PARTITION IDENTITIES—FROM EULER TO THE PRESENT

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1. Introduction. A *partition* of a positive integer n is defined as a way of writing n as the sum of positive integers. Two such ways of writing n in which the parts merely differ in the order in which they are written are considered the same partition. We shall denote by $p(n)$ the number of partitions of n. Thus, for example, since 5 can be expressed as the sum of positive integers by 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$, we have $p(5)=7$. An explicit formula for $p(n)$ valid for all positive integers n was discovered by Rademacher in 1937, but since it is a complicated infinite series and is not needed for the purposes of this paper, it will not be given here. On the other hand, there exists a simple generating function for $p(n)$, that is, a function which, when expanded into a power series $\sum_{n=0}^{\infty} c_n x^n$ has as its general coefficient $c_n = p(n)$.

THEOREM 1. *The generating function for $p(n)$ is given by*

$$f(x) = \frac{1}{\prod_{\nu=1}^{\infty} (1 - x^{\nu})}, \qquad \textit{where } |x| < 1. \tag{1}$$

Proof of Theorem 1. We have to show that the right hand side of (1), when expanded into a power series, has as its general coefficient $p(n)$. To do this, we rewrite the right hand side of (1) as

$$\begin{aligned} &(1 + x^1 + x^{1\cdot 2} + x^{1\cdot 3} + x^{1\cdot 4} + \cdots) \\ &\qquad \cdot(1 + x^2 + x^{2\cdot 2} + x^{2\cdot 3} + x^{2\cdot 4} + \cdots) \\ &\qquad \cdot(1 + x^3 + x^{3\cdot 2} + x^{3\cdot 3} + x^{3\cdot 4} + \cdots) \\ &\qquad \cdot(1 + x^4 + x^{4\cdot 2} + x^{4\cdot 3} + x^{4\cdot 4} + \cdots) \\ &\qquad \cdot(1 + x^5 + x^{5\cdot 2} + x^{5\cdot 3} + x^{5\cdot 4} + \cdots) \cdots . \end{aligned} \tag{2}$$

If we multiply this out and calculate, for example, the coefficient of x^5, we find that the term x^5 is obtained in the following ways:

$$1\cdot 1\cdot 1\cdot 1\cdot x^5\cdot 1 \cdots, \quad x^1\cdot 1\cdot 1\cdot x^4\cdot 1 \cdots, \quad 1\cdot x^2\cdot x^3\cdot 1 \cdots,$$
$$x^{1\cdot 2}\cdot 1\cdot x^3\cdot 1 \cdots, \quad x^1\cdot x^{2\cdot 2}\cdot 1 \cdots, \quad x^{1\cdot 3}\cdot x^2\cdot 1 \cdots, \quad x^{1\cdot 5}\cdot 1 \cdots .$$

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Alder has devoted enormous energy to serving the mathematical community. (Actually I first met him through our mutual interest in high school contests and visiting lecturer programs when I was in California in the fifties; he considerably influenced my participation in MAA activities.) Alder served as National President of Mu Alpha Theta from 1956–1959 and received its Distinguished Service Award in 1965. As Secretary of the MAA since 1960, he has probably contributed more than anyone else to the present vitality of this Association. *Editor.*

Each of these products corresponds to a partition of 5, indeed in exactly the order in which the partitions of 5 are listed above. Since there is a one-to-one correspondence between the number of times the term x^n is obtained in the product (2) and the number of partitions of n, the coefficient of x^n in (2) is $p(n)$.

The function $p(n)$ is also referred to as the number of *unrestricted partitions* of n, to make clear that no restrictions are imposed upon the way in which n is partitioned into parts. A very interesting—perhaps the most interesting—part of the theory of partitions concerns restricted partitions, that is, partitions in which some kind of restrictions is imposed upon the parts. The fascination in this study lies in the fact that there exist numerous surprising identities valid for all positive integers n of the general type

$$p'(n) = p''(n), \tag{3}$$

where $p'(n)$ is the number of partitions of n where the parts of n are subject to a first restriction and $p''(n)$ is the number of partitions of n where the parts of n are subject to an entirely different restriction. It is the object of this paper to give a survey of the existence and nonexistence of such identities as known up to date.

