Was Newton's Calculus a Dead End? The Continental Influence of Maclaurin's Treatise of Fluxions

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1. INTRODUCTION. Eighteenth-century Scotland was an internationally-recognized center of knowledge, "a modern Athens in the eyes of an enlightened world." [**74**, p. 40] [**81**] The importance of science, of the city of Edinburgh, and of the universities in the Scottish Enlightenment has often been recounted. Yet a key figure, Colin Maclaurin (1698–1746), has not been highly rated. It has become a commonplace not only that Maclaurin did little to advance the calculus, but that he did much to retard mathematics in Britain—although he had (fortunately) no influence on the Continent. Standard histories have viewed Maclaurin's major mathematical work, the two-volume *Treatise of Fluxions* of 1742, as an unread monument to ancient geometry and as a roadblock to progress in analysis. Nowadays, few people read the *Treatise of Fluxions*. Much of the literature on the history of the calculus in the eighteenth and nineteenth centuries implies that few people read it in 1742 either, and that it marked the end—the dead end—of the Newtonian tradition in calculus. [**9**, p. 235], [**49**, p. 429], [**10**, p. 187], [**11**, pp. 228–9], [**43**, pp. 246–7], [**42**, p. 78], [**64**, p. 144]

But can this all be true? Could nobody on the Continent have cared to read the major work of the leading mathematician in eighteenth-century Scotland? Or, if the work was read, could it truly have been "of little use for the researcher" [**42**, p. 78] and have had "no influence on the development of mathematics"? [**64**, p. 144]

We will show that Maclaurin's *Treatise of Fluxions* did develop important ideas and techniques and that it did influence the mainstream of mathematics. The Newtonian tradition in calculus did not come to an end in Maclaurin's Britain. Instead, Maclaurin's *Treatise* served to transmit Newtonian ideas in calculus, improved and expanded, to the Continent. We will look at what these ideas were, what Maclaurin did with them, and what happened to this work afterwards. Then, we will ask what by then should be an interesting question: why has Maclaurin's role been so consistently underrated? These questions will involve general matters of history and historical writing as well as the development of mathematics, and will illustrate the inseparability of the external and internal approaches in understanding the history of science.

2. THE STANDARD PICTURE. Let us begin by reviewing the standard story about Maclaurin and his *Treatise of Fluxions*. The calculus was invented independently by Newton and Leibniz in the late seventeenth century. Newton and Leibniz developed general concepts—differential and integral for Leibniz, fluxion and fluent for Newton—and devised notation that made it easy to use these concepts. Also, they found and proved what we now call the Fundamental Theorem of Calculus, which related the two main concepts. Last but not least, they successfully applied their ideas and techniques to a wide range of important problems.

[**9**, p. 299] It was not until the nineteenth century, however, that the basic concepts were given a rigorous foundation.

In 1734 George Berkeley, later Bishop of Cloyne, attacked the logical validity of the calculus as part of his general assault on Newtonianism. [**12**, p. 213] Berkeley's criticisms of the rigor of the calculus were witty, unkind, and—with respect to the mathematical practices he was criticizing—essentially correct. [**6**, v. 4, pp. 65–102] [**38**, pp. 33–34] [**82**, pp. 332–338] Maclaurin's *Treatise* was supposedly intended to refute Berkeley by showing that Newton's calculus was rigorous because it could be reduced to the methods of Greek geometry. [**10**, pp. 181–2, 187] [**9**, pp. 233, 235] Maclaurin himself said in this preface that he began the book to answer Berkeley's attack, [**63**, p. i] and also to rebut Berkeley's accusation that mathematicians were hostile to religion. [**78**, p. 50]

The majority of Maclaurin's treatise is contained in its first Book, which is called "The Elements of the Method of Fluxions, Demonstrated after the Manner of the Ancient Geometricians." That title certainly sounds as though it looks backward to the Greeks, not forward to modern analysis. And the text is full of words—lots of words. So much time is spent on preliminaries that it is not until page 162 that he can show that the fluxion of **ay** is **a** times the fluxion of **y**. Florian Cajori, whose writings have helped spread the standard story, compared Maclaurin to the German poet Klopstock who, Cajori said, was praised by all, read by none. [**10**, p. 188] While British mathematicians, bogged down with geometric baggage, studied and revered the work and notation of Newton and argued with Berkeley over foundations, Continental mathematicians went onward and upward analytically with the calculus of Leibniz. The powerful analytic results and techniques in eighteenth-century Continental mathematics were all that mathematicians like Cauchy, Riemann, and Weierstrass needed for their nineteenth-century analysis with its even greater power, together with its improved rigor and generality. [**9**, ch. 7] [**49**, p. 948] This story became so well known that it was cited by the literary critic Matthew Arnold, who wrote, "The man of genius [Newton] was continued by... completely powerless and obscure followers.... The man of intelligence [Leibniz] was continued by successors like Bernoulli, Euler, Lagrange, and Laplace—the greatest names in modern mathematics." [**1**, p. 54; cited by [**61**, p. 15]]

Now since I myself have contributed to the standard story, especially in delineating the links among Euler, Lagrange, and Cauchy, [**38**, chs. 3–6] I have a good deal of sympathy for it, but I now think that it must be modified. Maclaurin's *Treatise of Fluxions* is an important link between the calculus of Newton and Continental analysis, and Maclaurin contributed to key developments in the mathematics of his contemporaries. Let us examine the evidence for this statement.

3. THE NATURE OF MACLAURIN'S TREATISE OF FLUXIONS. Why—the standard story notwithstanding—might Maclaurin's *Treatise of Fluxions* have been able to transmit Newtonian calculus, improved and expanded, to the Continent? First, because the *Treatise of Fluxions* is not just one "Book," but two. While Book I is largely, though not entirely, geometric, Book II has a different agenda. Its title is "On the *Computations* in the Method of Fluxions." [my italics] Maclaurin began Book II by championing the power of symbolic notation in mathematics. [**63**, pp. 575–576] He explained, as Leibniz before him and Lagrange after him would agree, that the usefulness of symbolic notation arises from its generality. So, Maclaurin continued, it is important to demonstrate the rules of fluxions once again, this time from a more algebraic point of view. Maclaurin's appreciation of the algorithmic power of algebraic and calculus notation expresses a common eighteenth-century theme, one developed further by Euler and Lagrange in their pursuit of pure analysis detached from any kind of geometric intuition. To be sure, Maclaurin, unlike Euler and Lagrange, did not wish to detach the calculus from geometry. Nonetheless, Maclaurin's second Book in fact, as well as in rhetoric, has an algorithmic character, and most of its results may be read independently of their geometric underpinnings, even if Maclaurin did not so intend. (In his Preface to Book I, he even urged readers to look at Book II before the harder parts of Book I.) [**63**, p iii] The *Treatise of Fluxions*, then, was not foreign to the Continental point of view, and may have been written in part with a Continental audience in mind.

