt,\ f_2(x) = \int_{0}^{x} f_1(t)dt,\ f_3(x) = \int_{1/2}^{x} f_2(t)dt, \tag{2.2}$$

and generally

$$f_n(x) = \int_{\alpha_n}^{x} f_{n-1}(t)dt, \tag{2.3}$$

where

$$\alpha_n = \begin{cases} 0 \text{ if } n \text{ is even,} \\ 1/2 \text{ if } n \text{ is odd.} \end{cases} \tag{2.4}$$

LEMMA 4. *We have that*

$$f_n(x) \in \mathscr{S}_n, \qquad (n = 0,1,2,\cdots), \tag{2.5}$$

and

$$f_n(x) \in \begin{cases} P_0 \textit{ if } n \textit{ is odd,} \\ P_1 \textit{ if } n \textit{ is even.} \end{cases} \tag{2.6}$$

Proof: (2.5) is clear from (2.3) and an earlier remark that an integral of a spline is again a spline of a degree by one unit higher.

Also (2.6) follows from (2.3) and Lemma 3. Since $f_0(x) \in P_1$, we conclude that $f_1(x) \in P_0$ and therefore $f_2(x) \in P_1$ a.s.f. □

LEMMA 5.1. *In* $[0,1]$ *the functions* $f_{2k}(x)$ *are alternately strictly, convex or concave and vanish only at* $x = 0$ *and* $x = 1$.

2. *In* $[0,1]$ *the functions* $f_{2k-1}(x)$ *are alternately strictly increasing or decreasing and vanish at* $x = 1/2$ *only.*

In particular

$$(-1)^k f_{2k-1}(0) > 0,\ (-1)^k f_{2k}\left(\frac{1}{2}\right) > 0. \tag{2.7}$$

Proof: That $f_{2k}(x)$ vanishes at 0 follows from (2.3), (2.4), and its vanishing at 1 follows from (2.6), it being even about $x = 1/2$. (2.6) also implies that $f_{2k-1}(1/2) = 0$. The remaining statements follow from (2.3) by induction in n: $f_1(x)$ is strictly increasing, therefore $f_2(x)$ is strictly convex and therefore $f_3(x)$ is strictly decreasing. This implies the strict concavity of $f_4(x)$, a.s.f. □

LEMMA 6. *The functions defined by*

$$\mathscr{E}_{2k-1}(x) = f_{2k-1}(x)/f_{2k-1}(0) \tag{2.8}$$

and

$$\mathscr{E}_{2k}(x) = f_{2k}(x + \tfrac{1}{2})/f_{2k}(\tfrac{1}{2}) \tag{2.9}$$

are identical with the Euler splines as defined in §1B (see Figure 1).

Proof: Indeed, it should be clear that the newly defined functions enjoy the properties (1.10) and that they are bounded, since $|\mathscr{E}_n(x)| \leqq 1$ for all x. The unicity of the functions having these properties establishes the identity with the old definition. In any case for us (2.8) and (2.9) is the working definition of the Euler splines. □

If $f(x)$ is a bounded function defined on $\mathbb{R}$, we define its *norm* by

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|. \tag{2.10}$$

We shall be particularly concerned with the norm of $\mathscr{E}_n(x)$ and of its derivatives and write

$$\|\mathscr{E}_n^{(\nu)}\| = \gamma_{n,\nu}, \qquad (\nu = 0, 1, \cdots, n). \tag{2.11}$$

LEMMA 7.

$$\|\mathscr{E}_n^{(\nu)}\| = \begin{cases} |\mathscr{E}_n^{(\nu)}(0)| & \text{if } \nu \text{ is even,} \\ |\mathscr{E}_n^{(\nu)}(\tfrac{1}{2})| & \text{if } \nu \text{ is odd.} \end{cases} \tag{2.12}$$

Proof: (2.3) implies that

$$f_n^{(\nu)}(x) = f_{n-\nu}(x) \qquad (\nu = 0, \cdots, n). \tag{2.13}$$

Moreover, we easily show that

$$\|f_n\| = \begin{cases} |f_n(\tfrac{1}{2})| & \text{if } n \text{ is even,} \\ |f_n(0)| & \text{if } n \text{ is odd.} \end{cases} \tag{2.14}$$

Let $n = 2k$, and let $c = 1/f_{2k}(\tfrac{1}{2})$. By (2.9) and (2.13) we find

$$\mathscr{E}_{2k}^{(\nu)}(x) = c \cdot f_{2k}^{(\nu)}(x + \tfrac{1}{2}) = cf_{2k-\nu}(x + \tfrac{1}{2}).$$

By (2.14) this is seen to reach its largest absolute value at $x = 0$ if ν is even, and at $x = 1/2$ if ν is odd. Similarly, using (2.8), we establish (2.12) if n is odd. □

3. The connection with the Euler polynomials. Let us denote by $P_n(x)$ the polynomial of degree n that represents the spline function $f_n(x)$ in the interval $[0, 1]$. Thus

$$f_n(x) = P_n(x) \text{ if } 0 \leqq x \leqq 1, \; P_n(x) \in \pi_n. \tag{3.1}$$

Thus, from Figure 1 we find

$$P_0(x) = 1, \; P_1(x) = x - \tfrac{1}{2}, \; P_2(x) = \frac{x^2}{2} - \frac{x}{2}, \qquad \text{a.s.f.}$$

Clearly (2.3), (2.4) imply that

$$(3.2) \qquad P_n(x) = \int_{\alpha_n}^{x} P_{n-1}(t)dt, \quad \alpha_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1/2 & \text{if } n \text{ is odd.} \end{cases}$$

and therefore

$$(3.3) \qquad P_n'(x) = P_{n-1}(x).$$

A sequence of polynomials, like our $P_n(x)$, that is obtained by starting from $P_0(x) = 1$ and integrating successively, is called an *Appell sequence.* Integrating successively we obtain

$$P_1(x) = x + a_1, \; P_2(x) = \frac{x^2}{2!} + \frac{a_1}{1!}\frac{x}{1!} + \frac{a_2}{2!}, \cdots,$$

the nth polynomial being

$$(3.4) \quad P_n(x) = \frac{x^n}{n!} + \frac{a_1}{1!}\frac{x^{n-1}}{(n-1)!} + \frac{a_2}{2!}\frac{x^{n-2}}{(n-2)!} + \cdots + \frac{a_{n-1}}{(n-1)!}\frac{x}{1!} + \frac{a_n}{n!}.$$

Here $a_1/1!$, $a_2/2!, \cdots$ are the successive constants of integration.

Appell has observed that the infinite string of relations (3.4) can be described by a single relation involving series of powers of z. Indeed, multiplying the power series

$$(3.5) \qquad g(z) = \sum_0^\infty \frac{a_n}{n!} z^n \text{ and } e^{xz} = \sum_0^\infty \frac{x^n}{n!} z^n$$

and using (3.4) we find that

$$(3.6) \qquad g(z)e^{xz} = \sum_0^\infty P_n(x) z^n.$$

The left side is called the *generating function* of the polynomials $P_n(x)$.

Let us determine $g(z)$ for the particular sequence $P_n(x)$ defined by (3.1). By (3.2) we know that

$$P_{2k}(0) = 0, \; P_{2k-1}(\tfrac{1}{2}) = 0 \qquad (k = 1, 2, 3, \cdots).$$

Substituting into (3.6) the two values $x = 0$ and $x = \frac{1}{2}$, we conclude that $g(z) - 1$ is an odd function of z, and that $g(z)e^{z/2}$ is an even function of z. We therefore have the identities

$$g(z) - 1 = -g(-z) + 1 \text{ and } g(z)e^{z/2} = g(-z)e^{-z/2}.$$

Eliminating between them $g(-z)$ we obtain that

$$(3.7) \qquad g(z) = \frac{2}{e^z + 1}.$$

If we write

$$E_n(x) = n!P_n(x), \tag{3.8}$$

then (3.6) becomes

$$\frac{2e^{xz}}{e^z+1} = \sum_0^\infty \frac{E_n(x)}{n!} z^n. \tag{3.9}$$

This expansion shows that the $E_n(x)$ are the classical *Euler polynomials.* (See [9], [1, Chapter 23] also for further references.) Combining (3.1) and (3.8) we obtain

$$f_n(x) = E_n(x)/n! \text{ in } 0 \leqq x \leqq 1, \tag{3.10}$$

and therefore, by (2.8) and (2.9), that

$$\mathscr{E}_{2k-1}(x) = E_{2k-1}(x)/E_{2k-1}(0) \text{ in } 0 \leqq x \leqq 1, \tag{3.11}$$

$$\mathscr{E}_{2k}(x) = E_{2k}(x+\tfrac{1}{2})/E_{2k}(\tfrac{1}{2}) \text{ in } -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}. \tag{3.12}$$

The author could trace the spline function $n!f_n(x)$ to Nörlund's book [9, §16] where it is denoted by $\bar{E}_n(x)$, and where there are references to much earlier work by Hermite and Sonin (1896).

In concluding this section we mention the relations

$$\lim_{n\to\infty} L_n(x) = \frac{\sin \pi x}{\pi x} \tag{3.13}$$

and

$$\lim_{n\to\infty} \mathscr{E}_n(x) = \cos \pi x, \tag{3.14}$$

both of which hold uniformly for all real x. Concerning (3.13) see [12]. The relation (3.14) follows, via (2.8) and (2.9), from the beautiful Fourier series expansion of $f_n(x)$.

II. LANDAU'S PROBLEM FOR $\mathbb{R} = (-\infty, \infty)$. KOLMOGOROV'S THEOREM

4. Statement of Kolmogorov's theorem. Let $n \geqq 2$. We consider here the class of function $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ that are *bounded* and have a *bounded* nth *derivative* $f^{(n)}(x)$. This last condition needs some further explanations as follows: In the first place we assume that

$$f(x) \in C^{n-1}(\mathbb{R}) \tag{4.1}$$

and that

(4.2) $\qquad f^{(n-1)}(x)$ is piecewise continuously differentiable.

We interpret (4.2) to mean that the graph of $f^{(n-1)}(x)$ has a continuously turning tangent, except for corners with finite slopes for their right and left tangents, and that every finite interval contains at most a finite number of such corners. Finally, of course, $f^{(n)}(x)$ is to be bounded for all real x.