Perhaps the simplest identity of the above kind is given by the following theorem:

THEOREM 2. *The number of partitions of n into exactly μ parts (μ a given positive integer) is equal to the number of partitions of n into parts the largest of which is μ.*

Proof of Theorem 2. A partition of n into exactly μ parts can be represented graphically by μ lines of dots, the number of dots in each line equalling the part. Thus, the partition of 23 into the 5 parts $7+6+4+4+2$ can be represented by the following graph:

$$\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & & & & &
\end{array}$$

When read vertically by columns, this represents the partition of 23 into $5+5+4+4+2+2+1$, that is, into a partition, the largest part of which is 5. Thus, to each partition of n into μ parts corresponds a partition of n into parts the largest of which is μ, and, since this is a one-to-one correspondence, we have proved the theorem.

The following theorem follows immediately from Theorem 2.

THEOREM 3. *The number of partitions of n into at most μ parts (μ a given positive integer) is equal to the number of partitions of n with parts not exceeding μ.*

Theorem 2 was proved by means of a *combinatorial proof* in a direct way, that is, a one-to-one correspondence between the two types of restricted partitions was established. Many—or perhaps most—identities involving two kinds of restricted partitions are proved more easily, and up to now in some cases can be proved only by *analytical proofs*, that is, by showing that the generating functions for the two types of restricted partitions involved are identical. We shall now consider examples of such identities.

2. Restricted Partition Functions. In order to be able to prove identities of type (3) by use of generating functions, we need to know how to derive generating functions for certain restricted partitions. We shall list in the following table a few which will be needed later.

PARTITIONS INTO	GENERATING FUNCTION
Distinct parts	$\prod_{\nu=1}^{\infty}(1+x^{\nu})$
Odd parts	$1\Big/\prod_{\nu=0}^{\infty}(1-x^{2\nu+1})$
Parts not exceeding μ	$1\Big/\prod_{\nu=1}^{\mu}(1-x^{\nu})$
Parts taken from the set $\{a_1, a_2, \cdots\}$	$1\Big/\prod_{\nu=1}^{\infty}(1-x^{a_{\nu}})$

These generating functions can easily be derived by modifying the proof of Theorem 1 appropriately. Thus, since for distinct parts each positive integer is allowed no more than once, each of the infinite sums of (2) has to be reduced to its first two terms only. Similarly, for partitions with odd parts, the second, fourth, sixth, $\cdots$ infinite sums of (2) have to be deleted which immediately yields the generating function in the second line of the table above.

3. The Euler and Rogers-Ramanujan Identities. The most celebrated identity which is very easily proved by means of generating functions is due to Euler [11], who discovered it in 1748.

THEOREM 4. (Euler). *The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

Proof of Theorem 4. We have to show that the generating function for partitions into distinct parts, as given in line 1 of the above table, is equal to the generating function of *n* into odd parts, as given in line 2 of the above table. This is easily done as follows:

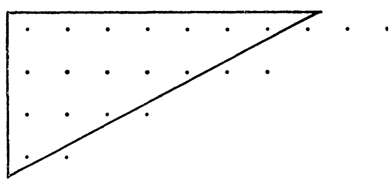
$$\prod_{\nu=1}^{\infty}(1+x^\nu)=(1+x)(1+x^2)(1+x^3)\cdots=\frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\cdots$$
$$=\frac{1}{\prod_{\nu=0}^{\infty}(1-x^{2\nu+1})}.$$

It is possible—although considerably more difficult—to prove this result by combinatorial methods, see, for example, [26].

Since the Euler identity involves partitions into distinct parts, that is, where parts must differ by at least 1, it is natural to ask whether there exists a corresponding identity involving partitions, where parts must differ by at least 2. Such an identity was discovered by Rogers and Ramanujan around the turn of the century.

THEOREM 5 (Rogers-Ramanujan). *The number of partitions of n into parts differing by at least* 2 *is equal to the number of partitions of n into parts which are congruent to* 1 *or* 4, *modulo* 5.

To express this theorem as an equality of the generating functions for the two kinds of restricted partitions involved, we need to derive the generating function for the number of partitions of n into parts differing by at least 2. We represent such a partition graphically, for example, $23=10+7+4+2$, as follows:



Since parts must differ by at least 2, each line must have at least 2 more dots than the one below. Thus, if the partition has exactly μ parts, the graph must have at least $1+3+5+\cdots+(2\mu-1)=\mu^2$ dots (in our graph, they are the dots inside the indicated triangle). Consequently a partition of n into μ parts differing by at least 2 can be graphically represented by a triangle with μ^2 dots and a partition of $n-\mu^2$ into at most μ parts. To obtain the number of all partitions of n into μ parts, we need the generating function for the number of partitions of $n-\mu^2$ into at most μ parts or, which is the same according to Theorem 2, the number of partitions of $n-\mu^2$ into parts not exceeding μ. From line 3 of the above table, we know that the generating function for the number of partitions of n into parts not exceeding μ is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^\mu)}.$$