Nor was this algebraic character a secret open only to the reader of English. There was a French translation in 1749 by the Jesuit R. P. Pézénas, including an extensive table of contents. [**62**] Lagrange, among others, seems to have used this French edition (since he cited it by the French title [**58**, p. 17] though he cited other English works in English [**58**, p. 18]). Pézénas' translation, moreover, was neither isolated nor idiosyncratic, but part of the activity of a network of Jesuits inter-
ested in mathematics and mathematical physics, especially work in English, with Maclaurin one of the authors of interest to them. [**84**, pp. 33, 221, 278, 517, 655] For instance, Pézénas himself translated other English works, including those by Desaguliers, Gardiner's logarithmic tables, and Seth Ward's *Young Mathematician's Guide* [**83**, pp. 571–2] Thus there was a well-worn path connecting English-language work with interested Continental readers. Furthermore, the two-fold character of the *Treaties of Fluxions* was noted, with special praise for Book II's treatment of series, by Silvestre-François Lacroix in the historical introduction to the second edition of his highly influential three-volume calculus textbook. [**52**, p. xxvii] Unfortunately, though, recognition of the two-fold character has been absent from the literature almost completely from Lacroix's time until the recent work by Sageng and Guicciardini. [**42**] [**78**] We shall address the reasons for this neglect in due course.

4. THE SOCIAL CONTEXT: THE SCOTTISH ENLIGHTENMENT. Another reason for doubting the standard picture comes from the social context of Maclaurin's career. Eighteenth-century Scotland, Maclaurin's home, was anything but an intellectual backwater. It was full of first-rate thinkers who energetically pursued science and philosophy and whose work was known and respected throughout Europe. One would expect Scotland's leading mathematician to share these connections and this international renown, and he did.

Although Scotland had been deprived of its independent national government by the Act of Union of 1707, it still retained, besides its independent legal system and its prevailing religion, its own educational system. The strength and energy of Scottish higher education in Maclaurin's time is owed in large part to the Scottish ruling classes, landowners and merchants alike, who saw science, mathematics, and philosophy as keys to what they called the "improvement" of their yet underdeveloped nation. [**65**, p. 254] [**80**, pp. 7–8, 10–11] [**17**, pp. 127, 132–3] Eighteenth-century Scotland, with one-tenth the population of England, had four major universities to England's two. [**80**, p. 116] Maclaurin, when he wrote the *Treatise of Fluxions*, was Professor of Mathematics at the University of Edinburgh. Edinburgh was about to become the heart of the Scottish Enlightenment, and Maclaurin until his death in 1746 was a leading figure in that city's cultural life.

Mathematics played a major role in the Scottish university curriculum. This was in part for engineers; Scottish military engineers were highly in demand even on the Continent. [**17**, p. 125] Maclaurin himself was actively interested in the applications of mathematics, and just before his untimely death had planned to write a book on the subject. [**36**] [**68**, p. xix] In addition, mathematics and Newtonian physics were part of the course of study for prospective clergyman. [**80**, p. 20] The influential "Moderate" party in the Church of Scotland appreciated the Newtonian reconciliation of science and religion. [**16**, pp. 53, 57]

Maclaurin's position in Edinburgh's cultural life was not just that of a technically competent mathematician. For instance, he was part of the Rankenian society, which met at Ranken's Tavern in Edinburgh to discuss such things as the philosophy of Bishop Berkeley; the society introduced Berkeley's philosophy to the Scottish university curriculum. [**24**, p. 222] [**17**, p. 133] [**65**, p. 197] Maclaurin and his physician friend Alexander Monro were the founders and moving spirits of the Edinburgh Philosophical Society. [**65**, p. 198] With Newton's encouragement, Maclaurin had become the chief spokesman in Scotland for the new Newtonian physics. His posthumously published book, *An Account of Sir Isaac Newton's Philosophical Discoveries*, was based on material Maclaurin used in his classes at Edinburgh, and the book was of great interest to philosophers. [**24**, p. 137] That book became well known on the Continent. It was translated into French almost as soon as it appeared, by Louis-Anne Lavirotte in 1749, and the first part appeared in Italian in Venice in 1762.

Anoher branch of Scottish science, namely medicine, also had many links with the Continent and was highly regarded there. Medical students went back and forth between Scotland, Holland, and France. [**17**, p. 135] [**80**, p. 7]

The best-known figures of eighteenth-century Scotland had major interactions with, and influence upon, Continental science and philosophy. [**39**] [**81**] Let it suffice to mention the names of four: the philosopher David Hume, who was a student at Edinburgh in Maclaurin's time; the geologist James Hutton, who attended and admired Maclaurin's lectures; [**34**, pp. 577–8] and, a bit after Maclaurin's time but still subject to his influence on Scottish higher education, the chemist Joseph Black and the economic and political philosopher Adam Smith. Maclaurin himself had twice won prizes from the Académie des Sciences in Paris, once in 1724 for a memoir on percussion, and then in 1740 (dividing the prize with Daniel Bernoulli, P. Antoine Cavalleri, and Leonhard Euler) for a memoir on the tides. [**79**, p. 611] [**39**, pp. 400–401]

Scotland in the eighteenth century nurtured first-rate intellectual work on mathematics, philosophy, science, medicine, and engineering, and did it all as part of a general European culture. [**39**, p. 412] [**81**, passim] The *Treatise of Fluxions* was the major mathematical work of a Scottish mathematician of considerable reputation on the Continent, a major work philosophically attuned to the enormously influential Newtonian physics and the Continentally popular algebraic symbolism. Such a work would certainly be of interest to Continental thinkers. Social considerations may not suffice to determine mathematical ideas, but they certainly affect the mathematician's ability to make a living, to get research support, and to promote contact and communication with other mathematicians and scientists at home and abroad. And so it was with Maclaurin.