Evidently, the Euler spline $\mathscr{E}_n(x)$ satisfies all these conditions. In fact we have already considered the norms (2.11) of its derivatives and Lemma 7 shows how to identify, by (2.12), the values of

$$\gamma_{n,\nu} = \| \mathscr{E}_n^{(\nu)} \|, \ (\nu = 0, 1, \cdots, n), \ \gamma_{n,0} = 1. \tag{4.3}$$

THEOREM OF KOLMOGOROV. *If $f(x)$ is such that*

$$\|f\| \leqq 1, \ \|f^{(n)}\| \leqq \gamma_{n,n} \tag{4.4}$$

then

$$\|f^{(\nu)}\| \leqq \gamma_{n,\nu} \ \textit{for} \ \nu = 1, 2, \cdots, n-1. \tag{4.5}$$

The constants $\gamma_{n,\nu}$ in (4.5) are *best* constants because the Euler spline $\mathscr{E}_n(x)$ satisfies (4.4) and furnishes the equality sign in (4.5), simultaneously for all values of ν. Complete proofs of this theorem, in the order of their appearance, are found in **[4]**, **[2]**, and **[13]**. As the title of this paper indicates we shall establish here only the cases $n = 2$, $n = 3$, and will indicate the general method of attack used in **[13]** by remarks concerning the problem for $n = 4$, $\nu = 1$, in §10.

In order to formulate the special cases that are to be established, we need the numerical values of the corresponding $\gamma_{n,\nu}$. From (2.8), (2.9), or by determining $f_2(x), f_3(x), f_4(x)$ directly by successive integrations from (2.3), we obtain

$$\mathscr{E}_2(x) = 1 - 4x^2 \ \text{in} \ \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{4.6}$$

$$\mathscr{E}_3(x) = 1 - 6x^2 + 4x^3 \ \text{in} \ [0, 1], \tag{4.7}$$

$$\mathscr{E}_4(x) = 1 - \frac{24}{5}x^2 + \frac{16}{5}x^4 \ \text{in} \ \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{4.8}$$

Using (2.11) and (2.12), we find that

$$\gamma_{2,0} = 1, \ \gamma_{2,1} = 4, \ \gamma_{2,2} = 8, \tag{4.9}$$

$$\gamma_{3,0} = 1, \ \gamma_{3,1} = 3, \ \gamma_{3,2} = 12, \ \gamma_{3,3} = 24, \tag{4.10}$$

$$\gamma_{4,0} = 1, \ \gamma_{4,1} = \frac{16}{5}, \ \gamma_{4,2} = \frac{48}{5}, \ \gamma_{4,3} = \frac{192}{5}, \ \gamma_{4,4} = \frac{384}{5}. \tag{4.11}$$

The first three cases of Kolmogorov's theorem may now be spelled out as follows.

THEOREM 1 (Landau). *If $f(x)$ is such that*

$$\|f\| \leqq 1, \ \|f''\| \leqq 8, \tag{4.12}$$

then

$$\|f'\| \leqq 4. \tag{4.13}$$

THEOREM 2. (G. E. Šilov). *If $f(x)$ is such that*

$$\|f\| \leqq 1, \ \|f'''\| \leqq 24, \tag{4.14}$$

then

$$\|f'\| \leqq 3, \ \|f''\| \leqq 12. \tag{4.15}$$

THEOREM 3. (G. E. Šilov). *If $f(x)$ is such that*

$$\|f\| \leqq 1, \ \|f^{(4)}\| \leqq \frac{384}{5}, \tag{4.16}$$

then

$$\|f'\| \leqq \frac{16}{5}, \ \|f''\| \leqq \frac{48}{5}, \ \|f'''\| \leqq \frac{192}{5}. \tag{4.17}$$

For a reference to Šilov's work see [**4**].

5. A kinematic interpretation: 1. It seems suggestive to think of x as time and of $f = f(x)$ as describing the motion of a point on the f-axis. The first inequality (4.12) means that the point f is forever moving on the segment $-1 \leqq f \leqq 1$. The second inequality (4.12) requires that the acceleration in absolute value should never exceed 8 cm/(sec)^2. The conclusion (4.13) states that the velocity will never exceed 4 cm/sec. We know that this value is reached for the motion $f = \mathscr{E}_2(x)$ which is periodic of period 2 cm (Figure 1). Likewise (4.14) means that the rate of change of the acceleration in absolute value is not to exceed 24 cm/(sec)^3. The conclusions concerning the velocity and acceleration are then described by the inequalities (4.15).

2. Let us consider the simple harmonic motion

$$f = \sin \omega x, \ (\omega \text{ positive constant}). \tag{5.1}$$

By differentiation we find that

$$\|f\| = 1, \ \|f'\| = \omega, \ \|f''\| = \omega^2, \ \|f'''\| = \omega^3. \tag{5.2}$$

We inforce (4.12) in the most advantageous way by choosing ω such that $\omega^2 = 8$, hence $\omega = 2\sqrt{2} = 2.83$. Thus $\|f''\| = 8$, while $\|f'\| = \omega = 2.83$ falls short of the optimal value 4 given by (4.13).

Assuming (4.14) and choosing $\omega^3 = 24$, hence $\omega = 2\sqrt[3]{3} = 2.88$ we find from (5.2) that $\|f'\| = \omega = 2.88$, $\|f''\| = \omega^2 = 8.29$, which are short of the optimal values 3 and 12, respectively, as given by (4.15).

6. A general formulation of Kolmogorov's theorem. Let $F(x)$ be a bounded function having a bounded nth derivative and let

$$\|F\| = M_0, \quad \|F^{(n)}\| = M_n. \tag{6.1}$$

What upper bound can we find for

$$\|F^{(\nu)}\| = M_\nu, \quad (0 < \nu < n)? \tag{6.2}$$

The best bound for M_ν is easily found as follows: Let a and b be positive constants and let

$$f(x) = aF(bx). \tag{6.3}$$

We shall now determine a and b such that $f(x)$ satisfies the conditions

$$\|f\| = 1, \quad \|f^{(n)}\| = \gamma_{n,n}. \tag{6.4}$$

Differentiating (6.3) and using (6.1) and (6.2), we find that

$$\|f\| = aM_0, \quad \|f^{(\nu)}\| = ab^\nu M_\nu, \quad \|f^{(n)}\| = ab^n M_n. \tag{6.5}$$

To insure (6.4) we determine a and b from the equations $aM_0 = 1$ and $ab^n M_n = \gamma_{n,n}$ and find the values

$$a = M_0^{-1}, \quad b = \gamma_{n,n}^{1/n} M_0^{1/n} M_n^{-1/n}. \tag{6.6}$$

For these values

$$\|f^{(\nu)}\| = ab^\nu M_\nu = M_0^{-1} \gamma_{n,n}^{\nu/n} M_0^{\nu/n} M_n^{-\nu/n} M_\nu. \tag{6.7}$$

The relations (6.4) show that $f(x)$ satisfies the assumptions (4.4) of Kolmogorov's theorem. We may therefore apply its conclusion to the effect that $\|f^{(\nu)}\| \leqq \gamma_{n,\nu}$. Using (6.7) we find that the following statement holds.

KOLMOGOROV'S GENERAL THEOREM. *The suprema* (6.1) *and* (6.2) *satisfy the inequality*

$$M_\nu \leqq C_{n,\nu} \cdot M_0^{1-(\nu/n)} M_n^{\nu/n}, \quad \text{where } C_{n,\nu} = \gamma_{n,\nu}\gamma_{n,n}^{-\nu/n} \quad (0 < \nu < n). \tag{6.8}$$

Notice that the factor $C_{n,\nu}$ is a numerical constant depending on n and ν, and that it is the best constant because we obtain equality in (6.8) for the function $F(x)$ obtained from (6.3) if we set there $f(x) = \mathscr{E}_n(x)$. This function is

$$F(x) = a^{-1}\mathscr{E}_n(b^{-1}x),$$

where a and b have the values (6.6).

Using the values (4.9) and (4.10), the inequalities (6.8) become

$$\text{for } n = 2\colon\ M_1 \leqq 2^{1/2} M_0^{1/2} M_2^{1/2}$$

and

$$\text{for } n = 3\colon\ M_1 \leqq (2^{-1}3^{2/3}) M_0^{2/3} M_3^{1/3}, \quad M_2 \leqq 3^{1/3} M_0^{1/3} M_3^{2/3}.$$

7. A few approximate differentiation formulae. Our immediate objective is to establish Theorems 1 and 2. For this purpose we assemble here a few simple tools.

LEMMA 8. *The following identities hold for functions $f(x)$ having appropriate derivatives which are integrable*:

$$\text{(A)}\qquad f'(\tfrac{1}{2}) = f(1) - f(0) + \int_0^1 K_1(x)f''(x)dx,$$

where

$$\text{(A')}\qquad K_1(x) = \begin{cases} x & \text{if } 0 \leqq x \leqq \tfrac{1}{2}, \\ x-1 & \text{if } \tfrac{1}{2} < x \leqq 1. \end{cases}$$

$$\text{(B)}\qquad f'(\tfrac{1}{2}) = f(1) - f(0) + \int_0^1 K_2(x)f'''(x)dx,$$

where

$$\text{(B')}\qquad K_2(x) = \begin{cases} -\tfrac{1}{2}x^2 & \text{if } 0 \leqq x \leqq \tfrac{1}{2} \\ -\tfrac{1}{2}(x-1)^2 & \text{if } \tfrac{1}{2} < x \leqq 1. \end{cases}$$

$$\text{(C)}\qquad f''(0) = f(1) - 2f(0) + f(-1) + \int_{-1}^1 K_3(x)f'''(x)dx,$$

where

$$\text{(C')}\qquad K_3(x) = \begin{cases} \tfrac{1}{2}(x+1)^2 & \text{if } -1 \leqq x \leqq 0 \\ -\tfrac{1}{2}(x-1)^2 & \text{if } 0 < x \leqq 1. \end{cases}$$

Proof: These formulae belong to those elementary parts of numerical analysis which deal with the approximate performance of the operations of Calculus (interpolation, differentiation, integration, a.s.f.). The fundamental tool in this field is Taylor's formula with Cauchy's integral remainder

$$\text{(7.1)}\qquad f(t) = f(a) + (t-a)f'(a) + \cdots + \frac{(t-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{1}{(n-1)!}\int_a^t (t-x)^{n-1}f^{(n)}(x)dx.$$

It is derived by integrating by parts the remainder n times.

A. We apply (7.1) for $n = 2$, $a = 1/2$ and the two values $t = 1$ and $t = 0$, obtaining

$$f(1) = f(\tfrac{1}{2}) + \tfrac{1}{2}f'(\tfrac{1}{2}) + \int_{1/2}^1 (1-x)f''(x)dx$$

$$f(0) = f(\tfrac{1}{2}) - \tfrac{1}{2}f'(\tfrac{1}{2}) + \int_{1/2}^0 (-x)f''(x)dx.$$

Subtracting we get

$$f(1)-f(0)=f'(\tfrac{1}{2})-\int_0^{1/2} xf''(x)dx-\int_{1/2}^1 (x-1)f''(x)dx$$

and this agrees with (A), (A′).

B. Observe that $K_2(x)$ is *continuous* and that

$$K_2'(x) = -K_1(x).$$

We may therefore integrate by parts the remainder of (A) to obtain

$$\int_0^1 K_1(x)f''(x)dx = -\int_0^1 f''(x)dK_2(x) = \int_0^1 K_2(x)f'''(x)dx,$$

because $K_2(0) = K_2(1) = 0$. This establishes (B) and (B′). Alternatively, we apply (7.1) for $n = 3$, $a = 1/2$ and the two values $t = 1$ and $t = 0$, and subtract the resulting relations.

