PERMANENTS

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1. Introduction. If A is an n-square matrix then the *permanent* of A is defined by

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1.1}$$

where the summation extends over S_n, the symmetric group of degree n. This matrix function appears naturally in any combinatorial setting where a count of the number of systems of distinct representatives of some configuration is required [A **14**]. In general the permanent is an appropriate invariant for matrices that arise in combinatorial investigations where the problem is essentially unaltered by a relabeling of the items under consideration. For example, the total number of derangements ("le problème des rencontres") of n distinct items is given by per $(J-I_n)$, where J is the n-square matrix with every entry equal to 1, and I_n is the n-square identity matrix. The initial ordering of the items is clearly immaterial. This is just a reflection of the more general fact that permutations of the rows and columns leave the permanent unaltered. In fact if T is a linear operator on the space of n-square matrices, $n>2$, then per $(T(A)) = \text{per } A$ for all A if and only if either $T(A) = DPAQL$ or $T(A) = DPA^TQL$, where P and Q are n-square permutation matrices, and D and L are n-square diagonal matrices satisfying per $(DL) = 1$. This theorem is a relatively recent discovery [B **18**].

As we shall subsequently see, the permanent is related to other more familiar matrix invariants, usually via inequalities. For example, Schur [B **51**] proved that per $A \geqq \det A$ for any positive semi-definite hermitian matrix. However, as Pólya observed [B **46**], no uniform affixing of $\pm$ signs to the elements of a matrix can convert the permanent into the determinant. In fact, quite a general result along these lines is known [B **19**]. Namely, there is no linear operation on matrices T: $A \to T(A)$ such that per $T(A) = \det A$ for all A (again except for $n=2$). This result, of course, terminates further efforts at investigating the permanent by relating it in some simple way to the more tractable determinant function.

The name "permanent" seems to have originated in Cauchy's memoir of 1812 [B **3**]. Cauchy's "fonctions symétriques permanentes" designate any symmetric function. Some of these, however, were permanents in the sense of the definition (1.1). Joachimstal [B **9**] points out that the sum $\sum \alpha_1\beta_2 \ldots \mu_m$ "tantum a determinante differt, quod omnes ejus termini sunt positivi." Hammond [B **7**] refers to the function (1.1) as an "alternate determinant." As far as we are aware the name "permanent" as defined in (1.1) was introduced by Muir [B **38**].

In the period 1855–1918 various results involving permanents were announced (see references). Many of these are very special identities involving permanents and determinants. Of these the one which apparently created the most interest is due to Borchardt [B **1**]: if $A=[a_{ij}]$, $a_{ij}=1/(t_i-\alpha_j)$, and $B=[b_{ij}]$, $b_{ij}=a_{ij}^2$, then $\det B=\det A \operatorname{per} A$.

Another interesting and more general identity is due to Muir [B **40**]. Let M be the totality of $(mn)!/(m!)^n n!$ permutations $\sigma\in S_{mn}$ for which

$$\sigma(m(t-1)+1)<\sigma(mt+1), \qquad t=1,\cdots,n-1,$$

and

$$\sigma(km+i)<\sigma(km+j), \quad k=0,\cdots,n-1, \quad 1\leqq i<j\leqq m.$$

Let P_σ denote the mn-square permutation matrix corresponding to σ. If X is an mn-square matrix partitioned into n^2 m-square nonoverlapping submatrices X_{ij} then define $B(X)$ to be the n-square matrix whose (r, s) entry is $\det X_{rs}$. Then for any mn-square matrix A

$$\det A=\sum_{\sigma\in M}\epsilon(\sigma)\det B(AP_\sigma) \qquad \text{if } m \text{ is odd},$$

$$\det A=\sum_{\sigma\in M}\epsilon(\sigma)\operatorname{per} B(AP_\sigma) \qquad \text{if } m \text{ is even}.$$

Much of the modern interest in the permanent stems from three results obtained in this century: Pólya's problem [B **46**], Schur's result [B **51**], both previously mentioned, and the unresolved conjecture due to van der Waerden [B **55**]. The current status of van der Waerden's conjecture is discussed in the next section.