5. MACLAURIN'S CONTINENTAL REPUTATION. An even better reason for not accepting the traditional view of Maclaurin is that his work demonstrably *was* read in the eighteenth century, and was read by the big names of Continental mathematics. He had a Continental acquaintance through travel and correspondence. Even before the *Treatise of Fluxions*, his reputation had been enhanced by his Académie prizes and by his books on geometry. He was thus a respected member of an international network of mathematicians with interests in a wide range of subjects, and the publication of the *Treatise of Fluxions* was eagerly anticipated on the Continent.

The *Treatise of Fluxions* of 1742 was Maclaurin's major work on analysis, incorporating and somewhat dwarfing what he had done earlier. It contains an exposition of the calculus, with old results explained and many new results introduced and proved. Maclaurin seems to have included almost everything he had done in analysis and its applications to Newtonian physics. In particular, the findings of his Paris prize paper on the tides were included and expanded. His other papers, the posthumous and relatively elementary *Algebra*, and his works on geometry as such—though highly regarded—do not concern us here, but his Continental reputation was enhanced by these as well.

Let us turn now to some specific evidence for the Continental reputation of Maclaurin's major work. In 1741, Euler wrote to Clairaut that, though he had not yet seen the Paris prize papers on the tides, "from Mr. Maclaurin I expect only excellent ideas." [**47**, p. 87] Euler added that he had heard from England (presumably from his correspondent James Stirling) that Maclaurin was bringing out a book on "differential calculus," and asked Clairaut to keep him posted about this. In turn, Clairaut asked Maclaurin later in 1741 about his plans for the book, [**66**, p. 348] which Clairaut wanted to see before publishing his own work on the shape of the earth. [**47**, p. 110] Euler did get the *Treatise of Fluxions*, and read enough of it quickly to praise it in a letter to Goldbach in 1743. [**48**, p. 179] Jean d'Alembert, in his *Traité de dynamique* of 1743, [**22**, sec. 37, n.] praised the rigor brought to calculus by the *Treatise of Fluxions*. D'Alembert's most recent biographer, Thomas Hankins, argues that Maclaurin's *Treatise*, appearing at this time, helped persuade d'Alembert that gravity could best be described as a continuous acceleration rather than a series of infinitesimal leaps. [**44**, p. 167] D'Alembert's general approach to the foundations of the calculus in terms of limits clearly was influenced by Newton's and Maclaurin's championing of limits over infinitesimals, in particular by Maclaurin's clear description of limits in one of the parts of his *Treatise of Fluxions* that explicitly responds to Berkeley's objections (and which incidentally may be the first explicit description of the tangent as the limit of secant lines; see Section 7). [**44**, p. 23] [**63**, pp. 422–3] Lagrange in his *Analytical Mechanics* [**55**, p. 243] said that Maclaurin, in the *Treatise of Fluxions*, was the first to treat Newton's laws of motion in the language of the calculus in a coordinate system fixed in space. Though C. Truesdell [**80**, pp. 250–3] has shown that Lagrange was wrong because Johann Bernoulli and Euler were ahead of Maclaurin on this, the fact that Lagrange believed this is one more piece of evidence for the Continental reputation of Maclaurin as mathematician and physicist.

6. MACLAURIN'S MATHEMATICS AND ITS IMPORTANCE. The previous points show that Maclaurin could have been influential, but not that he was. Five examples will reveal both the nature of Maclaurin's techniques and the scope of his influence: a special case of the Fundamental Theorem of Calculus; Maclaurin's treatment of maxima and minima for functions of one variable; the attraction of spheroids; what is now called the Euler-Maclaurin summation formula; and elliptic integrals.

a. Key Methods in the Calculus. Two methods were central to the study of real-variable calculus in the eighteenth and nineteenth centuries. One of these is studying real-valued functions by means of power-series representations. This tradition is normally thought first to flower with Euler; it is then most closely associated with Lagrange, and, later for complex variables, with Weierstrass. The second such method is that of basing the foundations of the calculus on the algebra of inequalities—what we now call delta-epsilon proof techniques—and using algebraic inequalities to prove the major results of the calculus; this tradition is most closely associated with the work of Cauchy in the 1820's. I have traced these traditions back to Lagrange and Euler in my work on the origins of Cauchy's calculus. [**38**, chs. 3–6] It is surprising, at least if one accepts the standard picture of the history of the calculus, that both of these methods—studying functions by power series, basing foundations on inequalities—were materially advanced by Maclaurin in the *Treatise of Fluxions*. It is especially striking that the importance of Maclaurin's work on series—work based, it is well to remember, on Newton's use of infinite series—was recognized and praised in 1810 by Lacroix, who also linked it with the series-based calculus of Lagrange. [**52**, p. xxxiii]

Maclaurin skillfully used algebraic inequalities in his proof of a special case of the Fundamental Theorem of Calculus. He showed, for a particular function, that if one takes the fluxion of the area under the curve whose equation is $y = f(x)$, one gets the function $f(x)$. In his proof, Maclaurin adapted the intuition underlying Newton's argument for this fact in *De Analysi* [**69**]—that the rate of change of the area under a curve is measured by the height of the curve—but Maclaurin's proof is more rigorous. Although Maclaurin's argument proceeds algebraically, the concepts involved resemble those of the Greek "method of exhaustion" (more precisely termed by Dijksterhuis "indirect passage to the limit"). [**26**, p. 130] A key step in this Greek work is first to assume that two equal areas or expressions for areas are *unequal*, and then to argue to a contradiction by using inequalities that hold among various rectilinear areas. Newton in the *Principia* had based proofs of new results about areas and curves on methods akin to those of the Greeks. Maclaurin carried this much further. It was Maclaurin's "conservative" allegiance to Archimedean *geometric* methods that led him to buttress the *kinematic* intuition of Newton's calculus with *algebraic* inequality proofs.