C. Apply (7.1) for $n = 3$, $a = 0$ and the two values $t = 1$ and $t = -1$ to obtain

$$f(1)=f(0)+f'(0)+\tfrac{1}{2}f''(0)+\tfrac{1}{2}\int_0^1 (1-x)^2 f'''(x)dx,$$

$$f(-1)=f(0)-f'(0)+\tfrac{1}{2}f''(0)+\tfrac{1}{2}\int_0^{-1} (-1-x)^2 f'''(x)dx.$$

Adding these we get

$$f(1)-2f(0)+f(-1)=f''(0)-\tfrac{1}{2}\int_{-1}^0 (x+1)^2 f'''(x)dx+\tfrac{1}{2}\int_0^1 (x-1)^2 f'''(x)dx$$

which is identical with (C) and (C′). □

8. Proofs of Theorems 1 and 2 and their extremizing functions in the strict sense. *Let us establish Theorem* 1 (§4): We consider the function

(8.1) $$f_0(x) = -\mathscr{E}_2(x).$$

From (4.6) and Figure 1 we see that it has the properties

(8.2) $$f_0(0) = -1,\ f_0(1) = 1,\ f_0'(\tfrac{1}{2}) = 4 \text{ and } f_0(x) = \begin{cases} 8 \text{ in } (0,\tfrac{1}{2}) \\ -8 \text{ in } (\tfrac{1}{2},1). \end{cases}$$

Applying the differentiation formula (A) of §7 to $f_0(x)$ we find by (8.2) and the explicit form (A′) of the kernel $K_1(x)$ that

(8.3) $$4 = f_0'(\tfrac{1}{2}) = 1+1+8\int_0^1 |K_1(x)|\,dx.$$

Let $f(x)$ be any function satisfying (4.12) and let us evaluate $f'(\tfrac{1}{2})$ by the formula (A). Moreover, we may assume that $f'(\frac{1}{2}) \geqq 0$, for if $f'(\frac{1}{2}) < 0$ then we could replace $f(x)$ by $-f(x)$. We now obtain

$$(8.4)\quad (0 \leqq) f'(\tfrac{1}{2}) = f(1) - f(0) + \int_0^1 K_1(x) f''(x) dx \leqq 1 + 1 + 8 \int_0^1 |K_1(x)| dx$$

the last inequality being a consequence of (4.12). Moreover, the last member is equal to 4 by (8.3) .Therefore

$$(8.5)\qquad |f'(\tfrac{1}{2})| \leqq 4.$$

This implies that $|f'(x_0)| \leqq 4$ no matter what x_0 may be. For also $f(x + x_0 - \frac{1}{2})$ satisfies all assumptions and applying (8.5) to it, we find that $|f'(x_0)| \leqq 4$. □

Let us assume now that

$$(8.6)\qquad f'(\tfrac{1}{2}) = 4$$

and see what the consequence are. Evidently (8.6) holds if and only if we have the equality sign in (8.4). Also, again in view of the conditions (4.12), we have equality in (8.4) if and only if $f(x)$ satisfies the conditions

$$(8.7)\qquad f(0) = -1,\ f(1) = 1,\ f''(x) = \begin{cases} 8 \text{ in } (0, \frac{1}{2}) \\ -8 \text{ in } (\frac{1}{2}, 1). \end{cases}$$

Moreover, $f(0) = -1$ and $\|f\| \leqq 1$ imply that $f'(0) = 0$ and $f(1) = 1$, with $\|f\| \leqq 1$, imply that $f'(1) = 0$. It clearly follows from (8.6) that

$$f(x) = -\mathscr{E}_2(x) \text{ in } (0,1).$$

We state this result as

THEOREM 4. *If*

$$(8.8)\qquad \|f\| \leqq 1,\ \|f''\| \leqq 8$$

and

$$(8.9)\qquad f'(\tfrac{1}{2}) = 4,$$

then

$$(8.10)\qquad f(x) = -\mathscr{E}_2(x) \text{ in the interval } [0,1].$$

Outside the interval $[0,1]$ there is little that we can say about the function $f(x)$ satisfying (8.8) and (8.9). Indeed, notice that there are many ways in which the function (8.10) can be extended to all reals and still satisfying (8.8) (of course with the *equality* sign in both inequalities). For beside the obvious extension

$$(8.11)\qquad f(x) = -\mathscr{E}_2(x) \text{ for all real } x,$$

we can also write

$$(8.12)\qquad f(x) = \begin{cases} 1 & \text{if } x > 1 \\ -\mathscr{E}_2(x) & \text{if } 0 \leqq x \leqq 1, \\ -1 & \text{if } x < 0, \end{cases}$$

and many similar modifications of the function (8.11).

A comment on the function $f(x)$ satisfying (8.8) and (8.9) is in order. We can call $f(x)$ an *extremizing function* in Theorem 1 because $f(x)$ satisfies (4.13) with the equality sign, hence

$$(8.13)\qquad \|f'\| = 4.$$

Moreover, we wish to call this $f(x)$ an extremizing function *in the strong* sense because the supremum of $|f'(x)|\,(=4)$ is *actually assumed for a real* x, *viz.* $x = \frac{1}{2}$. We shall see in §9 that there are numerous extremizing function $f(x)$ in Theorem 1, hence satisfying (8.13), such that

$$(8.14)\qquad |f'(x)| < 4 \text{ for all real } x.$$

Such functions may be called extremizing functions *in the weak sense.*

Let us establish Theorem 2 (§4): Let $f(x)$ satisfy (4.14) $\|f\| \leqq 1$, $\|f'''\| \leqq 24$, and let us show that (4.15) $\|f'\| \leqq 3$, $\|f''\| \leqq 12$.

We reproduce here the second differentiation formula

$$(\mathrm{B})\qquad f'(\tfrac{1}{2}) = f(1) - f(0) + \int_0^1 K_2(x) f'''(x)\,dx$$

of Lemma 8 and apply it to the function

$$(8.15)\qquad f_0(x) = -\mathscr{E}_3(x) = -1 + 6x^2 - 4x^3 \text{ in } [0,1].$$

This function has in $[0,1]$ the properties

$$(8.16)\qquad f_0(0) = -1,\ f_0(1) = 1,\ f'''(x) = -24.$$

In view of (B′), of Lemma 8, we know that $K_2(x) < 0$ in $(0,1)$, and from (B) we derive

$$(8.17)\qquad 3 = f_0'(\tfrac{1}{2}) = 1 + 1 + 24\int_0^1 |K_2(x)|\,dx.$$

If $f(x)$ is any function satisfying (4.14), let us evaluate $f'(\frac{1}{2})$ by (B), assuming that $f'(\frac{1}{2}) \geqq 0$ (otherwise we take $-f(x)$). We obtain

$$(8.18)\quad (0 \leqq) f'(\tfrac{1}{2}) = f(1) - f(0) + \int_0^1 K_2(x) f'''(x)\,dx \leqq 1 + 1 + 24\int_0^1 |K_2(x)|\,dx = 3,$$

by (8.17) and the first inequality (4.15) is thereby established.

At this point we interrupt our proof of Theorem 2 in order to see what we can say about $f(x)$ if

$$f'(\tfrac{1}{2}) = 3, \tag{8.19}$$

i.e., if equality holds in (8.18). From (4.14) we see that we have equality in (8.18) if and only if $f(x)$ has the properties

$$f(0) = -1, \; f(1) = 1, \text{ and } f'''(x) = -24 \text{ in } (0,1). \tag{8.20}$$

However, as before, we also have

$$f'(0) = f'(1) = 0 \tag{8.21}$$

and, of course, (8.19). The conditions (8.19), (8.20) and (8.21) are more than sufficient to imply

$$f(x) = -\mathscr{E}_3(x) \text{ in } [0,1]. \tag{8.22}$$

Let us record here this result as

COROLLARY 1. *If $f(x)$ satisfies* (4.14) *and* (8.19), *then* (8.22) *also holds.*

We now wish to establish the second inequality (4.15): For this purpose we need the third formula

$$f''(0) = f(1) - 2f(0) + f(-1) + \int_{-1}^{1} K_3(x) f'''(x)dx \tag{C}$$

of Lemma 8. We recall that by the formula (C′) of that lemma the kernel has the properties

$$K_3(x) > 0 \text{ in } (-1,0), \; K_3(x) < 0 \text{ in } (0,1). \tag{8.23}$$

We now apply (C) to the function

$$f_0(x) = -\mathscr{E}_3(x) = \begin{cases} -1 + 6x^2 + 4x^3 & \text{in } [-1,0] \\ -1 + 6x^2 - 4x^3 & \text{in } (0,1]. \end{cases} \tag{8.24}$$

This function has the properties

$$f_0(-1) = 1, \; f_0(0) = -1, \; f_0(1) = 1, \; f_0'''(x) = \begin{cases} 24 \text{ in } (-1,0), \\ -24 \text{ in } (0,1), \end{cases} \tag{8.25}$$

and (C), (8.23), and (8.25) show that

$$12 = f_0''(0) = 1 + 2 + 1 + 24\int_{-1}^{1} |K_3(x)|\, dx. \tag{8.26}$$

If $f(x)$ is any function satisfying (4.14), and assuming that $f''(0) \geqq 0$, an application of (C) shows that

$$(8.27)\quad \begin{aligned}(0 \leqq) f''(0) &= f(1) - 2f(0) + f(-1) + \int_{-1}^{1} K_3(x) f'''(x)dx \\ &\leqq 1 + 2 + 1 + 24 \int_{-1}^{1} |K_3(x)|\, dx = 12\end{aligned}$$

by (8.26). Applying this result to $f(x + x_0)$ we obtain that $|f''(x_0)| \leqq 12$, and Theorem 2 is established. □

Let us assume that $f(x)$, satisfying (4.14), is such that

$$(8.28)\qquad f''(0) = 12,$$

and let us examine the consequences of this assumption. Clearly (8.28) if and only if we have the equality sign in (8.27) and this turn holds if and only if

$$f(-1) = 1,\ f(0) = -1,\ f(1) = 1, \text{ and } f'''(x) = \begin{cases} 24 \text{ in } (-1, 0), \\ -24 \text{ in } (0, 1). \end{cases}$$

From this we conclude that

$$(8.29)\qquad f(x) = -\mathscr{E}_3(x) \text{ in } -1 \leqq x \leqq 1.$$

We have therefore established the

COROLLARY 2. *If $f(x)$ satisfies* (4.14) *and* (8.28) *holds, then also* (8.29) *holds.*

The following generalization follows by a change of origin:

COROLLARY 2′. *If $f(x)$ satisfies* (4.14) *and is such that*

$$(8.30)\qquad f''(a) = \pm 12$$

then

$$(8.31)\qquad f(x) = \mp \mathscr{E}_3(x - a) \textit{ if } a - 1 \leqq x \leqq a + 1.$$

We may now state our

THEOREM 5. *If* (4.14) $\|f\| \leqq 1$, $\|f'''\| \leqq 24$ *and if in one of the inequalities* (4.15) $\|f'\| \leqq 3$, $\|f''\| \leqq 12$, *we have the equality sign, the corresponding supremum being actually attained, then*

$$(8.32)\qquad f(x) = \mathscr{E}_3(x - c) \textit{ for all real } x,$$

for an appropriate constant c.