2. Properties of the permanent. In order to minimize the number of indices we introduce some notation for sequence sets. Let r and n be positive integers. The set $\Gamma_{r,n}$ is the totality of n^r sequences $\omega=(\omega_1,\cdots,\omega_r)$ for which $1\leqq\omega_i\leqq n$, $i=1,\cdots,r$. If $r\leqq n$ then $Q_{r,n}$ will denote the subset of $\Gamma_{r,n}$ consisting of those $\binom{n}{r}$ sequences ω for which $\omega_1<\omega_2<\cdots<\omega_r$. The set $G_{r,n}$ is the totality of $\binom{n+r-1}{r}$ sequences in $\Gamma_{r,n}$ for which $\omega_1\leqq\omega_2\leqq\cdots\leqq\omega_r$. If $A=[a_{ij}]$ is an n-square matrix and $\omega,\tau\in\Gamma_{r,n}$ then $A[\omega|\tau]$ is the r-square matrix whose (s, t) entry is $a_{\omega_s\tau_t}$. If $\omega,\tau\in Q_{r,n}$ then $A[\omega|\tau)$ is the $r\times(n-r)$ submatrix lying in rows ω and outside columns τ of A. Similarly for $A(\omega|\tau]$ and $A(\omega|\tau)$. If $\omega\in G_{r,n}$ then $\mu(\omega)$ will denote the product of the factorials of the multiplicities of the distinct integers in ω, e.g., $\mu(2,4,4,4,5,5,7)=3!2!$.

Permanents do possess some properties analogous to those of determinants:

(a) the permanent is a multilinear function of the rows and columns;
(b) $\operatorname{per} A^*=\overline{\operatorname{per} A}$;
(c) if P and Q are permutation matrices then $\operatorname{per} PAQ=\operatorname{per} A$;
(d) if D and G are diagonal matrices then $\operatorname{per} DAG=\operatorname{per} D \operatorname{per} A \operatorname{per} G$;
(e) (Laplace expansion theorem)

$$\operatorname{per} A=\sum_{\omega\in Q_{r,n}}\operatorname{per} A[1,\cdots,r\,|\,\omega]\operatorname{per} A(r+1,\cdots,n\,|\,\omega);$$

(f) (Cauchy-Binet theorem) if A is $m\times n$ and B is $n\times m$, $m\leqq n$,

$$\operatorname{per} AB = \sum_{\gamma\in G_{m,n}} \frac{1}{\mu(\gamma)} \operatorname{per} A[1, \cdots, m \mid \gamma] \operatorname{per} B[\gamma \mid 1, \cdots, m].$$

Properties (a)–(e) are immediate consequences of the definition (1.1). The proof of property (f) is somewhat more involved. The most useful property that determinants have is invariance under addition of a multiple of a row to another row. This property is conspicuous by its absence from the above list. Its failure makes the computation of any particular permanent difficult. The following formula due to Ryser [A **14**] makes the evaluation of certain permanents feasible by high speed computing devices. Let A be an n-square matrix and let A_r denote a matrix obtained from A by replacing some r columns of A by zeros. Let $S(X)$ be the product of the row sums of the matrix X. Then

$$\text{(2.1)}\quad \operatorname{per} A = S(A) - \sum S(A_1) + \sum S(A_2) - \cdots + (-1)^{n-1} \sum S(A_{n-1}),$$

where $\sum S(A_r)$ denotes the sum over all $\binom{n}{r}$ replacements of r of the columns of A by zeros. Formula (2.1) was used by Nikolai [B **45**] for computing permanents of incidence matrices for certain (v, k, λ)-configurations.

Let $\mathfrak{A}$ be a set of v distinct items $x_1, \cdots, x_v$ and suppose $\mathfrak{A}_1, \cdots, \mathfrak{A}_v$ are subsets of $\mathfrak{A}$. We define the v-square *incidence matrix* $A = [a_{ij}]$ for this configuration by $a_{ij}=1$ if $x_i\in\mathfrak{A}_j$ and $a_{ij}=0$ if $x_i\notin\mathfrak{A}_j$. A *system of distinct representatives* is an ordered v-tuple $(x_{\sigma(1)}, \cdots, x_{\sigma(v)})$, $x_{\sigma(t)}\in\mathfrak{A}_t$, $\sigma\in S_v$. Hence per A is just the number of such systems. In case each $\mathfrak{A}_i$ contains exactly k items and $\mathfrak{A}_i\cap\mathfrak{A}_j$, $i\neq j$, contains λ items, $0<\lambda<k<v$, then we have what is called a (v, k, λ)-configuration. The matrix A then satisfies $AA^T=A^TA=(k-\lambda)I_n+\lambda J$ and it can be shown that