What Maclaurin proved in the example under discussion is that, if the area under a curve up to x is given by x^n, the ordinate of the curve must be $y = nx^{n-1}$, which is known to be the fluxion of x^n. [**63**, pp. 752–754] Maclaurin's diagram for this is much like the one Newton gave in the *DeAnalysi*. [**69**, pp. 3–4] Maclaurin

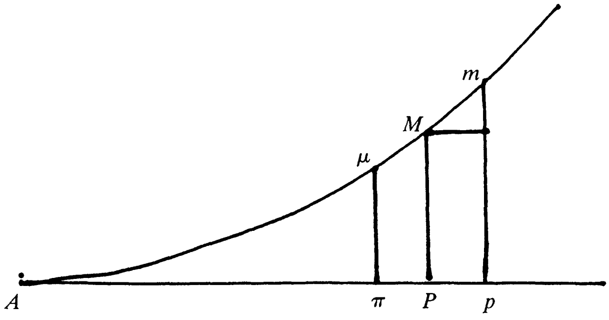

Figure 1

began by saying that, since x and y increase together, the following inequality holds between the areas shown:

$$x^n - (x-h)^n < yh < (x+h)^n - x^n. \tag{1}$$

(Maclaurin gave this inequality verbally; I have supplied the "<" signs; also, I use "h" for the increment where Maclaurin used "o".) Now Maclaurin recalled an algebraic identity he had proved earlier: [**63**, p. 583; inequality notation added]

$$\text{If } E < F, \text{ then } nF^{n-1}(E-F) < E^n - F^n < nE^{n-1}(E-F). \tag{2}$$

(It may strike the modern reader that, since nx^{n-1} is the derivative of x^n, this second inequality is a special case of the mean-value theorem for derivatives. I shall return to this point later.)

Now, letting $x - h$ play the role of F and x play the role of E, $E - F$ is h and the first inequality in (2) yields

$$n(x-h)^{n-1}h < x^n - (x-h)^n.$$

Similarly, if $F = x$ and $E = x + h$, then $E - F = h$ and the second inequality in (2) becomes

$$(x+h)^n - x^n < n(x+h)^n h.$$

Combining these with inequality (1) about the areas, Maclaurin obtained

$$n(x-h)^{n-1}h < yh < n(x+h)^{n-1}h.$$

Dividing by h produces

$$n(x-h)^{n-1} < y < n(x+h)^{n-1}. \tag{3}$$

Recall that, given that the area was x^n, Maclaurin was seeking an expression for y, the fluxion of that area. A modern reader, having reached the inequality (3), might stop, perhaps saying "let h go to zero, so that y becomes nx^{n-1}," or perhaps justifying the conclusion by appealing to the delta-epsilon characterization of limit. What Maclaurin did instead was what Archimedes might have done, a double reductio ad absurdum. But what Archimedes might have done geometrically and verbally, Maclaurin did algebraically. He assumed first that y is *not* equal to nx^{n-1}. Then, he said, it must be equal to $nx^{n-1} + r$ for some r. First, he considered the case when this r was positive. This will lead to a contradiction if h is chosen so that $y = n(x+h)^{n-1}$, since, he observed, inequality (3) will be violated when $h = (x^{n-1} + r/n)^{1/(n-1)}$. Similarly, he calculated the h that produces a contradiction when r is assumed to be negative. Thus there can be no such r, and $y = nx^{n-1}$. [**63**, p. 753]

Maclaurin introduced this proof by saying something surprising for a *Treatise of Fluxions*: that the use of the inequalities makes the demonstration of the value of y "independent of the notion of a fluxion." [**63**, p. 752] (Of course one would need the notion of fluxion to interpret y as the fluxion of the area function x^n, but the proof itself is algebraic.) This proof was presumably part of his agenda in writing the more algebraic Book II of the *Treatise* for an audience on the Continent, where fluxions were suspect as involving the idea of motion. Later Lagrange, in seeking his purely algebraic foundation for the calculus, explicitly said he wanted to free the calculus from fluxions and what he called the "foreign idea" of motion. It is thus striking that Lagrange's *Théorie des fonctions analytiques* (1797) gives a more general version of the kind of argument Maclaurin had given, applying to *any* increasing function that satisfies the geometric inequality expressed in (1). In place of the algebraic inequality (2), Lagrange used the mean-value theorem. [**58**, pp. 238–9] [**38**, pp. 156–158] The similarity of the two arguments does not prove influence, of course, but it certainly demonstrates that Maclaurin's work, which we know Lagrange read (e.g., [**58**, p. 17]), uses the algebra of inequalities in a way consistent with that used by Lagrange and his successors.

Maclaurin's argument exemplifies the way his *Treatise* reconciles the old and the new. The double *reductio ad absurdum* reflects his Archimedean agenda. Treating the area as generated by a moving vertical line, and then searching for the relationship between the area and its fluxion, are Newtonian. Maclaurin did not have a general proof of the Fundamental Theorem in this argument, but relied on an inequality based on the specific properties of a specific function. Nonetheless, he had the precise bounding inequalities for the area function used later by Lagrange, and he used an algebraic inequality proof in a manner that would not disgrace a nineteenth-century analyst.

Inequality-based arguments in the calculus as used by Lagrange and Cauchy owe a lot to the eighteenth-century study of algebraic approximations, and it once seemed to me that this was their origin. But the algebra of inequalities as used in Continental analysis, especially in d'Alembert's pioneering treatment of the tangent as the limit of secants in the article "Différentiel" in the *Encyclopédie*, [**19**] must owe something also to Maclaurin's translation of Archimedean geometry into algebraic dress to justify results in calculus. Throughout the eighteenth century, practitioners of the limit tradition on the Continent use inequalities; a clear line of influence connects Maclaurin's admirer d'Alembert, Simon L'Huilier (who was a foreign member of the Royal Society), the textbook treatment of limits by Lacroix, and, finally, Cauchy. [**38**, pp. 80–87]

Now let us turn to some of Maclaurin's work on series. There is, of course, the Maclaurin series, that is, the Taylor series expanded around zero. This result Maclaurin himself credited to Taylor, and it was known earlier to Newton and Gregory. It was called the Maclaurin series by John F. W. Herschel, Charles Babbage, and George Peacock in 1816 [**51**, pp. 620–21] and by Cauchy in 1823. [**14**, p. 257] Since it was obvious that Maclaurin had not invented it, the attribution shows appreciation by these later mathematicians for the way Maclaurin used the series to study functions. A key application is Maclaurin's characterization of maxima, minima, and points of inflection of an infinitely differentiable function by means of its successive derivatives. When the first derivative at a point is zero, there is a maximum if the second derivative is negative there, a minimum if it is positive. If the second derivative is also zero, one looks at higher derivatives to tell whether the point is a maximum, minimum, or point of inflection. These results can be proved by looking at the Taylor series of the function near the point in question, and arguing on the basis of the inequalities expressed in the definition of maximum and minimum. For instance (in modern [Lagrangian] notation), if $f(x)$ is a maximum, then

$$f(x) > f(x+h) = f(x) + hf'(x) + h^2/2!f''(x) + \cdots, \text{ and} \qquad (4)$$
$$f(x) > f(x-h) = f(x) - hf'(x) + h^2/2!f''(x) - \cdots$$

if h is small. If the derivatives are bounded, and if h is taken sufficiently small so that the term in h dominates the rest, the inequalities (4) can both hold only if $f'(x) = 0$. If $f'(x) = 0$, then the h^2 term dominates, and the inequalities (4) hold only if $f''(x)$ is negative. And so on.

