On the Sums $\sum_{k=-\infty}^{\infty}(4k+1)^{-n}$

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1. INTRODUCTION. Euler proved that the sums

$$\zeta(n) := \sum_{k=1}^{\infty} \frac{1}{k^n} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots \tag{1}$$

for even $n \geq 2$ and

$$L(n, \chi_4) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} = 1 - \frac{1}{3^k} + \frac{1}{5^k} - \frac{1}{7^k} + - \cdots \tag{2}$$

for odd $n \geq 3$ are rational multiples of π^n. (See Ayoub [**3**] for more on Euler's work on these and related sums.) This result can be stated equivalently as follows:

$$S(n) := \sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} \tag{3}$$

is a rational multiple of π^n for all integers $n = 2, 3, 4, \ldots$ for which the sum converges absolutely. This is equivalent to (1) and (2) because

$$S(n) = \begin{cases} (1 - 2^{-n})\,\zeta(n), & \text{if } n \text{ is even;} \\ L(n, \chi_4), & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

For future reference we tabulate for $n \leq 10$ the rational numbers $\pi^{-n}S(n)$, as well as $\pi^{-n}\zeta(n)$ for n even. To this end, we define

$$S(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}. \tag{5}$$

n	1	2	3	4	5	6
$\pi^{-n}S(n)$	1/4	1/8	1/32	1/96	5/1536	1/960
$\pi^{-n}\zeta(n)$		1/6		1/90		1/945

n	7	8	9	10
$\pi^{-n}S(n)$	61/184320	17/161280	277/8257536	31/2903040
$\pi^{-n}\zeta(n)$		1/9450		1/93555

One standard proof of the rationality of $\pi^{-n}S(n)$ is via the generating function

$$G(z) := \sum_{n=1}^{\infty} S(n) z^n = \sum_{n=1}^{\infty} \left(\sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} \right) z^n, \tag{6}$$

in which the inner sum is taken in the order $k = 0, -1, 1, -2, 2, -3, 3, \ldots$. The power series representing $G(z)$ converges for all z such that $|z| < 1$. Since the sum of the terms with $n > 1$ converges absolutely, we may interchange the order of summation in (6), obtaining

$$\sum_{k=-\infty}^{\infty}\left(\sum_{n=1}^{\infty}\frac{z^n}{(4k+1)^n}\right) = z\left(\frac{1}{1-z} - \frac{1}{3+z} + \frac{1}{5-z} - \frac{1}{7+z} + - \cdots\right). \tag{7}$$

Comparing the latter sum with the partial-fraction expansions for the tangent and cosecant, we find that

$$G(z) = \frac{\pi z}{4}\left(\sec\frac{\pi z}{2} + \tan\frac{\pi z}{2}\right). \tag{8}$$

Since the Taylor series of $z(\sec(z) + \tan(z))$ about $z = 0$ has rational coefficients, it follows from (8) that for each n the coefficient $S(n)$ of z^n in $G(z)$ is a rational multiple of π^n.

This also lets us compute the rational numbers $\pi^{-n}S(n)$. The numbers for even and odd n come from the even and odd parts $(\pi z/4)\tan(\pi z/2)$ and $(\pi z/4)\sec(\pi z/2)$ of $G(z)$. In the literature these are usually treated separately, and their coefficients are expressed in terms of Bernoulli and Euler numbers B_n and E_{n-1}, respectively, which are related to $S(n)$ by the formulas

$$B_{2m} = \frac{(-1)^{m-1}}{2^{2m-1}\pi^{2m}}\zeta(2m) = (-1)^{m-1}\frac{2}{2^{2m}-1}\pi^{-2m}S(2m), \tag{9}$$

$$E_{2m} = \frac{(-1)^m}{(2m)!\,2^{2m+2}}\pi^{-(2m+1)}S(2m+1). \tag{10}$$

(The formulas (9) and (10) specify only the Bernoulli and Euler numbers of positive even order. The odd-order ones all vanish except for $B_1 = -1/2$. This definition, as well as the initial value $B_0 = 1$, is needed for another important use of these numbers; namely, the formula

$$\sum_{k=1}^{N} k^{n-1} = \frac{1}{n}\sum_{m=1}^{n}\binom{n}{m}B_m N^{n-m} \tag{11}$$

for summing powers of the first N integers.) A short table of Euler and Bernoulli numbers of even order follows:

B_0	E_0	B_2	E_2	B_4	E_4	B_6	E_6	B_8	E_8	B_{10}
1	1	1/6	−1	−1/30	5	1/42	−61	−1/30	1385	5/66

It is known that all the Euler numbers are integers; we shall give a combinatorial interpretation of their absolute values $(-1)^m E_{2m}$ at the end of this paper.

The sums $S(n)$ continue to attract considerable interest in mathematical disciplines ranging from Fourier analysis to number theory. Euler's formulas predate the year 1750, and over the years since Euler's time, many new proofs of the rationality of $S(n)/\pi^n$ have been given. But it was only recently that Calabi[1] found a proof using only the formula for change of variables of multiple integrals. For instance, to prove

[1]The only paper by Calabi that contains this proof is one coauthored with Beukers and Kolk [**4**]. Nevertheless, the proof is due to Calabi alone; Beukers and Kolk's contribution to [**4**] concerns other aspects of that paper. They were shown this proof by Don Zagier, who also introduced me to it. Note that Kalman [**6**] also writes that he first learned of this proof in a talk by Zagier.

that $\zeta(2) = \pi^2/6$, or equivalently that $S(n) = \pi^2/8$ for $n = 2$, Calabi argues as follows: Write each term $(2k+1)^{-2}$ of the infinite sum in (1) as $\int_0^1\int_0^1 (xy)^{2k} dx\,dy$, and thus rewrite that sum as

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{k=0}^{\infty} \int_0^1 \int_0^1 (xy)^{2k}\, dx\, dy \tag{12}$$

$$= \int_0^1 \int_0^1 \left(\sum_{k=0}^{\infty} (xy)^{2k} \right) dx\, dy \tag{13}$$

$$= \int_0^1 \int_0^1 \frac{dx\, dy}{1-(xy)^2}. \tag{14}$$

(The interchange of sum and integral in (13) is readily justified, for instance by observing the positivity of each integrand.) Then perform the change of variable

$$x = \frac{\sin u}{\cos v}, \qquad y = \frac{\sin v}{\cos u}, \tag{15}$$

under which the integrand in (14) miraculously transforms to $1\, du\, dv$, and the region of integration $\{(x, y) \in \mathbf{R}^2 : 0 < x, y < 1\}$ is the one-to-one image of the isosceles right triangle $\{(u, v) \in \mathbf{R}^2 : u, v > 0,\ u + v < \pi/2\}$ (these assertions will be proved in greater generality later). Thus the value of the integral (14) is just the area $\pi^2/8$ of that triangle, Q.E.D.

In general, Calabi writes $S(n)$ as a definite integral over the n-cube $(0, 1)^n$ and transforms it to the integral representing the volume of the n-dimensional polytope

$$\Pi_n := \left\{ (u_1, u_2, \ldots, u_n) \in \mathbf{R}^n : u_i > 0,\ u_i + u_{i+1} < \frac{\pi}{2}\ (1 \le i \le n) \right\}. \tag{16}$$

Note that the u_i are indexed cyclically mod n, so

$$u_{n+1} := u_1, \tag{17}$$

here and henceforth. Since all the coordinates of each vertex of Π_n are rational multiples of π, the volume of Π_n must be a rational multiple of π^n. It turns out that there is another way to interpret $S(n)$ as the volume of Π_n; this alternative approach requires more analytical machinery, but better explains the appearance of the sum $S(n)$. We shall also give combinatorial interpretations of $S(n)$ by relating this volume to the enumeration of alternating permutations of $n + 1$ letters, and to the enumeration of cyclically alternating permutations of n letters when n is even. This leads to known formulas involving B_n and E_{n-1}. Our treatment via Π_n and another polytope relates those permutation counts directly to $S(n)$ without the intervention of Bernoulli and Euler numbers or their generating functions.