Consequently the coefficient of x^n in

$$\frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^\mu)} \tag{4}$$

equals the number of partitions of $n-\mu^2$ into parts not exceeding μ. To find the number of all partitions of n into parts differing by at least 2, we need to sum the coefficients of x^n in (4) for $\mu=1, 2, 3, \cdots$; that is, we determine the coefficient of x^n in

$$\sum_{\mu=1}^{\infty} \frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^\mu)}. \tag{5}$$

Accordingly, Theorem (5) is expressed as an identity of generating functions as follows:

$$\sum_{\mu=0}^{\infty} \frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^\mu)} = \prod_{\nu=0}^{\infty} \frac{1}{(1-x^{5\nu+1})(1-x^{5\nu+4})}. \tag{6}$$

The proof of (6) and consequently Theorem 5 is somewhat more lengthy than the proof of Euler's identity. The basic tool is the conversion of the infinite product appearing on the right hand side of (6) into the sum on the left hand side by use of Jacobi's identity

$$\prod_{\nu=0}^{\infty} (1-y^{2\nu+2})(1+y^{2\nu+1}z)(1-y^{2\nu+1}z^{-1}) = \sum_{\mu=-\infty}^{\infty} y^{\mu^2}z^{\mu}, \tag{7}$$

and the use of an auxiliary function for which a recurrence equation is derived.

The details of the proof will not be given here, but can be found in [**17**, Chap. 19] and [**20**].

Rogers and Ramanujan found a second identity in which the parts were not only required to differ by at least 2 but also to be all at least equal to 2.

THEOREM 6 (Rogers-Ramanujan). *The number of partitions of n into parts differing by at least* 2, *each part being greater than or equal to* 2, *is equal to the number of partitions of n into parts which are congruent to* 2 *or* 3, *modulo* 5.

To express this theorem as an identity of the generating functions of the two kinds of restricted partitions involved, we proceed exactly as in the derivation following Theorem 5 except that the triangle of that graph is replaced by a trapezoid, the bottom line of which contains 2 dots, the next to last 4 dots, etc., so that the trapezoid would contain inside a total of $2+4+6+\cdots+2\mu=\mu^2+\mu$ dots. Consequently, Theorem 6 is expressed as an identity of generating functions as follows:

$$\sum_{\mu=0}^{\infty} \frac{x^{\mu^2+\mu}}{(1-x)(1-x^2)\cdots(1-x^\mu)} = \prod_{\nu=0}^{\infty} \frac{1}{(1-x^{5\nu+2})(1-x^{5\nu+3})}. \tag{8}$$

4. The Nonexistence of Certain Other Identities of the Euler-Rogers-

Ramanujan Type. If we denote by $q_{d,m}(n)$ the number of partitions of n into parts differing by at least d, each part being greater than or equal to m, then the Euler and Rogers-Ramanujan identities are all of the type

$$q_{d,m}(n) = p_{d,m}(n), \tag{9}$$

where $p_{d,m}(n)$ is the number of partitions of n into parts taken from a fixed set $S_{d,m}$. Thus, in the case of the Euler identity ($d=1$, $m=1$), the set $S_{1,1}$ is the set of odd numbers; in the case of the first Rogers-Ramanujan identity, $S_{2,1}$ is the set of numbers congruent to 1 or 4, modulo 5.

It is natural to ask whether there are any more identities of the type (9). For $d=1$, such an identity exists for every m; that is, there exists the following generalization of the Euler identity:

THEOREM 7. *The number of partitions of n into distinct parts, each part being greater than or equal to m, is equal to the number of partitions of n into parts taken from the set $\{m, m+1, \cdots, 2m-1, 2m+1, 2m+3, \cdots\}$.*

The proof of this theorem is analogous to that of Theorem 4.

Aside from this generalization of the Euler identity and the two Rogers-Ramanujan identities, no other identities of type (9) can exist, so that we have the following theorem.