I have traced Cauchy's use of this technique back to Lagrange, and from Lagrange back to Euler. [**38**, pp. 117–118] [**37**, pp. 157–159] [**58**, pp. 235–6] [**29**, Secs. 253–254] But this technique is explicitly worked out in Maclaurin's *Treatise of Fluxions*. Indeed, it appears twice: once in geometric dress in Book I, Chapter IX, and then more algebraically in Book II. [**63**, pp. 694–696] Euler, in the version he gave in his 1755 textbook, [**20**] does not refer to Maclaurin on this point, but then he makes few references in that book at all. Still we might suspect, especially knowing that Stirling told Euler in a letter of 16 April 1738 [**91**] that Maclaurin had some interesting results on series, that Euler would have been particularly interested in looking at Maclaurin's applications of the Taylor series. Certainly Lacroix's praise for Maclaurin's work on series must have taken this set of results into account. [**52**, p. xxvii] Even more important, Lagrange, in unpublished lectures on the calculus from Turin in the 1750's, after giving a very elementary treatment of maxima and minima, referred to volume II of Maclaurin's *Treatise of Fluxions* as the chief source for more information on the subject. [**7**, p. 154] Since Lagrange did not mention Euler in this connection at all, Lagrange could well have not even have seen the *Institutiones calculi differentialis* of 1755 when he made this reference. This Taylor-series approach to maxima and minima (with the Lagrange remainder supplied for the Taylor series) plays a major role in the work of Lagrange, and later in the work of Cauchy. It is because Maclaurin thought of maxima and minima, and of convexity and concavity, in Archimedean geometrical terms that he was led to look at the relevant inequalities, just as the geometry of Archimedes helped Maclaurin formulate some of the inequalities he used to prove his special case of the Fundamental Theorem of Calculus.

b. Ellipsoids. We now turn to work in applied mathematics that constitutes one of Maclaurin's great claims to fame: the gravitational attraction of ellipsoids and the related problem of the shape of the earth. Maclaurin is still often regarded as the creator of the subject of attraction of ellipsoids. [**85**, pp. 175, 374] In the eighteenth century, the topic attracted serious work from d'Alembert, A.-C. Clairaut, Euler, Laplace, Lagrange, Legendre, Poisson, and Gauss. In the twentieth century, Subramanyan Chandrasekhar (later Nobel laureate in physics) devoted an entire chapter of his classic *Ellipsoidal Figures of Equilibrium* to the study of Maclaurin spheroids (figures that arise when homogeneous bodies rotate with uniform angular velocity), the conditions of stability of these spheroids and their harmonic modes of oscillation, and their status as limiting cases of more general figures of equilibrium. Such spheroids are part of the modern study of classical dynamics in the work of scientists like Chandrasekhar, Laurence Rossner, Carl Rosenkilde, and Norman Lebovitz. [**15**, pp. 77–100] Already in 1740 Maclaurin had given a "rigorously exact, geometrical theory" of homogeneous ellipsoids subject to inverse-square gravitational forces, and had shown that an oblate spheroid is a possible figure of equilibrium under Newtonian mutual gravitation, a result with obvious relevance for the shape of the earth. [**39**, p. 172] [**86**, p. xix] [**85**, p. 374]

Of particular importance was Maclaurin's decisive influence on Clairaut. Maclaurin and Clairaut corresponded extensively, and Clairaut's seminal 1743 book *La Figure de la Terre* [**18**] frequently, explicitly, and substantively cites his debts to Maclaurin's work. [**39**, pp. 590–597] A key result, that the attractions of two confocal ellipsoids at a point external to both are proportional to their masses and are in the same direction, was attributed to Maclaurin by d'Alembert, an attribution repeated by Laplace, Lagrange, and Legendre, then by Gauss, who went back to Maclaurin's original paper, and finally by Lord Kelvin, who called it "Maclaurin's splendid theorem." [**15**, p. 38] [**85**, pp. 145, 409] Lagrange began his own memoir on the attraction of ellipsoids by praising Maclaurin's treatment in the prize paper of 1740 as a masterwork of geometry, comparing the beauty and ingenuity of Maclaurin's work with that of Archimedes, [**57**, p. 619] though Lagrange, typically, then treated the problem analytically. Maclaurin's eighteenth- and nineteenth-century successors also credit him with some of the key methods used in studying the equilibrium of fluids, such as the method of balancing columns. [**39**, p. 597] Maclaurin's work on the attraction of ellipsoids shows how his geometric insights fruitfully influenced a subject that later became an analytic one.

c. The Euler-Maclaurin Formula. The Euler-Maclaurin formula expresses the value of definite integrals by means of infinite series whose coefficients involve what are now called the Bernoulli numbers. The formula shows how to use integrals to find the partial sums of series. Maclaurin's version, in modern notation, is:

$$\sum_{h=0}^{\infty} F(a+h) = \int_0^a F(x)\,dx + 1/2F(a) + 1/2F'(a)$$

$$- 1/720F'''(a) + 1/30240F^{(v)}(a) - \cdots$$

[**35**, pp. 84–86]