$$\text{(2.2)}\qquad |\det A| = k(k-\lambda)^{(v-1)/2}.$$

From (2.2) it is seen that $|\det A|$ depends only on the parameters v, k and λ and not on the configuration. The question posed and answered in the negative by Nikolai [B **45**] is whether per A similarly depends only on the parameters v, k, λ.

In answer to Montmort's "problème des rencontres" [A **14**] mentioned in section 1 we can compute

$$\operatorname{per}(J - I_v) = v!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^v \frac{1}{v!}\right).$$

In fact,

$$\operatorname{per}(zI_v + J) = v!\sum_{r=0}^{v} \frac{z^r}{r!}\cdot$$

The matrix $J-I_v$ is one of the few (0, 1)-circulants [B **35**] for which an explicit formula for the permanent is available. We describe a few other examples of permanents that can be directly computed. Let P_v be the v-square permutation matrix with ones in the first superdiagonal. Then of course per $P_v=1$. Also, per $(P_v+P_v^2)=2$ and [B 35]

$$\begin{aligned}\text{per}\,(P_v+P_v^2+P_v^3) &= \text{per}\,(P_{v-1}+P_{v-1}^2+P_{v-1}^3)+\text{per}\,(P_{v-2}+P_{v-2}^2+P_{v-2}^3)-2\\ &= \left(\frac{1+\sqrt{5}}{2}\right)^v+\left(\frac{1-\sqrt{5}}{2}\right)^v+2\\ &= \text{tr}\,(C_2^v)+2,\end{aligned}$$

where

$$C_2=\begin{pmatrix}0&1\\1&1\end{pmatrix}.$$

Moreover,

$$\begin{aligned}\text{per}\,(P_v+P_v^2+P_v^3+P_v^4) &= \text{per}\,(P_{v-1}+P_{v-1}^2+P_{v-1}^3+P_{v-1}^4)\\ &\quad+\text{per}\,(P_{v-2}+P_{v-2}^2+P_{v-2}^3+P_{v-2}^4)+\text{per}\,(P_{v-3}+P_{v-3}^2+P_{v-3}^3+P_{v-3}^4)-4\\ &= 2(\text{tr}\,(C_3^v)+1),\end{aligned}$$

where

$$C_3=\begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix}.$$

It is known that unfortunately no analogous recurrence formula for per $(\sum_{v=1}^k P_v^i)$, $k>4$, exists [B 35]. It is possible to show that if such a recurrence were to hold for $k>5$ then the van der Waerden conjecture would fail.

A remarkable result for (0, 1)-circulants was recently obtained by Tinsley [B 54]. Let A be a (0, 1)-circulant with k ones, $k\geqq 3$, in each row and column. Then

$$\text{per}\,A \geqq |\det A|,$$

with equality if and only if after suitable permutations of the rows and columns A can be reduced to a direct sum of 7-square (0, 1)-circulants of the form $I_7+P_7+P_7^3$. Then

$$\text{per}\,A = |\det A| = 24^{v/7}.$$

A *doubly stochastic* n-square matrix $A=[a_{ij}]$ is one with nonnegative entries each of whose row sums and column sums is 1. Thus $J_n=(1/n)J$ is doubly stochastic. These matrices appear in a wide variety of contexts ranging from probability theory to the theory of eigenvalue inequalities. The results on doubly stochastic matrices are discussed extensively in [A 7; B **11**; B **20**; B **37**]. Notice that the above (0, 1)-circulants are all multiples of doubly stochastic matrices as are the incidence matrices for the (v, k, λ)-configurations. There is not much known about upper and lower bounds for the permanent of a general doubly stochastic matrix. The following bounds were recently obtained and constitute examples of results relating the permanent with the more familiar matrix invariants.