Proof: 1. Let us assume that $\|f'\| = 3$. This supremum being assumed, we loose no generality by assuming that

$$f'(\tfrac{1}{2}) = 3. \tag{8.33}$$

Now Corollary 1 implies that

$$f(x) = -\mathscr{E}_3(x) \text{ in } [0,1]. \tag{8.34}$$

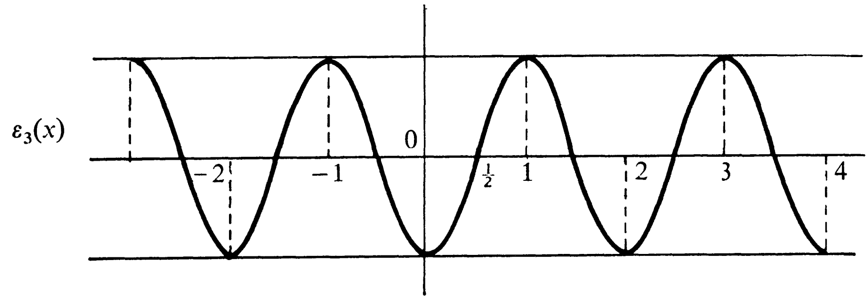

FIG. 2.

This in turn shows that $f''(1) = -12$ (see Figure 2) and now Corollary 2′ shows $f(x) = -\mathscr{E}_3(x)$ in $[0,2]$. But then surely $f''(2) = 12$ and Corollary 2′ implies that $f(x) = -\mathscr{E}_3(x)$ in $[1,3]$. We can continue in this way indefinitely and conclude that $f(x) = -\mathscr{E}_3(x)$ for $x \geqq 0$. However, the same reasoning works also to the left: From (8.34) we conclude that $f''(0) = +12$ and therefore (8.34) holds also in $[-1,1]$, hence $f''(-1) = -12$ and (8.34) holds in $[-2,0]$ a.s.f. Therefore $f(x) = -\mathscr{E}_3(x) = \mathscr{E}_3(x-1)$ holds for all real x.

2. If we have equality in the second inequality (4.15), we get the same conclusion by applying only Corollary 2′. □

9. The extremizing functions in the weak sense. In the present section we discuss only the cubic case of $n = 3$. Our last Theorem 5 has answered the question as to when we have the equality sign in one of the inequalities (4.15) for the case when the respective supremum is actually attained.

DEFINITION 4. *We say that $f(x)$ is an extremizing function in the weak sense for $n = 3$, provided that $f(x)$ satisfies the inequalities*

$$\|f\| \leqq 1, \quad \|f'''\| \leqq 24, \tag{9.1}$$

and therefore also

$$\|f'\| \leqq 3, \quad \|f''\| \leqq 12, \tag{9.2}$$

with the equality sign in one of the inequalities (9.2), *the corresponding supremum not being attained.*

This definition raises the following questions:

QUESTION 1. *Do extremizing functions in the weak sense exist?*

QUESTION 2. *Let us suppose that they do and let $f(x)$ be one such. Does then the equality sign hold in all four inequalities* (9.1), (9.2)?

We shall see that the answers to both questions are *affirmative.*

The affirmative answer to Question 1 is contained in

THEOREM 6. *There exist functions $f(x)$ such that*

$$\|f\| = 1,\ \|f'\| = 3,\ \|f''\| = 12,\ \|f'''\| = 24, \tag{9.3}$$

while

$$|f(x)| < 1,\ |f'(x)| < 3,\ |f''(x)| < 12,\ |f'''(x)| < 24 \text{ for all real } x. \tag{9.4}$$

Proof: We know that $f(x) = \mathscr{E}_3(x)$ satisfies (9.3), but not (9.4). To enforce both (9.3) *and* (9.4) we let the function "sag between $-\infty$ and $+\infty$" by passing to the new function

$$f(x) = \mathscr{E}_3(x)\phi(x) \tag{9.5}$$

with an appropriate positive function $\phi(x)$ to be constructed.

Using the known values (4.10) of $\gamma_{3,\nu} = \| \mathscr{E}^{(\nu)} \|$ we derive from (9.5) the inequalities

$$\begin{aligned} |f(x)| &= |\mathscr{E}\phi| \leqq \phi(x), \\ |f'(x)| &= |\mathscr{E}'\phi + \mathscr{E}\phi'| \leqq 3\phi(x) + |\phi'(x)|, \\ |f''(x)| &= |\mathscr{E}''\phi + 2\mathscr{E}'\phi' + \mathscr{E}\phi''| \leqq 12\phi(x) + 6|\phi'(x)| + |\phi''(x)|, \\ |f'''(x)| &= |\mathscr{E}'''\phi + 3\mathscr{E}''\phi' + 3\mathscr{E}'\phi'' + \mathscr{E}\phi'''| \leqq 24\phi(x) + 36|\phi'(x)| \\ &\qquad + 9|\phi''(x)| + |\phi'''(x)|. \end{aligned} \tag{9.6}$$

We shall therefore satisfy (9.4) if $\phi(x)$ is *positive* and such that

$$\begin{aligned} &\phi < 1, \\ &3\phi + |\phi'| < 3, \\ &12\phi + 6|\phi'| + |\phi''| < 12, \\ &24\phi + 36|\phi'| + 9|\phi''| + |\phi'''| < 24, \text{ for all real } x. \end{aligned}$$

These amount to

$$\begin{aligned} &1 - \phi > 0, \\ &1 - \phi > \frac{1}{3}|\phi'|, \\ &1 - \phi > \frac{1}{2}|\phi'| + \frac{1}{12}|\phi''|, \\ &1 - \phi > \frac{3}{2}|\phi'| + \frac{3}{8}|\phi''| + \frac{1}{24}|\phi'''|. \end{aligned}$$

Observe that the last inequality implies the previous ones. It suffices therefore to require that $\phi(x)$ *be positive and to satisfy the differential inequality*

$$1 - \phi(x) > \frac{3}{2}|\phi'(x)| + \frac{3}{8}|\phi''(x)| + \frac{1}{24}|\phi'''(x)| \text{ for all real } x. \tag{9.7}$$

In order to insure also the equations (9.3), it is clear that $\phi(x)$ *should also satisfy the boundary conditions*

$$\phi(x) \to 1,\ \phi'(x) \to 0,\ \phi''(x) \to 0,\ \phi'''(x) \to 0 \text{ as } x \to \pm\infty. \tag{9.8}$$

Indeed, Leibniz's formulae (see (9.6)!) and the periodicity of $\mathscr{E}(x)$ will then show that

$$\|f^{(\nu)}\| \geqq \overline{\lim_{x\to\pm\infty}} |(\mathscr{E}\phi)^{(\nu)}| = \|\mathscr{E}^{(\nu)}\|, \quad (\nu = 0,1,2,3).$$

Let

$$\psi(x) = 1 - e^{-\gamma x},\ (\gamma \text{ positive constant}). \tag{9.9}$$

A simple calculation shows that $\psi(x)$ will surely satisfy (9.7), provided that

$$1 > \frac{3}{2}\gamma + \frac{3}{8}\gamma^2 + \frac{1}{24}\gamma^3 \tag{9.10}$$

for which $0 < \gamma \leqq 1/2$ will certainly do. We now define

$$\phi(x) = \begin{cases} 1 - e^{-\gamma x} & \text{if } x \geqq 1, \\ \phi(-x) & \text{if } x \leqq -1. \end{cases} \tag{9.11}$$

Assuming (9.10), this function satisfies (9.7) outside the interval $(-1,1)$. Moreover, $\phi(x)$ is positive and also satisfies the boundary conditions (9.8).

There remains to bridge the gap between -1 and 1 and this we do by interpolation as follows. Let

$$P(x) = A + Bx^2 + Cx^4 \tag{9.12}$$

and

$$\phi(x) = P(x) \text{ in } -1 \leqq x \leqq 1. \tag{9.13}$$

We also require $P(x)$ to satisfy the interpolatory conditions

$$P(1) = \psi(1),\ P'(1) = \psi'(1),\ P''(1) = \psi''(1). \tag{9.14}$$

The functions $P(x)$ and $\phi(x)$ being both even, it is clear that the requirements (9.14) will insure that $\phi(x) \in C''(\mathbb{R})$.

We are yet to insure that $P(x)$ is positive and satisfies (9.7) in $[0,1]$, and therefore also in $[-1,1]$. From (9.14) we easily get for the coefficients of $P(x)$ the values

(9.15) $A = 1-(1+\frac{5}{8}\gamma+\frac{1}{8}\gamma^2)e^{-\gamma}$, $B = \frac{1}{4}\gamma(3+\gamma)e^{-\gamma}$, $C = -\frac{1}{8}\gamma(1+\gamma)e^{-\gamma}$.

1. *The positivity of* $P(x)$ *in* $[0,1]$: Dropping the positive term Bx^2

$$\begin{aligned} P(x) &= A + Bx^2 + Cx^4 > A + C \\ &= 1-(1+\frac{5}{8}\gamma+\frac{1}{8}\gamma^2)e^{-\gamma} - \frac{1}{8}\gamma(1+\gamma)e^{-\gamma} > 0 \end{aligned}$$

because the last inequality is equivalent to $e^\gamma > 1 + \frac{6}{8}\gamma + \frac{2}{8}\gamma^2$, which evidently holds.