James Stirling in 1738, congratulating Euler on his publication of that formula, told Euler that Maclaurin had already made it public in the first part of the *Treatise of Fluxions*, which was printed and circulating in Great Britain in 1737. [**47**, p. 88n] [**91**, p. 178] (On this early publication, see also [**63**, pp. iii, 691n]). P. L. Griffiths has argued that this simultaneous discovery rests on De Moivre's work on summing reciprocals, which also involves the so-called Bernoulli numbers. [**40**] [**41**, pp. 16–17] [**25**, p. 19] In any case, Euler and Maclaurin derived the Euler-Maclaurin formula in essentially the same way, from a similar geometric diagram and then by integrating various Taylor series and performing appropriate substitutions to find the coefficients. [**31**] [**32**] [**33**] Maclaurin's approach is no more Archimedean or geometric than Euler's; they are similar and independent. [**63**, pp. 289–293, 672–675] [**35**, pp. 84–93] [**67**] In subsequent work, Euler went on to extend and apply the formula further to many other series, especially in his *Introductio in analysin infinitorum* of 1748 and *Institutiones calculi differentialis* of 1755. [**35**, p. 127] But Maclaurin, like Euler, had applied the formula to solve many problems. [**63**, pp. 676–693] For instance, Maclaurin used it to sum powers of arithmetic progressions and to derive Stirling's formula for factorials. He also derived what is now called the Newton-Cotes numerical integration formula, and obtained what is now called Simpson's rule as a special case. It is possible that his work helped stimulate Euler's later, fuller investigations of these important ideas.

In 1772, Lagrange generalized the Euler-Maclaurin formula, which he obtained as a consequence of his new calculus of operators. [**53**] [**35**, pp. 169, 261] In 1834, Jacobi provided the formula with its remainder term, [**46**, pp. 263, 265] in the same paper in which he first introduced what are now called the Bernoulli polynomials. Jacobi, who called the result simply the Maclaurin summation formula, cited it directly from the *Treatise of Fluxions*. [**46**, p. 263] Later, Karl Pearson used the formula as an important tool in his statistical work, especially in analyzing frequency curves. [**72**, pp. 217, 262]

The Euler-Maclaurin formula, then, is an important result in the mainstream of mathematics, with many applications, for which Maclaurin, both in the eighteenth century and later on, has rightly shared the credit.

d. Elliptic Integrals. Some integrals (Maclaurin used the Newtonian term "fluents"), are algebraic functions, Maclaurin observed. Others are not, but some of these can be reduced to finding circular arcs, others to finding logarithms. By analogy, Maclaurin suggested, perhaps a large class of integrals could be studied by being reduced to finding the length of an elliptical or hyperbolic arc. [**63**, p. 652] By means of clever geometric transformations, Maclaurin was able to reduce the integral that represented the length of a hyperbolic arc to a 'nice' form. Then, by algebraic manipulation, he could reduce some previously intractable integrals to that same form. His work was translated into analysis by d'Alembert and then generalized by Euler. [**13**, p. 846] [**23**] [**27**, p. 526] [**28**, p. 258] In 1764, Euler found a much more elegant, general, and analytic version of this approach, and worked out many more examples, but cited the work of Maclaurin and d'Alembert as the source of his investigation. A.-M. Legendre, the key figure in the eighteenth-century history of elliptic integrals, credited Euler with seeing that, by the aid of a good notation, arcs of ellipses and other transcendental curves could be as generally used in integration as circular and logarithmic arcs. [**45**, p. 139] Legendre was, of course, right that "elliptic integrals" encompass a wide range of examples; this was exactly Maclaurin's point. Thus, although his successors accomplished more, Maclaurin helped initiate a very important investigation and was the first to appreciate its generality. Maclaurin's geometric insight, applied to a problem in analysis, again brought him to a discovery.

7. OTHER EXAMPLES OF MACLAURIN'S MATHEMATICAL INFLUENCE. The foregoing examples provide evidence of direct influence of the *Treatise of Fluxions* on Continental mathematics. There is much more. For instance, Lacroix, in his treatment of integrals by the method of partial fractions, called it "the method of Maclaurin, followed by Euler." [**52**, Vol. II, p. 10] [**63**, pp. 634–644] Of interest too is Maclaurin's clear understanding of the use of limits in founding the calculus, especially in the light of his likely influence on d'Alembert's treatment of the foundations of the calculus by means of limits in the *Encyclopédie*, which in turn influenced the subsequent use of limits by L'Huilier, Lacroix, and Cauchy, [**38**, chapter 3] (and on Lagrange's acceptance of the limit approach in his early work in the 1750's). [**7**] Although the largest part of Maclaurin's reply to Berkeley was the extensive proof of results in calculus using Greek methods, he was willing to explain important concepts using limits also. In particular, Maclaurin wrote, "As the tangent of an arch [arc] is the right line that limits the position of all the secants that can pass through the point of contact... though strictly speaking it be no secant; so a ratio may limit the variable ratios of the increments, though it cannot be said to be the ratio of any real increments." [**63**, p. 423] Maclaurin's statement answers Berkeley's chief objection—that the increment in a function's value is first treated as non-zero, then as zero, when one calculates the limit of the ratio of increments or finds the tangent to a curve. Maclaurin's statement is in the tradition of Newton's *Principia* (Book I, Scholium to Lemma XI), but is in a form much closer to the later work of d'Alembert on secants and tangents. [**20**] Maclaurin pointed out that most of the propositions of the calculus that he could prove by means of geometry "may be *briefly* demonstrated by this method [of limits]." [**63**, p. 87, my italics]

In addition, Maclaurin had considerable influence in Britain, on mathematicians like John Landen (whose work on series was praised by Lagrange), Robert Woodhouse (who sparked the new British interest in Continental work about 1800), and on Edward Waring and Thomas Simpson, whose names are attached to results well known today. [**42**] Going beyond the calculus, Maclaurin's purely geometric treatises were read and used by French geometers of the stature of Chasles and Poncelet. [**90**, p. 145] Thus, though Maclaurin may not have been the towering figure Euler was, he was clearly a significant and respected mathematician, and the *Treatise of Fluxions* was far more than an unread tome whose weight served solely to crush Bishop Berkeley.

8. WHY A TREATISE OF FLUXIONS? The *Treatise of Fluxions* was not really intended as a reply to Berkeley. Maclaurin could have refuted Berkeley with a pamphlet. It was not a student handbook either; this work is far from elementary. Nor was it merely written to glory in Greek geometry. Maclaurin wrote several works on geometry per se. But he was no antiquarian. Instead, the *Treatise of Fluxions* was the major outlet for Maclaurin's solution of significant research problems in the field we now call analysis. Geometry, as the examples I gave illustrate, was for Maclaurin a source of motivation, of insight, and of problem-solving power, as well as being his model of rigor.