If A is an n-square doubly stochastic matrix then

$$\text{per } A \leqq \left(\frac{\rho(A)}{n}\right)^{1/2}, \tag{2.3}$$

where $\rho(A)$ is the rank of A. Equality holds in (2.3) if and only if A is a permutation matrix [B 22]. If A happens to be normal as well then (2.3) can be improved to

$$\text{per } A \leqq \frac{\rho(A)}{n}, \tag{2.4}$$

where the inequality is strict unless A is a permutation matrix or $n=2$ and $A=J_2$ [B **22**].

Once again if A is doubly stochastic with h eigenvalues of modulus 1 then [B **23**]

$$\text{per } A \geqq \frac{1}{(n-h+1)^{n-h+1}}. \tag{2.5}$$

If A is also indecomposable then

$$\text{per } A \geqq \left(\frac{h}{n}\right)^n. \tag{2.6}$$

Equality holds in (2.5) or (2.6) if and only if A is a permutation matrix.

The best known conjecture on the lower bound for the permanent of an n-square doubly stochastic matrix is the van der Waerden conjecture:

$$\text{per } A \geqq \frac{n!}{n^n} \tag{2.7}$$

with equality if and only if $A=J_n$. We list six results that represent all that is currently known about (2.7). Let Ω_n denote the set of doubly stochastic n-square matrices and let Z_n be the subset of Ω_n consisting of those matrices with strictly positive entries.

I. *If* per $A=\min_{S\in\Omega_n}$ per S *then* $A^k\in Z_n$ *for some* k [B **26**].

II. *If* per $A=\min_{S\in\Omega_n}$ per A *then* per $A(i|j)=$ per A *if* $a_{ij}>0$ *and* per $A(i|j)$ $=$ per $A+\beta$ *if* $a_{ij}=0$, *where* $\beta\geqq 0$ *is independent of* i *and* j [B **26**].

III. *If* per $A=\min_{S\in\Omega_n}$ per S *and* $A\in Z_n$ *then* $A=J_n$ [B **26**].

IV. *If $A \neq J_n$ and A is in a sufficiently small neighborhood of J_n then* [B **26**]

$$\text{per } A > \text{per } J_n.$$

V. *For $n \leqq 3$,* $\min_{S \in \Omega_n}$ per $S = n!/n^n$ *and the minimum is assumed uniquely for* $S = J_n$ [B **26**].

VI. *If $A \in \Omega_n$ is positive semi-definite symmetric then* [B **28**; B **29**; B **33**; B **22**]

$$\text{per } A \geqq \frac{n!}{n^n}$$

with equality if and only if $A = J_n$.

To conclude we quote a recently obtained upper bound for the permanent of a general (0, 1)-matrix. If A is a (0, 1)-matrix with row sums $r_1, \cdots, r_n$ then [B **34**]

$$\text{per } A \leqq \prod_{i=1}^{n} \left(\frac{r_i + 1}{2}\right),$$

with equality if and only if A is a permutation matrix.

3. Recent developments. In this section we describe a relatively new approach to the permanent function. The central idea is to represent the permanent as an inner product on the symmetry class of completely symmetric tensors. At the outset this may seem like an unnecessarily sophisticated approach to a matrix function which does not appear much more complicated than the determinant. In order that the reader may appreciate the difficulties involved we list a number of results recently obtained by these techniques. It seems unlikely that a frontal attack on any of these inequalities would succeed.

THEOREM 1. *If A is an n-square positive semi-definite hermitian matrix with row sums $r_1, \cdots, r_n$ and if $r = \sum_{i=1}^{n} r_i \neq 0$ then*

$$\text{per } A \geqq n! \prod_{i=1}^{n} |r_i|^2/r^n. \tag{3.1}$$

Equality holds in (3.1) *if and only if* (i) *A has a zero row or* (ii) $\rho(A) = 1$, [B **22**].

Notice that as a direct corollary to Theorem 1 we obtain the result VI at the end of section 2.

THEOREM 2 (Permanent analogue of the Hadamard determinant theorem [B **21**; B **14**; B **15**]). *If A is an n-square positive semi-definite hermitian matrix then*

$$\text{per } A \geqq \prod_{i=1}^{n} a_{ii} \tag{3.2}$$

with equality if and only if A is a diagonal matrix or A has a zero row.