THEOREM 8. *The number $q_{d,m}(n)$ of partitions of n into parts differing by at least d, each part being greater than or equal to m, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever unless $d=1$ or $d=2$, $m=1, 2$.*

This theorem was proved for the case $m=1$ by D. H. Lehmer [18] in 1946, and for the general case by this writer [1] in 1948. To prove it, we note that by a slight generalization of the argument used to derive the generating function (5), the one for the number $q_{d,m}(n)$ of partitions into parts differing by at least d, each part being greater than or equal to m, is given by

$$\sum_{\mu=0}^{\infty} \frac{x^{m\mu+d\mu(\mu-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{\mu})}, \tag{10}$$

while the generating function for the number of partitions of n into parts taken from a fixed set $\{a_1, a_2, \cdots\}$ is given in the last line of the above table. The proof then consists of showing that no matter how the a_i are chosen, the latter generating function cannot equal that given by (10).

This then proves that the Euler identity and its generalization, given in Theorem 7, together with the two Rogers-Ramanujan identities are indeed the set of all identities of type (9) which can exist.

The question may be raised whether identities of type (9) are possible if $p_{d,m}(n)$ is the number of partitions of n into *distinct* parts taken from a fixed set $S_{d,m}(n)$. That this is not possible is stated in the following theorem:

THEOREM 9. *The number $q_{d,m}(n)$ of partitions of n into parts differing by at least d, each part being greater than or equal to m, is not equal to the number of partitions of n into distinct parts taken from any set of integers whatsoever unless $d=1$.*

The proof of this theorem, also given in [1], consists of showing that for no choice of the elements a_i of the set $S_{d,m}$ the generating function for the number of partitions of n into distinct parts taken from that set, namely

$$\prod_{\nu=1}^{\infty}(1+x^{a_\nu}),$$

can equal the generating function given by (10).

5. Early Combinatorial Generalizations of the Euler Identity. Although, in accordance with Section 4, certain generalizations of the Euler identity cannot exist, it was already proved in the last century that others do exist. The first remarkable result in this direction was proved by Glaisher [12] in 1883.

THEOREM 10 (Glaisher). *The number of partitions of n into parts not divisible by d is equal to the number of partitions of n of the form $n=n_1+n_2+\cdots+n_s$, where $n_i \geqq n_{i+1}$ and $n_i \geqq n_{i+d-1}+1$.*

For $d=2$, Theorem 10 clearly reduces to Euler's Theorem since the last inequality then requires each part of a partition, when written in nonincreasing order of the parts, to be at least 1 greater than the next one. For $d=3$, the last inequality requires that in any set of three consecutive parts of a partition the first is greater by at least 1 than the last one, thus permitting in this case two consecutive parts to be equal.

Another generalization of Euler's Theorem in an entirely different direction was discovered by Sylvester [26] in 1882.

THEOREM 11 (Sylvester). *The number of partitions of n into odd parts, where exactly k distinct parts appear, is equal to the number of partitions of n into distinct parts, where exactly k sequences of consecutive integers appear.*

Note that in this theorem, a sequence of k consecutive integers may consist of a single integer if $k=1$.

Thus, for $n=13$, the partitions into odd parts where exactly 3 distinct parts appear are $9+3+1$, $7+5+1$, $7+3+1+1+1$, $5+3+3+1+1$, $5+3+1+1+1+1+1$, so that we have 5 partitions of this kind, while the number of partitions of 13 into distinct parts, where exactly 3 sequences of consecutive integers appear, are $9+3+1$ (which consists of the three sequences of 1 integer each), $8+4+1$, $7+5+1$, $7+4+2$, $6+4+2+1$ (which consists of two sequences of 1 integer each and one sequence, namely 2, 1, of two consecutive integers) so that again we have 5 such partitions.

Euler's Theorem is a direct consequence of Theorem 11 by summing over all values of k.

This theorem—and consequently also Euler's Theorem—was proved arithmetically by Sylvester [26, Section 46]. It was proved by the use of generating functions by Andrews [5] in 1966.