For Maclaurin, rigor was not an end in itself, or a goal pursued for purely philosophical reasons. It was motivated by his research goals in analysis. For instance, Maclaurin developed his theory of maxima, minima, points of inflection, convexity and concavity, orders of contact, etc., because he wanted to study curves of all types, including those that cross over themselves, loop around and are tangent to themselves, and so on. He needed a sophisticated theory to characterize the special points of such curves. Again, in problems as different as studying the attraction of ellipsoids and evaluating integrals approximately, he needed to use infinite series and know how close he was to their sum. Thus, rigor, to Maclaurin, was not merely a tool to defend Newton's calculus against Berkeley—though it was that—nor just a response to the needs of a professor to present his students a finished subject—though it may have been that as well. In many examples, Maclaurin's rigor serves the needs of his research.

Moreover, the *Treatise of Fluxions* contains a wealth of applications of fluxions, from standard physical problems such as curves of quickest descent to mathematical problems like the summation of power series—in the context of which, incidentally, Maclaurin gave what may be the earliest clear definition of the sum of an infinite series: "There are progressions of fractions which may be continued at pleasure, and yet the sum of the terms be always less than a certain finite number. If the difference betwixt their sum and this number decrease in such a manner, that by continuing the progression it may become less than any fraction how small soever that can be assigned, this number is the *limit of the sum of the progression*, and is what is understood by the value of the progression when it is supposed to be continued indefinitely." [**63**, p. 289] Thus, though eighteenth-century Continental mathematicians did not care passionately about foundations, [**38**, pp. 18–24] they could still appreciate the *Treatise of Fluxions* because they could mine it for results and techniques.

9. WHY THE TRADITIONAL VIEW? If the reader is convinced by now that the traditional view is wrong, that Maclaurin's Treatise did not mark the end of the Newtonian tradition, and that not all of modern analysis stems solely from the work of Leibniz and his school, the question arises, how did that traditional view come to be, and why it has been so persistent?

Perhaps the traditional view could be explained as follows. Consider the approach to mathematics associated with Descartes: symbolic power, not debates over foundations; problem-solving power, not axioms or long proofs. The Cartesian approach to mathematics is clearly reflected in the work and in the rhetoric of Leibniz, Johann Bernoulli, Euler, Lagrange—especially in the historical prefaces to his influential works—and even Cauchy. These men, the giants of their time, are linked in a continuous chain of teachers, close colleagues, and students. Some topics, like partial differential equations and the calculus of variations, were developed mostly on the Continent. Moreover, the Newton-Leibniz controversy helped drive English and Continental mathematicians apart. Thus the Continental tradition can be viewed as self-contained, and the outsider sees no need for eighteenth-century Continental mathematicians to struggle through 750 pages of a *Treatise of Fluxions*, which is at best in the Newtonian notation and at worst in the language of Greek geometry. Lagrange's well-known boast that his *Analytical Mechanics* [**55**] had (and needed) no diagrams, thus opposing analysis to geometry at the latter's expense, reinforced these tendencies and enshrined them in historical discourse. But the explanation we have just given does not suffice to explain the strength, and persistence into the twentieth century, of the standard interpretation. The traditional view of Maclaurin's lack of importance has been reinforced by some other historiographical tendencies that deserve our critical attention.

The traditional picture of Maclaurin's *Treatise of Fluxions* radically separates his work on foundations, which it regards as geometric, sterile, and antiquarian, from his important individual results, which often are mentioned in histories of mathematics but are treated in isolation from the purpose of the *Treatise*, in isolation from one another, and in isolation from Maclaurin's overall approach to mathematics. Strangely, both externalist and internalist historians, each for different reasons, have reinforced this picture.

For instance, in the English-speaking world, viewing the *Treatise* as only about Maclaurin's foundation for the calculus, and thus as a dead end, has been perpetuated by the "decline of science in England" school of the history of eighteenth-century science, stemming from such early nineteenth-century figures as John Playfair, and, especially, Charles Babbage. [**77**] [**2**] [**4**] Babbage felt strongly about this because he was a founder of the Cambridge Analytical Society, which fought to introduce Continental analysis into Cambridge in the early nineteenth century. This group had an incentive to exaggerate the superiority of Continental mathematics and downgrade the British, as is exemplified by their oft-quoted remark that the principles of "pure d-ism" should replace what they called the "dot-age" of the University. [**5**, ch. 7] [**10**, p. 274] The pun, playing on the Leibnizian and Newtonian notation in calculus, may be found in [**2**, p. 26]. These views continued to be used in the attempt by Babbage and others to reform the Royal Society and to increase public support for British science.

It is both amusing and symptomatic of the misunderstanding of Maclaurin's influence that Lacroix's one-volume treatise on the calculus of 1802, [**50**] translated into English by the Cambridge Analytical Society with added notes on the method of series of Lagrange, [**51**] was treated by them, and has been considered since, as a purely "Continental" work. But Lacroix's short treatise was based on the concept of limit, which was Newtonian, elaborated by Maclaurin, adapted by d'Alembert and L'Huilier, and finally systematized by Lacroix. [**38**, pp. 81–86] Moreover, the translators' notes by Babbage, Herschel, and Peacock supplement the text by studying functions by their Taylor series, thus using the approach that Lacroix himself, in his multi-volume treatise of 1810, had attributed to Maclaurin. This is, of course, not to deny the overwhelming importance of the contributions of Euler and Lagrange, both to the mathematics taught by the Analytical Society and to that included by Lacroix in his 1802 book, nor to deny the Analytical Society's emphasis on a more abstract and formal concept of function. But all the same, Babbage, Herschel, and Peacock were teaching some of Maclaurin's ideas without realizing this.