2. $P(x)$ *satisfies* (9.7) *in* $[0,1]$: We are to find γ such that

$$1 - A - Bx^2 - Cx^4 > \frac{3}{2}|2Bx + 4Cx^3| + \frac{3}{8}|2B + 12Cx^2| + \frac{1}{24}|24Cx| \tag{9.16}$$

holds in $0 \leqq x \leqq 1$. Dropping on the left the positive term $-Cx^4$, cancelling the common factor $e^{-\gamma}$, and taking on the left side all terms with their negative values for $x = 1$ and on the right with their positive values for $x = 1$, we easily find, after rearrangements that the inequality (9.16) is surely satisfied if the inequality $1 > \frac{1}{8}(35\gamma + 25\gamma^2)$ holds. This is the case if

$$0 < \gamma < \frac{1}{5}. \tag{9.17}$$

To summarize: Let γ satisfy (9.17) and $P(x)$ be defined by (9.12) and (9.15). Finally let

$$\phi(x) = \begin{cases} 1 - e^{-\gamma|x|} & \text{if } |x| \geqq 1 \\ P(x) & \text{if } -1 < x < 1. \end{cases}$$

Then $f(x)$, defined by (9.5), satisfies the conditions (9.3) and (9.4) of Theorem 6. □

The second question is answered affirmatively by

THEOREM 7. *Let* $f(x)$ *be such that*

$$\|f\| \leqq 1, \quad \|f'''\| \leqq 24 \tag{9.18}$$

and therefore

$$\|f'\| \leqq 3, \quad \|f''\| \leqq 12. \tag{9.19}$$

If the equality sign holds in one of the inequalities (9.19), *then the equality sign holds in all four inequalities.*

Proof: 1. Let us assume that

$$\|f'\| = 3. \tag{9.20}$$

If the supremum $\|f'\|$ is assumed, then we know by Theorem 5 that (8.32) holds and we are through. We may therefore assume that

$$|f'(x)| < 3 \text{ for all real } x. \tag{9.21}$$

Let (x_ν), $(\nu = 1, 2, \cdots)$, be a sequence of points such that

$$\lim_{\nu \to \infty} f'(x_\nu) = 3, \tag{9.22}$$

the reasoning to be applied being similar if this limit should be -3. It should be clear that the sequence (x_ν) can not have a finite limit point ξ, for we could then conclude from the continuity of $f'(x)$ that $f'(\xi) = 3$, in contradiction to (9.21). We may therefore assume that $x_\nu \to +\infty$, or perhaps $-\infty$. Let us assume that

$$\lim_{\nu \to \infty} x_\nu = +\infty. \tag{9.23}$$

By the formula (B) of Lemma 8 we may write

$$f'(x_\nu) = f(x_\nu + \tfrac{1}{2}) - f(x_\nu - \tfrac{1}{2}) + \int_0^1 K_2(x) f'''(x + x_\nu - \tfrac{1}{2})dx, \tag{9.24}$$

while (8.17) shows that

$$3 = 1 + 1 + \int_0^1 |K_2(x)| \cdot 24dx. \tag{9.25}$$

From (9.22) we conclude that

$$\begin{aligned} &f(x_\nu + \tfrac{1}{2}) - f(x_\nu - \tfrac{1}{2}) + \int_0^1 |K_2(x)| \{-f'''(x + x_\nu - \tfrac{1}{2})\}dx \\ &\to 1 + 1 + \int_0^1 |K_2(x)| \cdot 24dx \qquad \text{as } \nu \to \infty. \end{aligned} \tag{9.26}$$

From this relation we shall derive all that we need.

We observe first that

$$f(x_\nu + \tfrac{1}{2}) - f(x_\nu - \tfrac{1}{2}) + \int_0^1 |K_2| \{-f'''\}dx > 1 + 1 + \int_0^1 |K_2| \cdot 24dx - \varepsilon$$

if $\nu > N(\varepsilon)$, while

$$\int_0^1 |K_2| 24dx \geqq \int_0^1 |K_2| \{-f'''\}dx$$

and $1 \geqq -f(x_\nu - \tfrac{1}{2})$ hold anyway. Adding these three inequalities we find that $f(x_\nu + \tfrac{1}{2}) > 1 - \varepsilon$ if $\nu > N(\varepsilon)$ and therefore

$$\lim_{\nu \to \infty} f(x_\nu + \tfrac{1}{2}) = 1. \tag{9.27}$$

Similarly we find that

$$\lim_{\nu\to\infty} f(x_\nu - \tfrac{1}{2}) = -1, \tag{9.28}$$

and finally, from (9.26), that

$$\lim_{\nu\to\infty} \int_0^1 |K_2(x)| \{-f'''(x + x_\nu - \tfrac{1}{2})\} dx = \int_0^1 |K_2(x)| \cdot 24 dx.$$

If we write

$$\phi_\nu(x) = 24 + f'''(x + x_\nu - \tfrac{1}{2}), \qquad (0 \leqq x \leqq 1), \tag{9.29}$$

we know that this sequence of piece-wise continuous functions has the properties

$$0 \leqq \phi_\nu(x) \leqq 48 \text{ in } [0,1], \tag{9.30}$$

and

$$\lim_{\nu\to\infty} \int_0^1 |K_2(x)| \phi_\nu(x) dx = 0. \tag{9.31}$$

From (B′) of §7 we know that $|K_2(x)|$ vanishes at 0 and 1, that it increases in $[0,\frac{1}{2}]$ and decreases in $[\frac{1}{2},1]$. Also that $K_2(x) = K_2(1-x)$. We choose α such that $0 < \alpha < \frac{1}{2}$ and may write

$$\int_0^1 |K_2(x)| \phi_\nu(x) dx \geqq |K_2(\alpha)| \int_\alpha^{1-\alpha} \phi_\nu(x) dx \geqq |K_2(\alpha)| \cdot \inf_{[\alpha,1-\alpha]} \phi_\nu(x). \tag{9.32}$$

Now (9.31) implies that

$$\inf_{[\alpha,1-\alpha]} \phi_\nu(x) \to 0 \text{ as } \nu \to \infty. \tag{9.33}$$

Selecting ξ_ν in $[\alpha, 1-\alpha]$ such that $\phi_\nu(\xi_\nu) < \inf \phi_\nu(x) + 2^{-\nu}$, we conclude from (9.33) that $\phi_\nu(\xi_\nu) \to 0$. Finally, returning to f''' by (9.29) we have shown that

$$\lim_{\nu\to\infty} f'''(\xi_\nu + x_\nu - \tfrac{1}{2}) = -24. \tag{9.34}$$

Evidently (9.27), or (9.28), and (9.34) show that

$$\|f\| = 1, \ \|f'''\| = 24. \tag{9.35}$$

There remains to show that

$$\|f''\| = 12. \tag{9.36}$$

From (9.31) and (9.32) we conclude that

$$\lim_{\nu\to\infty} \int_\alpha^{1-\alpha} \phi_\nu(x) dx = 0.$$

However, this integral can be evaluated by (9.29) and we obtain

$$24(1-2\alpha)+f''(\eta_\nu)-f''(\xi_\nu)\to 0,$$

where $\eta_\nu = 1-\alpha+x_\nu-\frac{1}{2}$, $\xi_\nu = \alpha + x_\nu - \frac{1}{2}$. Therefore $f''(\xi_\nu)-f''(\eta_\nu)\to 24(1-2\alpha)$ as $\nu\to\infty$, hence $f''(\xi_\nu)-f''(\eta_\nu) > 24-48\alpha-\varepsilon$ if $\nu > N(\varepsilon)$. Adding to this the inequality $f''(\eta_\nu)\geqq -12$ we obtain that $f''(\xi_\nu) > 12-48\alpha-\varepsilon$ if $\nu > N(\varepsilon)$. Since $\xi_\nu\to+\infty$ and α is arbitrarily small, we conclude that

$$\overline{\lim_{x\to+\infty}}\, f''(x) \geqq 12.$$

This, together with $\|f''\|\leqq 12$, shows that (9.36) holds.

2. A similar method, this time using the approximate differentiation formula (C) of Lemma 8, allows to show that (9.36) implies the equality sign in all other inequalities (9.18) and (9.19). However, we omit further details. □

3. There are theorems analogous to Theorems 6 and 7 for the case when $n = 2$ and they are easier to derive. Also for $n = 2$ there are extremizing functions in the weak sense, i.e., satisfying (8.8), (8.13), and (8.14). The details may be left to the reader.

10. How is Theorem 3 established? As its title indicates, this paper is devoted to the elementary cases of Landau's problem. However, Theorem 3 is no longer an elementary case. The ideas underlying its proof are just as simple as before, but the necessary tools, i.e., the required approximate differentiation formulae, are more complicated.

Let us sketch, with a minimum of detail, a proof of the first inequality

$$\|f'\|\leqq 16/5 \tag{10.1}$$

of Theorem 3, assuming that

$$\|f\|\leqq 1,\ \|f^{(4)}\|\leqq 384/5. \tag{10.2}$$

The approximate differentiation formula that we need is

$$\begin{aligned} f'(\tfrac{1}{2}) &= \mu f(1)+\mu\lambda f(2)+\mu\lambda^2 f(3)+\cdots \\ &\quad -\mu f(0)-\mu\lambda f(-1)-\mu\lambda^2 f(-2)\cdots+\int_{-\infty}^{\infty}K(x)f^{(4)}(x)dx, \end{aligned} \tag{10.3}$$

where

$$\mu = \frac{-60+12\sqrt{30}}{5} = 1.14534,\ \lambda = -11+2\sqrt{30} = -.045\,548. \tag{10.4}$$

The kernel $K(x)$ is a cardinal cubic spline, i.e., having its knots at the integers, except that at $x=\frac{1}{2}$ it has a discontinuity in its second derivative. It satisfies $K(x) = -K(1-x)$ and is therefore odd about the point $x = 1/2$. $K(x)$ decays exponentially as $x \to \pm\infty$ so that $K(x)$ is absolutely integrable on the real axis. Moreover

$$K(\nu + \tfrac{1}{2}) = 0 \quad \text{for all integer } \nu, \tag{10.5}$$

and $K(x)$ vanishes nowhere else. Finally

$$K(x) < 0 \text{ if } -\tfrac{1}{2} < x < \tfrac{1}{2} \tag{10.6}$$

and it changes sign at each $\nu + \frac{1}{2}$ (see Figure 3).

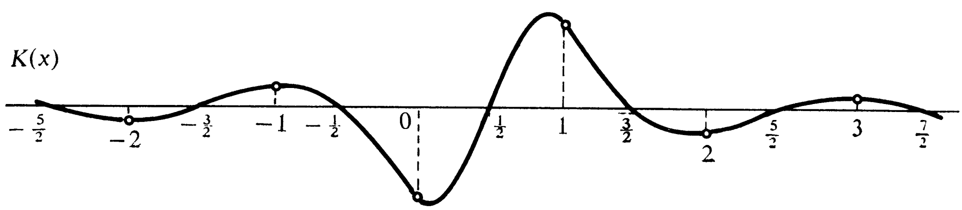

FIG. 3.