6. Analytic Generalizations of the Rogers-Ramanujan Identities. If we consider the Rogers-Ramanujan identities merely as functional equations and disregard their interpretations in terms of partitions, then it is possible to generalize the Rogers-Ramanujan identities. Since Jacobi's identity (7) which plays a vital role in the proof of the Rogers-Ramanujan identities converts an infinite product consisting of *three* different terms into a certain infinite sum, it seems natural to consider the set involved in the first Rogers-Ramanujan identity as the set of numbers *not* congruent to 0, ± 2 (mod 5), rather than the set of numbers congruent to 1 or 4 (mod 5). The first Rogers-Ramanujan identity (6) involving the case where parts are *not* congruent to 0, ± 2 (mod 5) can then be generalized to the case where the parts are not congruent to 0, $\pm k$ (mod $2k+1$) as follows:

THEOREM 12. *The following identity holds*:

$$\prod_{\nu=0}^{\infty} \frac{(1 - x^{(2k+1)\nu+k})(1 - x^{(2k+1)\nu+k+1})}{(1 - x^{(2k+1)\nu+1})(1 - x^{(2k+1)\nu+2}) \cdots (1 - x^{(2k+1)\nu+2k})} = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)}{(1 - x)(1 - x^2) \cdots (1 - x^\mu)}, \tag{11}$$

where the left side is the generating function for the number of partitions into parts not congruent to 0, $\pm k$ (mod $2k+1$) *and the* $G_{k,\mu}(x)$ *are polynomials in* x *and reduce to the monomials* x^{μ^2} *for* $k=2$, *that is, for the Rogers-Ramanujan case.*

The second Rogers-Ramanujan identity (8) involving the case where parts are congruent to 2 or 3, modulo 5, or, what is the same, parts *not* congruent to 0, ± 1 (mod 5) can be generalized to the case where parts are not congruent to 0, ± 1 (mod $2k+1$) as follows:

THEOREM 13. *The following identity holds*:

$$\prod_{\nu=0}^{\infty} \frac{1}{(1 - x^{(2k+1)\nu+2})(1 - x^{(2k+1)\nu+3}) \cdots (1 - x^{(2k+1)\nu+(2k-1)})} = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)x^\mu}{(1 - x)(1 - x^2) \cdots (1 - x^\mu)}, \tag{12}$$

where the $G_{k,\mu}(x)$ *are the same polynomials as those of Theorem* 12.

More generally, it can be shown that identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(k-r)$ (mod $2k+1$) exist for each r within the range $0 \leqq r \leqq k-1$, so that for a given modulus $2k+1$, there are always k such identities. This agrees with our knowledge that for the modulus 5 two such identities exist.

Theorems 12 and 13 were proved by this writer [2] in 1954 by means of generalizing the proof of the Rogers-Ramanujan identities, again making use of Jacobi's identity (7) and using a generalization of the auxiliary function of the proof of the Rogers-Ramanujan identities. This more general function and a recurrence formula involving it were originally introduced by Selberg in 1936 [22, p. 4, equation 3]. Another proof of Theorems 12 and 13 was given by Singh in 1957 [23]. In [2] some properties of the polynomials $G_{k,\mu}(x)$ were given. Further properties of these polynomials were derived in two papers by Singh in 1957 [24] and in 1959 [25]. An explicit formula for these polynomials was given by Carlitz [9] in 1960. While Theorems 12 and 13 show that it is possible to generalize the Rogers-Ramanujan identities, it has not been possible to describe the right hand sides of (11) and (12) as generating functions for certain types of restricted partitions.

7. Combinatorial Generalizations of the Rogers-Ramanujan Identities. The first success in attempts to generalize the Rogers-Ramanujan identities in a way in which the generalization states an equality between two kinds of restricted partitions was achieved by Gordon in 1961 [15]. This generalization extends the Rogers-Ramanujan identities in a way similar to that in which Glaisher's Theorem 10 extended the Euler identity:

THEOREM 14 (Gordon). *Let $p_{k,r}(n)$ denote the number of partitions of n into parts not congruent to* 0, $\pm r$ (mod $2k+1$), *where* $1 \leqq r \leqq k$. *Let $q_{k,r}(n)$ denote the number of partitions of n of the form $n = n_1 + n_2 + \cdots + n_s$ where $n_i \geqq n_{i+1}$, $n_i \geqq n_{i+k-1} + 2$ and with* 1 *appearing as a part at most $r-1$ times, then*

$$p_{k,r}(n) = q_{k,r}(n). \tag{13}$$

This theorem reduces to the first Rogers-Ramanujan identity for $k=2$, $r=2$ and to the second for $k=2$, $r=1$. As in the case of the analytic generalizations of the Rogers-Ramanujan identities discussed in Section 6, there are also in this case for a given modulus $2k+1$ exactly k identities.