In any case, the views expressed by Babbage and others have strongly influenced Cambridge-oriented writers like W. W. Rouse Ball, who said that the history of eighteenth-century English mathematics "leads nowhere." [**5**, p. 98] H. W. Turnbull, though he wrote sympathetically about Maclaurin's mathematics on one occasion, [**88**] blamed Maclaurin on another occasion for the decline: "When Maclaurin produced a great geometrical work on fluxions, the scale was so heavily loaded that it diverted England from Continental habits of thought. During the remainder of the century, British mathematics were relatively undistinguished." [**89**, p. 115]

Historians of Scottish thought, working from their central concerns, have also unintentionally contributed to the standard picture. George Elder Davie, arguing from social context to a judgment of Maclaurin's mathematics, held that the Scots, unlike the English, had an anti-specialist intellectual tradition, based in philosophy, and emphasizing "cultural and liberal values." Wishing to place Maclaurin in this context, Davie stressed what he called Maclaurin's "mathematical Hellenism," [**24**, p. 112] and was thus led to circumscribe the achievement of the *Treatise of Fluxions* as having based the calculus "on the Euclidean foundations provided by [Robert] Simson," [**24**, p. 111] who had made the study of the writings of the classical Greek geometers the "national norm" in Scotland. The "Maclaurin is a geometer" interpretation among Scottish historians has been further reinforced by a debate in 1838 over who would fill the Edinburgh chair in mathematics. Phillip Kelland, a candidate from Cambridge, was seen as the champion of Continental analysis, while the partisans of Duncan Gregory argued for a more geometrical approach. Wishing to enlist the entire Scottish geometric tradition on the side of Gregory, Sir William Hamilton wrote, "The great Scottish mathematicians, . . . even Maclaurin, were decidedly averse from the application of the mechanical procedures of algebra." [**24**, p. 155] Though Kelland eventually won the chair, the dispute helped spread the view that Maclaurin had been hostile to analysis. More recently, Richard Olson has characterized Scottish mathematics after Maclaurin as having been conditioned by Scottish common-sense philosophy to be geometric in the extreme. [**70**, pp. 4, 15] [**71**, p. 29] But in emphasizing Maclaurin's influence on this development, Olson, like Davie, has overstated the degree to which Maclaurin's approach was geometric.

By contrast, consider internalist historians. The treatment of Maclaurin's results as isolated reflects what Herbert Butterfield called the Whig approach to history, viewing the development of eighteenth-century mathematics as a linear progression toward what we value today, the collection of results and techniques which make up classical analysis. Thus, mathematicians writing about the history of this period, from Moritz Cantor in the nineteenth century to Hermann Goldstine and Morris Kline in the twentieth, tell us what Maclaurin did with specific results, some named after him, for which they have mined the *Treatise of Fluxions*. [**13**, pp. 655–63] [**35**, pp. 126ff, 167–8] [**49**, pp. 522–3, 452, 442] They either neglect the apparently fruitless work on foundations, or, viewing it as geometric, see it as a step backward. It is of course true that many Continental mathematicians used Maclaurin's results without accepting the geometrical and Newtonian insights that Maclaurin used to produce them. But without those points of view, Maclaurin would not have produced those results.

Both externalist and internalist historians, then, have treated Maclaurin's work in the same way: as a throwback to the Greeks, with a few good results that happen to be in there somewhat like currants in a scone. Further, the fact that Maclaurin's book, especially its first hundred pages, is very hard to read, especially for readers schooled in modern analysis, has encouraged historians who focus on foundations to read only the introductory parts. The fact that there is so much material has encouraged those interested in results to look only at the sections of interest to them. And the fact that the first volume is so overwhelmingly geometric serves to reinforce the traditional picture once again whenever anybody opens the *Treatise*. The recent Ph.D. dissertation by Erik Sageng [**78**] is the first example of a modern scholarly study of Maclaurin's *Treatise* in any depth. The standard picture has not yet been seriously challenged in print.

10. SOME FINAL REFLECTIONS. Maclaurin's work had Continental influence, but with an important exception—his geometric foundation for the calculus. Mastering this is a major effort, and I know of no evidence that any eighteenth-century Continental mathematician actually did so. Lagrange perhaps came the closest. In the introduction to his *Théorie des fonctions analytiques*, Lagrange could say only, Maclaurin did a good job basing calculus on Greek geometry, so it can be done, but it is very hard. [**58**, p. 17] In an unpublished draft of this introduction, Lagrange said more pointedly: "I appeal to the evidence of all those with the courage to read the learned treatise of Maclaurin and with enough knowledge to understand it: have they, finally, had their doubts cleared up and their spirit satisfied?" [**73**, p. 30]

Something else may have blunted people's views of the mathematical quality of Maclaurin's *Treatise*. The way the book is constructed partly reflects the Scottish intellectual milieu. The Enlightenment in Britain, compared with that on the Continent, was marked less by violent contrast and breaks with the past than by a spirit of bridging and evolution. [**75**, pp. 7–8, 15] Similarly, Scottish reformers operated less by revolution than by the refurbishment of existing institutions. [**16**, p. 8] These trends are consistent with the two-fold character of the *Treatise of Fluxions*: a synthesis of the old and the new, of geometry and algebra, of foundations and of new results, a refurbishment of Newtonian fluxions to deal with more modern problems. This contrasts with the explicitly revolutionary philosophy of mathematics of Descartes and Leibniz, and thus with the spirit of the *mathématicien* of the eighteenth century on the Continent.

Of course Scotland was not unmarked by the conflicts of the century. During the Jacobite rebellion in 1745, Maclaurin took a major role in fortifying Edinburgh against the forces of Bonnie Prince Charlie. When the city was surrendered to the rebels, Maclaurin fled to York. Before his return, he became ill, and apparently never really recovered. He briefly resumed teaching, but died in 1746 at the relatively young age of forty-eight. Nonetheless, the Newtonian tradition in the calculus was not a dead end. Maclaurin in his lifetime, and his *Treatise of Fluxions* throughout the century, transmitted an expanded and improved Newtonian calculus to Continental analysts. And Maclaurin's geometric insight helped him advance analytic subjects.

We conclude with the words of an eighteenth-century Continental mathematician whose achievements owe much to Maclaurin's work. [**39**, pp. 172, 412–425, 590–597] The quotation [**66**, p. 350] illustrates Maclaurin's role in transmitting the Newtonian tradition to the Continent, the respect in which he was held, and the eighteenth-century social context essential to understanding the fate of his work.

In 1741, Alexis-Claude Clairaut wrote to Colin Maclaurin, "If Edinburgh is, as you say, one of the farthest corners of the world, you are bringing it closer by the number of beautiful discoveries you have made."

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