THEOREM 3 (Schur [B **51**]). *If A is an n-square positive semi-definite hermitian matrix then*

$$\text{per } A \geqq \det A \tag{3.3}$$

with equality if and only if A is diagonal or A has a zero row.

THEOREM 4. *If A is an n-square normal matrix with eigenvalues $\alpha_1, \cdots, \alpha_n$ then* [B **22**]

$$|\text{per } A| \leqq \frac{1}{n} \sum_{i=1}^{n} |\alpha_i|^n. \tag{3.4}$$

THEOREM 5. *If A is an $m \times n$ matrix and B is an $n \times m$ matrix then*

$$|\text{per } AB|^2 \leqq \text{per } AA^* \text{ per } B^*B. \tag{3.5}$$

Equality holds in (3.5) *if and only if either* (i) *A has a zero row or* (ii) *B has a zero column or* (iii) *$A = DPB^*$, where D is a diagonal matrix and P is a permutation matrix* [B **29**].

We introduce the preliminary material on tensor spaces that will suffice to exhibit the technique of proof used in establishing these results.

Let V be an n-dimensional unitary space with inner product (x, y). We let $V^{(m)}$ denote the space of *m-contravariant tensors* [A **8**, Chap. 16]. This is most easily defined as follows. Let $M_m(V)$ be the space of m-multilinear functionals on V, i.e., the space of complex valued functions $\phi(x_1, \cdots, x_m)$, $x_i \in V$, $i = 1, \cdots, m$, linear in each x_i separately. For example, if V is the space of complex n-tuples and $m = n$, $x_i = (x_{i1}, \cdots, x_{in})$, then the multilinear functional ϕ defined by $\phi(x_1, \cdots, x_n) = \det(x_{ij})$ is in $M_n(V)$. The space $V^{(m)}$ is now defined as the dual space of $M_m(V)$, i.e., $V^{(m)}$ is the space of all complex-valued linear functionals on $M_m(V)$. There are certain distinguished m-contravariant tensors, the *decomposable* tensors, that arise from elements of V: if $x_i \in V$, then $f = x_1 \otimes \cdots \otimes x_m$ is the element of $V^{(m)}$ whose value on any $\phi \in M_m(V)$ is given by

$$f(\phi) = \phi(x_1, \cdots, x_m).$$

The decomposable tensor f is called the *tensor product* of the x_i, $i = 1, \cdots, m$.

It is easy to show that if $e_1, \cdots, e_n$ is a basis of V then the n^m tensor products $e_{\omega_1} \otimes \cdots \otimes e_{\omega_m}$, $\omega \in \Gamma_{m,n}$, constitute a basis of $V^{(m)}$. An inner product can be introduced in $V^{(m)}$ in terms of the inner product in V. The defining relation is given in terms of its values on pairs of tensor products,

$$(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) = \prod_{i=1}^{m} (x_i, y_i). \tag{3.6}$$

Certain special operators $P(\sigma)$ on $V^{(m)}$, called *permutation operators*, are defined in terms of elements $\sigma \in S_m$:

$$P(\sigma)x_1 \otimes \cdots \otimes x_m = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}. \tag{3.7}$$

The *completely symmetric operator* on $V^{(m)}$ is now defined by

$$T_m = \frac{1}{m!} \sum_{\sigma \in S_m} P(\sigma). \tag{3.8}$$

Thus T_m maps $V^{(m)}$ into a subspace of itself. The range of T_m is denoted by $V_{(m)}$ and is called the *symmetry class* of completely symmetric tensors [A **4**, p. 217]. If $x_1, \cdots, x_m$ are in V we set

$$x_1 * \cdots * x_m = T_m(x_1 \otimes \cdots \otimes x_m); \tag{3.9}$$

$x_1 * \cdots * x_m$ is called the *symmetric* product of the x_i, $i=1, \cdots, m$. It is clear from the definition of the symmetric product that if $\sigma \in S_m$ then

$$P(\sigma)x_1 * \cdots * x_m = x_1 * \cdots * x_m.$$

The operator T_m is hermitian with respect to the inner product (3.6) and is also idempotent:

$$T_m = T_m^2 = T_m^*. \tag{3.10}$$

We are now ready to bring the permanent into this picture. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$ be arbitrary vectors in V and set $a_{ij} = (x_i, y_j)$. Then by (3.9), (3.10), (3.6), and (1.1) in succession we compute

$$\begin{aligned}
(x_1 * \cdots * x_m, y_1 * \cdots * y_m) &= (T_m x_1 \otimes \cdots \otimes x_m, T_m y_1 \otimes \cdots \otimes y_m) \\
&= (T_m^* T_m x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) \\
&= (T_m x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} (P(\sigma)x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} (x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}, y_1 \otimes \cdots \otimes y_m) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^{m} (x_{\sigma^{-1}(i)}, y_i) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^{m} (x_i, y_{\sigma(i)}) \\
&= \frac{1}{m!} \operatorname{per} A.
\end{aligned}$$

We have then

$$(x_1 * \cdots * x_m, y_1 * \cdots * y_m) = \frac{1}{m!} \operatorname{per} A, \tag{3.11}$$

where $A = (a_{ij}) = ((x_i, y_j))$.

To illustrate the use of the identity (3.11) we prove Theorem 1. Set $m=n$. Since A is positive semi-definite hermitian it is a Gram matrix based on some set of vectors $x_1, \cdots, x_n$: $a_{ij}=(x_i, x_j)$. From (3.11) and the Cauchy-Schwarz inequality applied to the inner product in $V^{(m)}$ we have

$$(3.12)\qquad \frac{1}{n!}\operatorname{per} A = (x_1 * \cdots * x_n, x_1 * \cdots * x_n) \geqq \left|\left(x_1 * \cdots * x_n, \frac{u}{\|u\|}\right)\right|^2,$$

where u is any nonzero vector in $V^{(m)}$. Let $v=\sum_{i=1}^n x_i$ so that

$$(v, v) = \sum_{i,j=1}^{n} (x_i, x_j) = \sum_{i,j=1}^{n} a_{ij} = r.$$

Since $r\neq 0$, $v\neq 0$ and in fact $\|v\|^2=r>0$. We set $u=v*\cdots*v$ and compute, by (3.11), that

$$\|u\|^2 = (v*\cdots*v, v*\cdots*v) = \frac{1}{n!}\operatorname{per} B,$$

where $b_{ij}=r$, $i, j=1, \cdots, n$. Hence $\|u\|^2=r^n$. Returning to (3.12) we have

$$\frac{1}{n!}\operatorname{per} A \geqq |(x_1*\cdots*x_n, v*\cdots*v)|^2/r^n = \left|\frac{1}{n!}\operatorname{per}((x_i, v))\right|^2/r^n.$$

Now $(x_i, v)=(x_i, \sum_{j=1}^n x_j)=\sum_{j=1}^n (x_i, x_j)=\sum_{j=1}^n a_{ij}=r_i$. Hence $\operatorname{per}((x_i, v)) =n!\prod_{i=1}^n r_i$ and therefore

$$\frac{1}{n!}\operatorname{per} A \geqq \prod_{i=1}^{n} |r_i|^2/r^n,$$

the inequality (3.1). The discussion of the cases of equality is a bit involved and we omit it. Notice that if A is doubly stochastic then $r_1=\cdots=r_n=1$, $r=n$, and (3.1) implies VI in section 2.

We refer the reader to [B **29**] in which a very similar argument is used to prove Theorem 5. We illustrate the use of (3.5) in proving (3.3). Since A is positive semi-definite hermitian it can be factored, $A=TT^*$, in which T is upper triangular. Notice that, for such a T, per $T=\det T$. Thus

$$\begin{aligned}\det A &= \det TT^* = \det T \det T^*\\ &= \operatorname{per} T \operatorname{per} T^*\\ &\leqq |\operatorname{per} TI_n|\,|\operatorname{per} I_nT^*|\\ &\leqq \sqrt{\operatorname{per} TT^*}\sqrt{\operatorname{per} TT^*} = \operatorname{per} A.\end{aligned}$$

The inequality (3.3) could have been obtained from (3.2) and the Hadamard determinant theorem. However the proof of (3.2) is quite involved [B **14**; B **15**] and will be omitted.