To keep this paper self-contained, we first review Calabi's transformation [**4**] that proves $S(n) = \mathrm{Vol}(\Pi_n)$. This elegant proof remains little-known (except possibly for the case $n = 2$ shown earlier, which was the second of Kalman's six proofs of $\zeta(2) = \pi^2/6$ [**6**]), and deserves wider exposure. We then give the analytic interpretation of both $S(n)$ and $\mathrm{Vol}(\Pi_n)$ as the trace of T^n for a certain compact self-adjoint operator T on the Hilbert space $\mathrm{L}^2(0, \pi/2)$. (I thank the referee for bringing to my attention a similar evaluation of an integral studied by Kubilius [**7**].) Finally, we relate $S(n)$ and polytope volumes to the enumeration of alternating and cyclically alternating permutations.

2. EVALUATING $S(n)$ BY CHANGE OF VARIABLES. Following Calabi [**4**], we generalize (15) to the n-variable transformation

$$x_1 = \frac{\sin u_1}{\cos u_2}, \quad x_2 = \frac{\sin u_2}{\cos u_3}, \quad \ldots, \quad x_{n-1} = \frac{\sin u_{n-1}}{\cos u_n}, \quad x_n = \frac{\sin u_n}{\cos u_1}, \tag{18}$$

some of whose properties are established in the following two lemmas.

Lemma 1. *The Jacobian determinant of the transformation (18) is*

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} = 1 \pm (x_1 x_2 \cdots x_n)^2, \tag{19}$$

the sign $-$ or $+$ chosen according to whether n is even or odd.

Proof. The partial derivatives $\partial x_i / \partial u_j$ are given by

$$\frac{\partial x_i}{\partial u_j} = \begin{cases} (\cos u_i)/(\cos u_{i+1}), & \text{if } j = i; \\ (\sin u_i \sin u_{i+1})/(\cos^2 u_{i+1}), & \text{if } j \equiv i + 1 \bmod n; \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Thus the expansion of the Jacobian determinant has only two nonzero terms, one coming from the principal diagonal $j = i$, one from the cyclic off-diagonal $j \equiv i + 1 \bmod n$. The product of the principal diagonal entries $(\cos u_i)/(\cos u_{i+1})$ simplifies to 1, and always appears with coefficient $+1$. The product of the off-diagonal entries is

$$\prod_{i=1}^{n} \frac{\sin u_i \sin u_{i+1}}{\cos^2 u_{i+1}} = \prod_{i=1}^{n} x_i^2, \tag{21}$$

and appears with coefficient $(-1)^{n-1}$, the sign of an n-cycle in the permutation group. Therefore the Jacobian determinant is given by (19), as claimed. ∎

Lemma 2. *The transformation (18) maps the polytope Π_n one-to-one to the open unit cube $(0, 1)^n$.*

Proof. Certainly if u_i, u_{i+1} are positive and $u_i + u_{i+1} < \pi/2$ then

$$0 < x_i = \frac{\sin u_i}{\cos u_{i+1}} < \frac{\sin(\pi/2 - u_{i+1})}{\cos u_{i+1}} = 1. \tag{22}$$

Likewise we see that, given arbitrary x_i in $(0, 1)$, any $(u_1, \ldots, u_n)$ in $(0, \pi/2)^n$ satisfying (18) must lie in Π_n. It remains to show that there exists a unique such solution $(u_1, \ldots, u_n)$. Rewrite the equations (18) as

$$u_1 = f_{x_1}(u_2), \quad u_2 = f_{x_2}(u_3), \quad \ldots, \quad u_{n-1} = f_{x_{n-1}}(u_n), \quad u_n = f_{x_n}(u_1), \tag{23}$$

where f_x $(0 < x < 1)$ is the map

$$f_x(u) := \sin^{-1}(x \cos u) \tag{24}$$

of the interval $(0, \pi/2)$ to itself. Since

$$\left| \frac{d}{du} f_x(u) \right| = \left| -\frac{x \sin u}{\sqrt{1 - x^2 \cos^2 u}} \right| < \frac{x \sin u}{\sqrt{x^2 - x^2 \cos^2 u}} = 1, \tag{25}$$