Gordon in his paper [15] gives a combinatorial proof of Theorem 14 which, therefore, contains as a special case a combinatorial proof of the Rogers-Ramanujan identities. Andrews [6] in 1966 gave an analytic proof of Theorem 14 along the lines of Ramanujan's proof of his identities [20] using the auxiliary function and its recurrence formula introduced by Selberg [22] and also used in the proofs of Theorems 12 and 13 [2].

For some purposes, it is advisable to note that $q_{k,r}(n)$ can also be thought of as the number of partitions of n of the form

$$n = \sum_{i=1}^{\infty} f_i \cdot i,$$

where f_i denotes the number of times the part i appears in the partition, $f_i + f_{i+1} \leqq k-1$, and with 1 appearing as a part at most $r-1$ times. The condition $f_i + f_{i+1} \leqq k-1$ implies that the total number of appearances of two consecutive integers i and $i+1$ in a partition is at most $k-1$, so that in any set of k consecutive parts in a partition, arranged in nonincreasing order of parts, the first and last part must differ by at least 2, that is, $n_i \geqq n_{i+k-1}+2$.

Using this latter interpretation, Andrews [4] in 1965 was able to generalize Gordon's Theorem further as follows:

THEOREM 15 (Andrews). *Let $p_{d,k,r}(n)$ denote the number of partitions of n into parts not congruent to* 0, $\pm dr$ (mod $d(2k+1)$), *where* $d \geqq 1$, $1 \leqq r \leqq k$. *Let $q_{d,k,r}(n)$ denote the number of partitions of n of the form $n=\sum_{i=1}^{\infty} f_i \cdot i$, where if $f_i \equiv \alpha$* (mod d) ($0 \leqq \alpha \leqq d-1$) *then $f_i+f_{i+1} \leqq dk+\alpha-1$, and where* 1 *appears as a part at most $dr-1$ times.*

The above is a corrected version of the abstract which appeared in the *Notices* of the AMS. If $d=1$, Theorem 15 reduces to Gordon's Theorem 14. If $k=1$, $r=1$, Theorem 15 reduces to Glaisher's Theorem 10. This theorem consequently is the first theorem which contains *both* the Euler identity *as well as* the Rogers-Ramanujan identities as special cases.

8. Schur's Identity. If in the Euler identity (see Theorem 4), we replace "odd parts" by "parts congruent to ± 1, modulo 4" and in the first Rogers-Ramanujan identity (see Theorem 5) "parts which are congruent to 1 or 4, modulo 5" by "parts which are congruent to ± 1, modulo 5," the similarity of these two identities becomes even more striking. Let us, therefore, define $p_d(n)$ as the number of partitions of n into parts congruent to ± 1, modulo $d+3$; then the Euler identity can be written as $q_{1,1}(n)=p_1(n)$ and the first Rogers-Ramanujan identity as $q_{2,1}(n)=p_2(n)$. It follows as a consequence of Theorem 8 that $q_{3,1}(n)$ cannot equal $p_3(n)$, but the obvious question arises as to whether there is a relationship between the two. Schur proved that $q_{3,1}(n)-p_3(n)$ is the number of partitions of n into parts differing by at least 3 and containing at least 2 consecutive multiples of 3 so that we have the following theorem, stated in a slightly different form:

THEOREM 16 (Schur). *The number of partitions of n into parts differing by at least* 3 *among which no two consecutive multiples of* 3 *appear is equal to the number of partitions of n into parts which are congruent to* 1 *or* 5, *modulo* 6.

Thus, for example, the number of partitions of 15 into parts differing by at least 3 among which no two consecutive multiples of 3 appear are 15,

$$14+1, 13+2, 12+3, 11+4, 10+5, 10+4+1, 9+5+1, 8+5+2,$$

so that the number of partitions of this kind is 9 which is the number $q_{3,1}(15)$ of all partitions of 15 into parts differing by at least 3, namely 10, minus the number among these partitions in which two consecutive multiples of 3 appears, namely 1 in this case (that is, $9+6$). As the reader may verify, this number, 9, is also the number of partitions of 15 into parts congruent to 1 or 5, modulo 6.

Theorem 16 was proved by Schur in 1926 [21] by means of a lemma concerning recurrence relations for certain polynomials. In 1928, Gleissberg [**13**] gave an intricate arithmetic proof of this theorem. A shorter proof was given by Andrews [**8**] in 1967, based on Appell's Comparison Theorem [**10**, p. 101].