4. Conjectures and problems. In this section we list several conjectures and problems that involve the permanent function. The conjectures in this field, that have been announced at various times, seem to separate into two quite unsatisfactory classes. In the first class are those statements that sound plausible but appear at present beyond reach. In the second class are those conjectures that seemed for a time undeniably true and for which counterexamples were eventually discovered. We list some conjectures that are as yet unclassified.

CONJECTURE 1 (van der Waerden). *If A is an n-square doubly stochastic matrix then*

$$\operatorname{per} A \geqq \frac{n!}{n^n}$$

with equality if and only if $A = J_n$ [B **55**; B **26**; B **28**; B **33**].

CONJECTURE 2 (H. J. Ryser). *If A is an n-square doubly stochastic matrix then*

$$\operatorname{per} AA^T \leqq \operatorname{per} A. \tag{4.1}$$

Until just recently it was conjectured [A **14**; p. 59] that

$$\operatorname{per} AB \leqq \min \{\operatorname{per} A, \operatorname{per} B\} \tag{4.2}$$

when both A and B are n-square doubly stochastic matrices. We are indebted to the referee for making available to us a surprising counter-example to (4.2) discovered by W. B. Jurkat of Syracuse University:

$$A = \frac{1}{24}\begin{bmatrix} 11 & 5 & 8 \\ 13 & 11 & 0 \\ 0 & 8 & 16 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$24^3 \operatorname{per} A = 3808, \qquad 24^3 \operatorname{per} AB = 3840.$$

Of course, if (4.1) were known then Conjecture 1 would follow immediately from VI in section 2. (Morris Newman recently communicated to us a counter-example to Conjector 2.)

CONJECTURE 3 (H. J. RYSER). *If A is an mk-square $(0, 1)$-matrix with k ones in each row and column then*

$$\operatorname{per} A \leqq \sum_1^m {}^{\cdot} J, \tag{4.3}$$

where $\sum_1^m \cdot J$ denotes the direct sum of J taken m times and J is k-square [A **14**].

CONJECTURE 4 (H. Minc). *If A is an n-square $(0, 1)$-matrix with row sums r_i, $i = 1, \cdots, n$, then*

$$\operatorname{per} A \leqq \prod_{i=1}^n (r_i!)^{1/r_i}. \tag{4.4}$$

Clearly (4.4) implies (4.3) [B **34**].

CONJECTURE 5 (H. J. Ryser). *If the totality of v-square* (0,1)*-matrices with k ones in each row and column contains incidence matrices of* (v, k, λ)*-configurations then the permanent is minimal in this totality for one of these incidence matrices.*

CONJECTURE 6 (H. Minc). *Let* R_k *denote the class of all v-square* (0, 1)*-matrices with k ones in each row and column. Then for a fixed v*

$$\min_{A\in R_k} \operatorname{per}\left(\frac{1}{k} A\right)$$

is monotone decreasing in k.

CONJECTURE 7 (M. Marcus and M. Newman). *If A is n-square doubly stochastic then*

$$\operatorname{per}(I_n - A) \geqq 0.$$

This is known to be true in case A is symmetric as well [B **29**]. (R. A. Brualdi and Morris Newman announced the affirmative resolution of Conjecture 7.)

CONJECTURE 8 (M. Marcus and M. Newman). *If A is positive semi-definite n-square hermitian and* $1 \leqq k \leqq n$ *then*

$$\operatorname{per} A \geqq \operatorname{per} A[1, \cdots, k \mid 1, \cdots, k] \operatorname{per} A[k+1, \cdots, n \mid k+1, \cdots, n].$$

This is known for $k=1$ [B **14**, B **15**].

CONJECTURE 9 (M. Marcus). *Let A be an mk-square positive semi-definite hermitian matrix partitioned as follows:*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$$

in which each A_{ij} *is k-square. Let B be the m-square matrix whose* (i, j) *entry is* per A_{ij}. *Then*

$$\operatorname{per} A \geqq \operatorname{per} B. \tag{4.5}$$

If A is positive definite then equality holds in (4.5) *if and only if* $A = \sum_{i=1}^{m} {}^{\cdot} A_{ii}$.