If we substitute into (10.3) the function

$$f_0(x) = -\mathscr{E}_4(x),$$

we find that $K(x)f_0^{(4)}(x)$ is positive for all x, except that it vanishes if $x = \nu + \frac{1}{2}$ by (10.5). Since $f_0^{(4)}(x) = \pm 384/5$ we obtain the result

$$\frac{16}{5} = f_0'\left(\frac{1}{2}\right) = 2\mu \sum_0^\infty |\lambda|^\nu + \frac{384}{5}\int_{-\infty}^{\infty} |K(x)|\,dx. \tag{10.7}$$

If $f(x)$ is any function satisfying (10.2), and assuming that $f'(\frac{1}{2}) \geqq 0$, we obtain from (10.3) and (10.2) the estimate

$$0 \leqq f'\left(\frac{1}{2}\right) \leqq 2\mu \sum_0^\infty |\lambda|^\nu + \frac{384}{5}\int_{-\infty}^{\infty} |K(x)|\,dx = \frac{16}{5},$$

by (10.7). By reasonings used before, the equality sign is seen to hold only if $f(x) = -\mathscr{E}_4(x)$. This establishes (10.1), except that we have not proved the identity (10.3), nor do we propose to do so. However, let me say the following: The formula (10.3) is exact, i.e., its remainder vanishes, whenever $f(x)$ is a cubic polynomial. This clearly does not characterize the formula. However, (10.3) can be shown to be exact if $f(x)$ is a cardinal cubic spline with knots at $\nu + \frac{1}{2}$ that grows at most like a power of $|x|$ as $x \to \pm\infty$, and this condition characterizes the formula (10.3) and allows to derive it.

A last remark: The question arises whether (10.3) could be replaced in the above application by some appropriate *finite* formula that involves only finitely many of the ordinates $f(\nu)$. The answer is no: It can be shown that no finite differentiation formula exists that will serve the same purpose. For further details we refer to [**13**].

III. LANDAU'S PROBLEM FOR $\mathbb{R}_+ = [0, \infty)$.

11. The case $n = 2$. Landau's problem for the halfline $\mathbb{R}_+$ is similar to the problem solved by Kolmogorov's theorem, the difference being that now the competition is open only for functions from $\mathbb{R}_+$ to $\mathbb{R}$. Accordingly, the role of the previous norm $\|f\|$ is now taken over by the *halfline norm*

$$\|f\|_+ = \sup|f(x)| \text{ for } x \geqq 0. \tag{11.1}$$

To facilitate the comparison with the results of Part II, we choose the same normalization as in Kolmogorov's theorem, namely

$$\|f\|_+ \leqq 1, \ \|f^{(n)}\|_+ \leqq \gamma_{n,n}, \tag{11.2}$$

the objective being to find within this class those functions that maximize the norms

$$\|f^{(\nu)}\|_+, \text{ for } \nu = 1, 2, \cdots, n-1. \tag{11.3}$$

The transition to other normalizations, such as the one used in [**11**], can be achieved by means of the trivial transformation (6.3) used in §6. In §13 the $\mathbb{R}_+$-analogue of Kolmogorov's theorem will be mentioned. In the meantime we turn to the first of the two elementary cases of the problem.

We assume that

$$\|f\|_+ \leqq 1, \ \|f''\|_+ \leqq 8, \tag{11.4}$$

and seek a function $f_0(x)$ such that $\|f_0'\|_+ \geqq \|f'\|_+$ for all functions $f(x)$ satisfying (11.4).

The function $\mathscr{E}_2(x)$ satisfies (11.4) and we also know that $\|\mathscr{E}_2'\|_+ = 4$ (this is where the constant 4 of Theorem 1 came from). Now we can do better! Indeed, let us consider $\mathscr{E}_2(x)$ for $x \geqq -\frac{1}{2}$, and let us remove the knot at $x = -\frac{1}{2}$ and continue the quadratic $y = 1 - 4x^2$ (see (4.6)) also for values of $x < -\frac{1}{2}$, until we reach the point where the parabolic graph of $y = 1 - 4x^2$ intersects the horizontal line $y = -1$. We find that this happens for $x = -1/\sqrt{2}$. We consider the function

$$g(x) = \begin{cases} 1 - 4x^2 & \text{if } -1/\sqrt{2} \leqq x \leqq 0, \\ \mathscr{E}_2(x) & \text{if } 0 \leqq x < \infty, \end{cases}$$

and shift the origin to $-1/\sqrt{2}$ to define

$$f_0(x) = g(x - 1/\sqrt{2}) \text{ for } x \geqq 0, \text{ (see Figure 4).}$$

Clearly

$$\|f_0\|_+ = 1, \ \|f_0\|_+ = 8. \tag{11.5}$$

However, it should also be clear from Figure 4 that $\|f_0'\|_+$ is reached by $|f_0'(x)|$ for $x = 0$ so that

$$\|f_0'\|_+ = f_0'(0) = g'(-1/\sqrt{2}) = -8x|_{x=-1/\sqrt{2}} = 4\sqrt{2} = 5.65684.$$

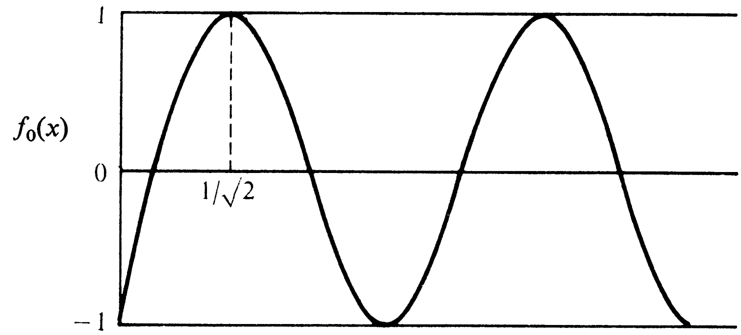

FIG. 4.

Therefore

$$\|f_0'\|_+ = f_0'(0) = 4\sqrt{2}. \tag{11.6}$$

We see that the conditions (11.4) no longer imply that $\|f'\|_+ \leqq 4$, as in Theorem 1, but allow considerably larger values such as $\|f'\|_+ = 4\sqrt{2}$. This, however, is the largest value, a fact which we state as

THEOREM 8. (Landau's theorem). *If*

$$\|f\|_+ \leqq 1, \quad \|f''\|_+ \leqq 8, \tag{11.7}$$

then

$$\|f'\|_+ \leqq 4\sqrt{2}. \tag{11.8}$$

Here $4\sqrt{2}$ *is the best constant because it is reached for the above function* $f_0(x)$.

Proof: As in the case of Theorem 1, we need a differentiation formula. By Taylor's formula (7.1), for $n = 2$, $a = 0$ and $t = 1/\sqrt{2}$, we have

$$f(1/\sqrt{2}) = f(0) + (1/\sqrt{2})f'(0) + \int_0^{1/\sqrt{2}} (1/\sqrt{2} - x)f''(x)dx.$$

Solving for $f'(0)$ we obtain

$$f'(0) = \sqrt{2}f(1/\sqrt{2}) - \sqrt{2}f(0) - \int_0^{1/\sqrt{2}} (1 - x\sqrt{2})f''(x)dx. \tag{11.9}$$

Applying this to $f_0(x)$ we find that

$$4\sqrt{2} = f_0'(0) = \sqrt{2} + \sqrt{2} + 8\int_0^{1/\sqrt{2}} (1 - x\sqrt{2})dx. \tag{11.10}$$

If $f(x)$ is any function satisfying (11.7), and assuming $f'(0) \geqq 0$ (else we work with $-f$), and estimating $f'(0)$ from (11.9) and (11.7) we obtain

$$0 \leqq f'(0) \leqq \sqrt{2} + \sqrt{2} + 8 \int_0^{1/\sqrt{2}} (1 - x\sqrt{2})dx = 4\sqrt{2}$$

by (11.10). Therefore

$$|f'(0)| \leqq 4\sqrt{2}. \tag{11.11}$$

However, if $f(x)$ satisfies (11.7), also $f(x + x_0)$, with $x_0 \geqq 0$, will satisfy (11.7). We may therefore apply to $f(x + x_0)$ our previous conclusion (11.11) to infer that $|f'(x_0)| \leqq 4\sqrt{2}$. Therefore $\|f'\|_+ \leqq 4\sqrt{2}$. □

We turn now to the *extremizing functions*. Unlike the situation discussed in §9 (for $n = 3$) there are no extremizing functions in the weak sense in the present case of $\mathbb{R}_+$. In fact we have the following very precise theorem.

THEOREM 9. *If* $f(x)$ *satisfies* (11.7) *and if*

$$\|f'\|_+ = 4\sqrt{2} \tag{11.12}$$

then

$$f(x) = \pm f_0(x) \text{ in the interval } 0 \leqq x \leqq 1/\sqrt{2}. \tag{11.13}$$

REMARK. Beyond (11.13) there is little that can be said about the extremizing function $f(x)$. Indeed, the function (11.13) can be continued from $1/\sqrt{2}$ to $+\infty$ in various ways, such as $f(x) = \pm 1$ if $x > 1/\sqrt{2}$, or else by $f(x) = \pm \mathscr{E}_2(x - (1/\sqrt{2}))$, without violating the basic condition (11.7).

Proof: We distinguish two cases depending on whether the supremum $\|f'\|_+$ is attained or not.

1. Let us assume that it is attained and that

$$f'(\xi) = 4\sqrt{2}, \tag{11.14}$$

for if this value were $-4\sqrt{2}$ we could work with $-f(x)$. Let us write

$$K(x) = 1 - x\sqrt{2}, \ (0 \leqq x \leqq 1/\sqrt{2}) \tag{11.15}$$

for the kernel in the formula (11.9). By (11.9) and (11.10) we conclude that the equation (11.14) is equivalent to

$$\begin{aligned} \sqrt{2}f\left(\xi + \frac{1}{\sqrt{2}}\right) - \sqrt{2}f(\xi) - \int_0^{1/\sqrt{2}} K(x)f''(x + \xi)dx \\ = \sqrt{2} + \sqrt{2} + 8 \int_0^{1/\sqrt{2}} K(x)dx. \end{aligned} \tag{11.16}$$

Because $K(x)$ is positive in $[0, 1/\sqrt{2})$, (11.16) and (11.7) imply that

(11.17) $f(\xi) = -1, f\left(\xi + \frac{1}{\sqrt{2}}\right) = 1$, and $f''(x+\xi) = -8$ in$\left(0, \frac{1}{\sqrt{2}}\right)$.

Clearly $\xi = 0$, for if ξ were positive, then $f(\xi) = -1$ and $\|f\|_+ \leqq 1$, would imply that $f'(\xi) = 0$, in contradiction to the assumption (11.14). Now (11.17) reduce to

$$f(0) = -1,\ f(1/\sqrt{2}) = 1,\ f''(x) = -8 \text{ in } (0, 1/\sqrt{2}),$$

and this already implies that $f(x) = -1 + 4\sqrt{2}x - 4x^2 = f_0(x)$ in $[0, 1/\sqrt{2}]$. Therefore (11.13) is established for this case.