9. The Nonexistence of Certain Generalizations of Schur's Identity. Using the notation of Section 8, we know that the difference $q_{d,2}(n)-p_d(n)$ is equal to 0 for $d=1$ (Euler identity) and for $d=2$ (the first Rogers-Ramanujan identity) and that for $d=3$ this difference represents the number of partitions of n into parts differing by at least 3 and containing at least 2 consecutive multiples of 3. For $d\geqq 4$, however, there seems to be no simple interpretation of the difference $q_{d,1}(n)-p_d(n)$, even if it could be shown to be nonnegative. The following theorem shows in particular that there cannot be an interpretation exactly like that for $d=3$:

THEOREM 17. *The number of partitions of n into parts differing by at least d, among which no two consecutive multiples of d appear, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever if* $d>3$.

This theorem was proved in 1948 by this writer [**1**]. The method used in that proof can also be used to prove that there cannot exist a dual to Schur's Theorem in the sense that the second of the Rogers-Ramanujan identities is a dual to the first one, so that we have the following theorem:

THEOREM 18. *The number of partitions of n into parts differing by at least* 3, *no part being equal to* 1, *among which no two consecutive multiples of* 3 *appear, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever.*

No results concerning $q_{d,1}(n)-p_d(n)$ for $d\geqq 4$ are available, not even whether this difference is always greater than or equal to 0 although this question was posed by the author as a research problem in the Bulletin of the Amer. Math. Soc. [**3**].

10. Combinatorial Generalizations of Schur's Identity for the Case where $d=2$. Two theorems, both of which are very similar in nature to Schur's Theorem in that they consider partitions of n into parts differing by at least 2 among which no two consecutive multiples of 2 appear (that is, the 3 of Schur's Theorem is replaced by 2) were discovered independently by Göllnitz [**14**] in 1960 and Gordon [**16**] in 1965.

THEOREM 19. (Göllnitz-Gordon). *The number of partitions of n into parts differing by at least* 2 *among which no two consecutive even numbers appear is equal to the number of partitions of n into parts which are congruent to* 1, 4, 7, *modulo* 8.

Thus, for example, the number of partitions of 11 into parts differing by at least 2 and containing no two consecutive even integers are 11, 10+1, 9+2, 8+3, and 7+3+1, so that there are 5 partitions of this kind, while the number of partitions of 11 into parts congruent to 1, 4, 7, modulo 8, are $9+1+1$, $7+4$, $7+1+1+1+1$, $4+4+1+1+1$, and $1+1+\cdots+1$, again 5 such partitions.

THEOREM 20 (Göllnitz-Gordon). *The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear and with each part being at least equal to 3 is equal to the number of partitions of n into parts which are congruent to* 3, 4, 5, *modulo* 8.

11. A Combinatorial Generalization of the Göllnitz-Gordon Identities. Andrews [7] in 1967 generalized the Göllnitz-Gordon identities in the same manner that Gordon's Theorem 14 generalizes the Rogers-Ramanujan identities.

THEOREM 21 (Andrews). *Let $p_{k,r}(n)$ denote the number of partitions of n into parts not congruent to* 2 (mod 4) *and not congruent to* 0, $\pm(2r-1)$(mod $4k$), *where* $1\leqq r\leqq k$. *Let $q_{k,r}(n)$ denote the number of partitions of n of the form $n=\sum_{i=1}^{\infty} f_i\cdot i$, where $f_1+f_2\leqq r-1$ and for all $i\geqq 1$*

$$f_{2i-1}\leqq 1 \quad \textit{and} \quad f_{2i}+f_{2i+1}+f_{2i+2}\leqq k-1.$$

Then

$$p_{k,r}(n) = q_{k,r}(n). \tag{14}$$

Theorem 21 reduces to Theorem 19 for $k=2$, $r=2$ and to Theorem 20 for $k=2$, $r=1$. Let us consider the case $k=3$, $r=3$, then the partitions enumerated by $p_{3,3}(7)$ are $6+1$, $5+2$, $4+3$, $3+3+1$, and $3+1+1+1+1$, so that $p_{3,3}(7)=5$, while the partitions enumerated by $q_{3,3}(7)$ are 7, $6+1$, $5+2$, $4+3$, and $4+2+1$, so that $q_{3,3}(7)=5$.

12. Some Other Identities of the Schur Type. In 1967 Andrews [8] proved a theorem which is similar to Schur's identity, but involves as modulus a multiple of 4.