Note that Conjecture 9 implies Conjecture 8. For suppose (4.5) holds. Assume without loss of generality in Conjecture 8 that $k \leqq n-k$. Let C be the $2(n-k)$-square matrix: $C = I_{n-2k} \dot{+} A$. Partition C as follows:

$$C = \begin{bmatrix} I_{n-2k} \dot{+} A_k & C_{12} \\ C_{21} & A_{n-k} \end{bmatrix},$$

where $A_k = A[1, \cdots, k \mid 1, \cdots, k]$ and $A_{n-k} = A[k+1, \cdots, n \mid k+1, \cdots, n]$. Clearly C is poritive semi-definite hermitian and (4.5) implies

$$\begin{aligned} \text{per } A &= \text{per } C \geqq \text{per} \begin{bmatrix} \text{per } (I_{n-2k} + A_k) & \text{per } C_{12} \\ \text{per } C_{21} & \text{per } A_{n-k} \end{bmatrix} \\ &= \text{per } A_k \text{ per } A_{n-k} + |\text{per } C_{12}|^2 \\ &\geqq \text{per } A_k \text{ per } A_{n-k}. \end{aligned}$$

CONJECTURE 10 (M. Marcus). *Let $A = [a_{ij}]$ and $a_{ij} > 0$, $i, j = 1, \cdots, n$. If the $n!$ terms $\prod_{i=1}^n a_{i\sigma(i)}$ in the expansion of* per A *take on at most r different values then $\rho(A) \leqq r$.*

This conjecture is known to be true for $n < 5$ [unpublished results of H. Minc and R. Westwick].

CONJECTURE 11 (M. Marcus and M. Newman). *Let $A \in \Omega_n$. There exists no positive number β, independent of i, j, such that*

$$\text{per } A(i \mid j) = \text{per } A, \qquad a_{ij} \neq 0$$

and

$$\text{per } A(i \mid j) = \text{per } A + \beta, \qquad a_{ij} = 0.$$

(See section 2, II.)

CONJECTURE 12 (M. Marcus). *Let Δ_n be the group of n-square nonsingular matrices of the form PD, where P is a permutation matrix and D is a diagonal matrix. Show that Δ_n is a maximal group on which the permanent is multiplicative. In other words, Δ_n cannot be a proper subgroup of a group G in which* per AB $=$ per A per B *for all $A, B \in G$.*

CONJECTURE 13 (M. Marcus). *If A is doubly stochastic and $f(z) =$* per $(zI_n - A)$ *then the zeros of $f(z)$ lie in or on the boundary of the unit disc $|z| \leqq 1$.*

PROBLEM 1. *Find the maximum value of $f(U) =$* per (U^*AU) *as U runs over all n-square unitary matrices. Here A is a fixed n-square positive semi-definite hermitian matrix* [B **29**].

It is known [B **22**] that $f(U) \leqq \text{tr } (A^n)/n$, but in general this bound is not achievable.

PROBLEM 2. *Let H be a subgroup of S_n and let χ be a character of degree* 1 *of H. Under what conditions on χ and H will the inequality*

$$\sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \leqq \text{per } A$$

hold for all positive semi-definite hermitian A? [B **51**].

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THE CONSTRUCTION OF ORTHOGONAL AND QUASI-ORTHOGONAL NUMBER SETS

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1. Introduction. There has always been great interest in the discovery of inversion relations for series and associated orthogonality relations. The references at the end of this paper will give some idea of current activity in this field and each bears an intimate relationship to our work here. The reader will find it of interest to compare some of our results with the relations in Riordan's recent paper [6]. Several results of Riordan were influenced by the orthogonality relations found by the writer [2]. Theorem 3 below is in turn based on an extension of ideas in [2] which are not unlike some new results found by Carlitz [1].

We shall distinguish between *orthogonality* and *quasi-orthogonality*. Such a distinction has been made by S. Tauber [9] and we shall follow his nomenclature. Associated with two number sets A_n^m and B_n^m are the polynomials

$$A_n(x) = \sum_{k=0}^{n} A_n^k x^k, \qquad B_n(x) = \sum_{k=0}^{n} B_n^k x^k, \tag{1.1}$$