2. Let us assume that

$$|f'(x)| < 4\sqrt{2} \text{ for } x \geqq 0, \tag{11.18}$$

and *let us show that this can not possibly happen.*

Indeed, the assumption (11.12) implies the existence of an infinite sequence (x_ν) of points of $\mathbb{R}_+$, such that

$$\lim_{\nu \to \infty} f'(x_\nu) = 4\sqrt{2}, \tag{11.19}$$

where on the right we have chosen the positive sign without loss of generality. On the other hand we have the following: If

$$x \geqq \tfrac{1}{2}, \tag{11.20}$$

then the formula (A) of Lemma 8, the relation (8.3), and the assumptions (11.7) show that

$$|f'(x)| = \left|f(x+\tfrac{1}{2}) - f(x-\tfrac{1}{2}) + \int_0^1 K_1(t) f''(t+x-\tfrac{1}{2})dt\right|$$
$$\leqq 1 + 1 + 8\int_0^1 |K_1(t)|\, dt = 4.$$

Thus (11.20) implies that

$$|f'(x)| \leqq 4. \tag{11.21}$$

From (11.19) we now conclude (observe that $4 < 4\sqrt{2}$!) that

$$0 \leqq x_\nu \leqq \tfrac{1}{2} \text{ for } \nu \text{ sufficiently large, } \nu > N \text{ say.} \tag{11.22}$$

From the Bolzano-Weierstrass theorem we infer that the sequence (x_ν) has a limit point ξ in $[0, \frac{1}{2}]$ and therefore

$$\lim_{\nu' \to \infty} x_{\nu'} = \xi, \tag{11.23}$$

where ν' is an appropriate increasing sequence of integers. The continuity of $f'(x)$ now implies that

$$4\sqrt{2} = \lim f'(x_{\nu'}) = f'(\lim x_{\nu'}) = f'(\xi).$$

Therefore $f'(\xi) = 4\sqrt{2}$, in contradiction to (11.18). The second possibility therefore never arises and Theorem 9 is established. □

12. The case $n = 3$. As in the previous case we retain the conditions (4.14) of Theorem 2 but this time for $\mathbb{R}_+$, hence

$$\|f\|_+ \leqq 1, \quad \|f'''\|_+ \leqq 24, \tag{12.1}$$

and wish to find $f_0(x)$ satisfying (12.1) and having the largest possible value for the norm $\|f_0'\|_+$ of its first derivative. We also seek (perhaps another) $f_0(x)$ satisfying (12.1) and maximizing $\|f_0''\|_+$. We shall see that one and the same function $f_0(x)$ will do both. From (4.3) and (4.10) we know that

$$\|\mathscr{E}_3\|_+ = 1, \quad \|\mathscr{E}_3'\|_+ = 3, \quad \|\mathscr{E}_3''\|_+ = 12, \quad \|\mathscr{E}_3'''\|_+ = 24, \tag{12.2}$$

so that $\mathscr{E}_3(x)$ satisfies (12.1). However, by an appropriate modification of $\mathscr{E}_3(x)$ we can increase considerably the norms of f' and f''.

To obtain the modified function $f_0(x)$ we remove the knot $x = 0$ of $\mathscr{E}_3(x)$, and continue its cubic polynomial branch (4.7), hence $1 - 6x^2 + 4x^3$, for negative values of x until it intersects the line $y = -1$. This happens for $x = -\frac{1}{2}$ and we define the function

$$g(x) = \begin{cases} 1 - 6x^2 + 4x^3 & \text{if } -\frac{1}{2} \leqq x \leqq 0, \\ \mathscr{E}_3(x) & \text{if } x \geqq 0. \end{cases} \tag{12.3}$$

For technical reasons we shift the origin to the point $x = -\frac{1}{2}$ and define

$$f_0(x) = g(x - \tfrac{1}{2}) \quad \text{(see Figure 5).} \tag{12.4}$$

Notice that $f_0(x)$ is a cubic spline in $\mathbb{R}_+$ having no longer a knot at $x = \frac{1}{2}$, in fact

$$f_0(x) = -1 + 9x - 12x^2 + 4x^3 \quad \text{if } 0 \leqq x \leqq 3/2. \tag{12.5}$$

Clearly

$$\|f_0\|_+ = 1, \quad \|f_0'''\| = 24. \tag{12.6}$$

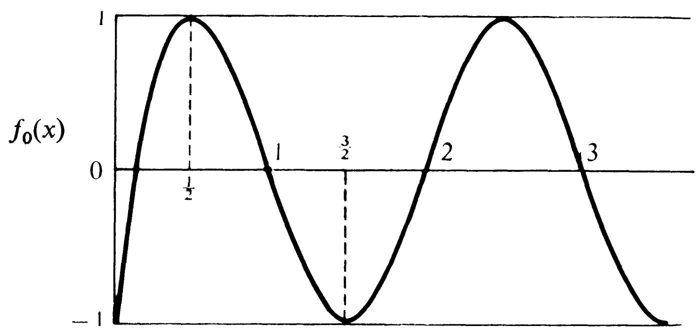

FIG. 5.

Moreover, we verify easily from (12.5) that $|f_0'(x)|$ reaches its largest value for $x = 0$, hence

$$\|f_0'\|_+ = f_0'(0) = 9. \tag{12.7}$$

Similarly we find that also $|f_0''(x)|$ reaches its largest value for $x = 0$. From (12.5) we read off this value to be

$$\|f_0''\| = -f_0''(0) = 24. \tag{12.8}$$

Comparing (12.7) and (12.8) with (4.15), we see that $f_0(x)$ surpasses by far the corresponding, bounds of Theorem 2. These, however, are the largest possible values, as stated by

THEOREM 10. (A. P. Matorin). *If*

$$\|f\|_+ \leqq 1, \ \|f'''\|_+ \leqq 24, \tag{12.9}$$

then

$$\|f'\|_+ \leqq 9, \ \|f''\|_+ \leqq 24. \tag{12.10}$$

In (12.10) *the constants* 9 *and* 24 *are the best constants because they are reached by the above function* $f_0(x)$, (see [8]).

Proof: We need two differentiation formulae that we get from (7.1). Applying (7.1) for $n = 3$, $a = 0$, and the two values $t = \frac{1}{2}$ and $t = 3/2$, we obtain

$$f\left(\frac{1}{2}\right) = f(0) + \frac{1}{2}f'(0) + \frac{1}{8}f''(0) + \frac{1}{2}\int_0^{1/2}\left(\frac{1}{2} - x\right)^2 f'''(x)dx,$$

$$f\left(\frac{3}{2}\right) = f(0) + \frac{3}{2}f'(0) + \frac{9}{8}f''(0) + \frac{1}{2}\int_0^{3/2}\left(\frac{3}{2} - x\right)^2 f'''(x)dx.$$

Solving these equations for $f'(0)$ and $f''(0)$ we obtain the two formulae

$$f'(0) = -\frac{8}{3}f(0) + 3f\left(\frac{1}{2}\right) - \frac{1}{3}f\left(\frac{3}{2}\right) + \int_0^{3/2} K_4(x)f'''(x)dx, \tag{12.11}$$

where

$$K_4(x) = \begin{cases} x\left(1 - \frac{4}{3}x\right) & \text{if } 0 \leqq x \leqq \frac{1}{2}, \\ \frac{1}{6}\left(\frac{3}{2} - x\right)^2 & \text{if } \frac{1}{2} < x \leqq \frac{3}{2}, \end{cases} \tag{12.12}$$

and

$$f''(0) = \frac{8}{3}f(0) - 4f\left(\frac{1}{2}\right) + \frac{4}{3}f\left(\frac{3}{2}\right) + \int_0^{3/2} K_5(x)f'''(x)dx, \tag{12.13}$$

where

$$K_5(x) = \begin{cases} \frac{4}{3}\left(x^2 - \frac{3}{4}\right) & \text{if } 0 \leqq x \leqq \frac{1}{2}, \\ -\frac{2}{3}\left(x - \frac{3}{2}\right)^2 & \text{if } \frac{1}{2} < x \leqq \frac{3}{2}. \end{cases} \tag{12.14}$$

Notice that in each of these formulae the coefficients of $f(0)$, $f\left(\frac{1}{2}\right)$, and $f\left(\frac{3}{2}\right)$ alternate in sign and that

$$K_4(x) > 0 \text{ and } K_5(x) < 0 \text{ in } 0 < x < \frac{3}{2}. \tag{12.15}$$

We now return to the function $f_0(x)$ defined by (12.4) and graphed in Figure 5. From Figure 5 and (12.5) we gather the following properties:

$$f_0(0) = -1,\ f_0\left(\frac{1}{2}\right) = 1,\ f_0\left(\frac{3}{2}\right) = -1, \tag{12.16}$$

$$f_0'(0) = 9,\ f_0''(0) = -24, \tag{12.17}$$

$$f_0''''(x) = 24 \text{ in } \left[0, \frac{3}{2}\right). \tag{12.18}$$

Applying the identities (12.11) and (12.13) to $f_0(x)$, we obtain by (12.15) the relations

$$f_0'(0) = 9 = \frac{8}{3} + 3 + \frac{1}{3} + 24\int_0^{3/2} |K_4(x)|\,dx, \tag{12.19}$$

$$-f_0''(0) = 24 = \frac{8}{3} + 4 + \frac{4}{3} + 24\int_0^{3/2} |K_5(x)|\,dx. \tag{12.20}$$

If $f(x)$ is a function satisfying the conditions (12.9), we can estimate its derivatives at the origin by (12.11) and (12.13), and obtain

$$|f'(0)| \leqq \frac{8}{3} + 3 + \frac{1}{3} + 24\int_0^{3/2} |K_4(x)|\,dx,$$

$$|f''(0)| \leqq \frac{8}{3} + 4 + \frac{4}{3} + 24\int_0^{3/2} |K_5(x)|\,dx.$$

The right hand sides being equal to 9 and 24, respectively, in view of (12.19) and (12.20), we conclude that

$$|f'(0)| \leqq 9,\ |f''(0)| \leqq 24.$$

Applying this result to $f(x + x_0)$, where $x_0 > 0$, we obtain (12.10). □

We shall now investigate the extremizing functions in Matorin's Theorem 10 and shall see that extremizing functions *in the weak sense* do not exist. We begin with

LEMMA 9. 1. *If* $f(x)$ *satisfies* (12.9) *and*

(12.21) $$|f'(\xi)| = 9 \text{ for some } \xi \geqq 0,$$

then necessarily $\xi = 0$ *and*

(12.22) $$f(x) = \pm f_0(x), \ (x \geqq 0),$$

where $f_0(x)$ *is the function defined by* (12.4) (Figure 5).

2. *The same conclusions* ($\xi = 0$ *and* (12.22)) *hold if*

(12.23) $$|f''(\xi)| = 24 \text{ for some } \xi \geqq 0.$$

Proof: 1. Let us first *assume* that $\xi = 0$ hence

(12.24) $$f'(0) = 9.$$

By an oft repeated argument we conclude from (12.11) and (12.19), that (12.24) is equivalent to the relation

(12.25) $$-\frac{8}{3}f(0)+3f\left(\frac{1}{2}\right)-\frac{1}{3}f\left(\frac{3}{2}\right)+\int_0^{3/2} K_4(x)f'''(x)dx = \frac{8}{3}+3+\frac{1}{3}+24\int_0^{3/2}|K_4(x)|dx,$$

and that this implies that

(12.26) $$f(0) = -1, f\left(\frac{1}{2}\right) = 1, f\left(\frac{3}{2}\right) = 1, f'''(x) = 24 \text{ in } \left(0, \frac{3}{2}\right).$$

This information already suffices to conclude that

(12.27) $$f(x) = f_0(x) \text{ in } [0, 3/2].$$

But then $f''(3/2) = 12$ (see Figure 5). By Corollary 2′ we now conclude that the identity (12.27) can be extended to [1/2, 5/2]. Continuing in this manner we see that (12.27) holds for $x \geqq 0$.