THEOREM 22 (Andrews). *Let $p_r(n)$ denote the number of partitions of n into parts which are either even and not congruent to $4r-2$ (mod $4r$) or odd and congruent to $2r-1$ or $4r-1$ (mod $4r$), where $r\geqq 2$. Let $q_r(n)$ denote the number of partitions of n of the form $n=n_1+n_2+\cdots+n_s$, where $n_i\geqq n_{i-1}$ and, if n_i is odd, $n_i-n_{i+1}\geqq 2r-1$ for $1\leqq i\leqq s$, where we define $n_{s+1}=0$. Then*

$$p_r(n) = q_r(n). \tag{15}$$

In the proof of this theorem, Andrews used a generalization of the method used in his proof of Schur's Theorem.

A student of the author, Elmo Moore [19], proved in 1968 that Theorem 22 is also valid for $r=1$, so that we have the following theorem:

THEOREM 23 (Moore). *Let $p_1(n)$ be the number of partitions of n into parts which are either divisible by* 4 *or odd. Let $q_1(n)$ denote the number of partitions of n of the form $n_1+n_2+\cdots+n_s$, where $n_i\geqq n_{i+1}$ and, if n_i is odd, $n_i-n_{i+1}\geqq 1$ for $1\leqq i\leqq s$. Then*

$$p_1(n) = q_1(n). \tag{16}$$

Thus, for example, for $n=8$, the partitions of n into parts which are either divisible by 4 or odd are 8, 7+1, 5+3, 5+1+1+1, 4+4, 4+3+1, 4+1+1+1+1, 3+3+1+1, 3+1+1+ $\cdots$ +1, and 1+1+ $\cdots$ +1, so that $p_1(8)=10$, while the partitions of 8 enumerated by $q_1(8)$ are 8, 7+1, 6+2, 5+3, 5+2+1, 4+4, 4+3+1, 4+2+2, 3+2+2+1, and 2+2+2+2, so that $q_1(8)=10$.

For $r=2$, Theorem 22 leads to the following interesting result, as noted by Andrews [8].

THEOREM 24 (Andrews). *Let $p_2(n)$ be the number of partitions of n into parts which are congruent to* 0, 2, 3, 4, 7 (mod 8). *Let $q_2(n)$ denote the number of partitions of n of the form $n_1+n_2+\cdots+n_s$, where $n_i \geqq n_{i+1}$, $n_s \geqq 2$, and, if n_i is odd, $n_i - n_{i+1} \geqq 3$. Then*

$$p_2(n) = q_2(n). \tag{17}$$

13. Conclusion. The above indicates that, after some period of inactivity, the 1960's have brought a considerable increase in interest in partition identities and that new types of identities have been discovered, such as those stated in Section 12, as well as identities which are generalizations of those already known, such as Andrews' Theorem 15 which contains both the Euler identity and the Rogers-Ramanujan identities as special cases.

There is every indication that further research in both of these directions will lead to more surprising results. Perhaps the ultimate objective might be the discovery of an identity which contains most or all of the partition identities discussed in this paper as special cases, or perhaps, more modestly, at least the Euler, Rogers-Ramanujan and Schur identities.

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SOME MATHEMATICIANS I HAVE KNOWN

GEORGE PÓLYA, Stanford University

Professor Klee, Ladies and Gentlemen,

The occasion requires that I should make a speech. Yet I am very old, my days of invention are over. The little mathematical remarks I have made lately

Prof. Pólya received his Univ. Budapest degree in 1912 and holds honorary degrees from the E. T. H. Zürich, Univ. Alberta, and Univ. Wisconsin. He taught at the E. T. H. until 1940 and has been at Stanford Univ. since. His numerous visiting posts include Cambridge, Oxford, Paris, Göttingen, and Princeton. He is a Correspondent of the Paris Academy of Sciences and holds honorary membership in the Council of the Soc. Math. de France, the London Math. Soc. and the Swiss Math. Soc. Prof. Pólya received the M. A. A. Distinguished Service Award in 1963 and the 1968 N. Y. Film Festival top Blue Ribbon for "Let us teach guessing."

The scientific contributions of George Pólya include over 230 research papers and the books, *Inequalities* (with Hardy and Littlewood), *How to Solve It*, *Isoperimetric Inequalities* (with Szegö), *Mathematics and Plausible Reasoning* (2 v.), and *Mathematical Discovery* (2 v.).

Prof. Pólya's personal influence on three generations of mathematicians has been enormous. Perhaps no book in existence has influenced the direction of thinking of young mathematicians more than his two volume masterpiece with G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*. *Editor.*