Let us now show that ξ must vanish. Indeed, if

(12.28) $$\xi > 0 \text{ and } f'(\xi) = 9 \text{ (say)},$$

then as above we conclude, as in (12.26), that $f(\xi) = -1$, a.s.f. But then we must have $f'(\xi) = 0$ (or else $\|f\|_+ \leqq 1$ would be violated!), which contradicts the assumption $f'(\xi) = 9$.

2. If (12.23) holds, we apply similar reasonings using formulae (12.13) and (12.20). □

THEOREM 11. *Let*

$$\|f\|_+ \leqq 1, \quad \|f'''\|_+ \leqq 24, \tag{12.29}$$

and therefore

$$\|f'\|_+ \leqq 9, \quad \|f''\|_+ \leqq 24. \tag{12.30}$$

If the equality sign holds in one of the inequalities (12.30), *then*

$$f(x) = \pm f_0(x) \text{ for } x \geqq 0, \tag{12.31}$$

where $f_0(x)$ *is the function defined by* (12.4) (Figure 5).

Proof: 1. Let us suppose that

$$\|f'\|_+ = 9. \tag{12.32}$$

If this supremum is assumed, hence (12.21) holds, then the conclusion (12.31) is already assured by Lemma 9. We may therefore assume that

$$|f'(x)| < 9 \text{ for } x \geqq 0, \tag{12.33}$$

and let us show that *this can not happen* by reaching a contradiction.

By (12.32) and (12.33), there exists an infinite sequence (x_ν) of points of $\mathbb{R}_+$ such that

$$\lim_{\nu \to \infty} f'(x_\nu) = 9. \tag{12.34}$$

(If this limit were -9 we could work with $-f(x)$). In the interval $x \geqq 1/2$ we can apply the differentiation formula (B) of Lemma 8, in the form

$$f'(x) = f(x + \tfrac{1}{2}) - f(x - \tfrac{1}{2}) + \int_0^1 K_2(t) f'''(t + x - \tfrac{1}{2}) dt$$

to conclude from (8.17) that $|f'(x)| \leqq 1 + 1 + 24 \int_0^1 |K_2(t)|\, dt = 3$. Thus

$$|f'(x)| \leqq 3 \text{ if } x \geqq \tfrac{1}{2}. \tag{12.35}$$

Confronting (12.34) with (12.35) we conclude that $0 \leqq x_\nu \leqq \frac{1}{2}$ if $\nu > N$. The Bolzano-Weierstrass theorem insures the existence of an appropriate infinite sequence of increasing integers (ν') such that

$$\lim_{\nu' \to \infty} x_{\nu'} = \xi, \text{ for some } \xi \text{ within } [0, \tfrac{1}{2}]. \tag{12.36}$$

Using the continuity of $f'(x)$, we conclude from (12.36) and (12.34) that $f'(\xi) = 9$, which contradicts our assumption (12.33).

2. If $\|f''\|_+ = 24$, we may use entirely similar arguments. If the supremum is assumed, we use Lemma 9. That the supremum is always assumed is shown by

contradiction as above: Formula (C) of Lemma 8 shows that $|f''(x)| \leqq 12$ if $x \geqq 1$ (here we use (8.26)!), and the continuity of $f''(x)$ takes care of the rest. □

13. The case $n = 4$ is not elementary. Our success in attacking the Landau problem for $\mathbb{R}_+$ for $n = 2$ and $n = 3$ with the modified Euler splines $f_0(x)$ seems surprising, to say the least. However, for $n = 4$ this approach does not work anymore. To make it clear why, let us try to do it. Our problem is to study functions satisfying

$$\|f\|_+ \leqq 1, \quad \|f^{(4)}\|_+ \leqq 384/5 = 76.8 \tag{13.1}$$

and to determine within this class the best, or least, constants $\gamma^+_{4,\nu}$ such that

$$\|f^{(\nu)}\|_+ \leqq \gamma^+_{4,\nu} \qquad (\nu = 1,2,3). \tag{13.2}$$

Stated equivalently: *Within the class of functions satisfying* (13.1) *we wish to maximize each of the three norms on the left side of* (13.2).

We start from $\mathscr{E}_4(x)$. From (4.8) we know that

$$P(x) = 1 - \frac{24}{5}x^2 + \frac{16}{5}x^4 = \mathscr{E}_4(x) \text{ if } -\frac{1}{2} \leqq x \leqq \frac{1}{2}.$$

We consider $\mathscr{E}_4(x)$ for $x \geqq -\frac{1}{2}$ only, and remove its knot at $x = -\frac{1}{2}$ to continue the graph of the quartic $P(x)$ for $x \leqq -\frac{1}{2}$. We find that it has a minimum value at $x = -\sqrt{3}/2 = -.866$, where it assumes the value $-4/5$, and thereafter increases to $+\infty$ as $x \to -\infty$. The new function so obtained satisfies the second condition (13.1). However, to satisfy also the first condition (13.1), we must cut it off at the point where it intersects the line $y = 1$. This is found to take place at $x = -\sqrt{6}/2 = -1.225$. Accordingly we define

$$g(x) = \begin{cases} 1 - \frac{24}{5}x^2 + \frac{16}{5}x^4 & \text{in } [-\sqrt{6}/2, 0], \\ \mathscr{E}_4(x) & \text{in } [0, \infty). \end{cases}$$

As before, we shift the origin to $-\sqrt{6}/2$ and define the function

$$f_0(x) = g\left(x - \frac{\sqrt{6}}{2}\right) \text{ for } x \geqq 0 \text{ (see Figure 6).} \tag{13.3}$$

We find that

$$\begin{aligned} \|f_0'\|_+ &= -f_0'(0) = 48\sqrt{6}/10 = 11.7576 \\ \|f_0''\|_+ &= f_0''(0) = 48 \\ \|f_0'''\|_+ &= -f_0'''(0) = 384\sqrt{6}/10 = 94.0604. \end{aligned} \tag{13.4}$$

These values are surely *lower bounds* for the best constants $\gamma^+_{4,\nu}$ of (13.2). However, our $f_0(x)$ is *certainly not an extremizing function*. This can be seen from Figure 6

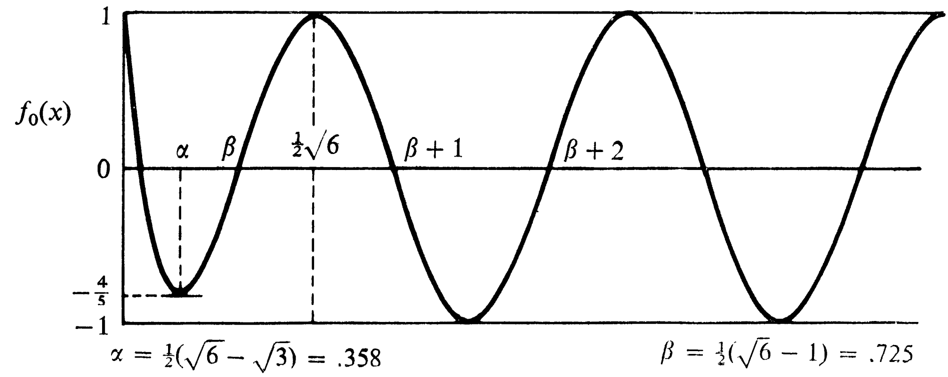

FIG. 6.

because *the first minimum value of* $f_0(x)$ *is* $= -4/5$ *and thereby fails to reach down to the line* $y = -1$. However, I do not know any *explicitly defined* function $f(x)$, satisfying (13.1), whose norms are superior to the norms (13.4) of $f_0(x)$.

At this point we state (see [**11**])

THE $\mathbb{R}_+$-ANALOGUE OF KOLMOGOROV'S THEOREM. *Let* $n \geqq 2$. *There is a spline function* $e_n(x)$ *of degree n, satisfying* $\| e_n \|_+ = 1$, $\| e_n^{(n)} \|_+ = \gamma_{n,n}$, *with the following property: If*

$$\|f\|_+ \leqq 1, \quad \|f^{(n)}\|_+ \leqq \gamma_{n,n}, \tag{13.5}$$

then

$$\|f^{(\nu)}\|_+ \leqq \| e_n^{(\nu)} \|_+ = | e_n^{(\nu)}(0) |, \qquad (\nu = 1, 2, \cdots, n-1). \tag{13.6}$$

These are the best constants because we have equalities if $f(x) = e_n(x)$. *If* $n \geqq 3$, *then* $\pm e_n(x)$ *are the only functions with these properties.*

We call $e_n(x)$ the *one-sided Euler spline of degree n*. Just like $\mathscr{E}_n(x)$, also $e_n(x)$ has the property that $e_n^{(n)}(x)$ is a step-function assuming the values $\pm \gamma_{n,n}$ only. Figures 4 and 5 show the graphs of $e_2(x)$ and $e_3(x)$, respectively. The knots of $f_0(x)$ (Figure 6) are at its zeros $\beta + 1, \beta + 2, \cdots$. The graph of $e_4(x)$ looks much like the graph of $f_0(x)$ (Figure 6), except that also its first minimum is $= -1$. However, the knots of $e_4(x)$ do not agree with its zeros, but approach them in the limit as we approach $+\infty$.

No explicit expressions are known for $e_n(x)$ ($n \geqq 4$). Rather $e_n(x)$ is defined in [**11**] as the limit of a sequence of spline functions of degree n, that are themselves defined by minimum properties. In deriving the numerical results of [**11**] good approximation of $e_n(x)$, for $n = 4, 5, 6$, are used. These approximations furnish for $n = 4$ the values of the best constants in (13.2):

$$\gamma_{4,1}^{+} = -e_4'(0) = 12.695$$
$$\gamma_{4,2}^{+} = e_4''(0) = 50.393$$
$$\gamma_{4,3}^{+} = -e_4'''(0) = 96.197.$$

In conclusion let me say the following. The Landau problems are *extremum problems.* Faced with an extremum problem we are often well on the way to its solution, provided that we are lucky enough to guess what the extremizing function is. The extremizing functions $\mathscr{E}_n(x)$ of the $\mathbb{R}$-problem are beautiful, simple, and easily computable functions. This is decidedly not the case of $e_n(x)$, if $n \geqq 4$, and this is the reason why the $\mathbb{R}_+$-problem was more difficult to solve.

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