each f_{x_i} is a contraction map of the interval $[0, \pi/2]$; hence so is their composite, which thus has a unique fixed point in that interval. This point cannot be at either endpoint, because $f_x(\pi/2) = 0$ and $f_x(0) = \sin^{-1}(x)$ belongs to $(0, \pi/2)$ for all x in the interval $(0, 1)$. Therefore $f_{x_1} \circ f_{x_2} \circ \cdots \circ f_{x_{n-1}} \circ f_{x_n}$ has a unique fixed point in $(0, \pi/2)$, and the simultaneous equations (23) have a unique solution $(u_1, \ldots, u_n)$ for each $(x_1, \ldots, x_n)$, as claimed. ■

Thus we see that the volume of the polytope Π_n is

$$\underset{\substack{u_i>0 \\ u_i+u_{i+1}<\pi/2}}{\int \cdots \int} 1\, du_1 \cdots du_n = \int_0^1 \cdots \int_0^1 \frac{dx_1 \cdots dx_n}{1 \pm (x_1 \cdots x_n)^2} \tag{26}$$

$$= \int_0^1 \cdots \int_0^1 \sum_{k=0}^{\infty} (-1)^{nk} (x_1 \cdots x_n)^{2k}\, dx_1 \cdots dx_n. \tag{27}$$

Note that when n is even, the second integral in (26) is improper due to the singularity at $(x_1, \ldots, x_n) = (1, \ldots, 1)$, but the change of variable remains valid because the integrand is everywhere positive. By absolute convergence we may now interchange the sum and multiple integral in (27), obtaining

$$\sum_{k=0}^{\infty} (-1)^{nk} \int_0^1 \cdots \int_0^1 (x_1 \cdots x_n)^{2k}\, dx_1 \cdots dx_n = \sum_{k=0}^{\infty} \frac{(-1)^{nk}}{(2k+1)^n} = S(n). \tag{28}$$

We have thus proved:

Theorem 1. *The volume of the polytope Π_n is $S(n)$ for all $n \geq 2$.*

Corollary 1.1. *$S(n)$ is a rational multiple of π^n for all $n \geq 2$.*

Indeed, the volume of Π_n is $(\pi/2)^n$ times the volume of the polytope

$$\frac{2}{\pi}\Pi_n = \left\{(v_1, v_2, \ldots, v_n) \in \mathbf{R}^n : v_i > 0,\ v_i + v_{i+1} < 1\ (1 \leq i \leq n)\right\}, \tag{29}$$

which is clearly a rational number. ■

Remark. These results hold also when $n = 1$, though a bit more justification is needed because the alternating sum (2) no longer converges absolutely when $n = 1$. In that case Π_n reduces to the line segment $0 < u_1 < \pi/4$, and the change of variable (18) simplifies to $x_1 = \tan u_1$, so we recover the evaluation of

$$S(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots \tag{30}$$

as the arctangent integral

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(1) = \frac{\pi}{4}. \tag{31}$$

3. RELATING $S(n)$ TO Π_n VIA LINEAR OPERATORS.

For u, v in $(0, \pi/2)$, define $K_1(u, v)$ to be the characteristic function of the isosceles right triangle $\{(u, v) \in \mathbf{R}^2 : u, v > 0,\ u + v < \pi/2\}$ encountered in the introduction, that is,

$$K_1(u, v) = \begin{cases} 1, & \text{if } u + v < \pi/2; \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

We may then rewrite the volume of the polytope Π_n as

$$\int_0^{\pi/2} \cdots \int_0^{\pi/2} \prod_{i=1}^{n} K_1(u_i, u_{i+1})\, du_1\, du_2 \cdots du_n \tag{33}$$

$$= \int_0^{\pi/2} K_n(u, u)\, du \tag{34}$$

(recall that $u_{n+1} = u_1$), where

$$K_n(u, v) = \int_0^{\pi/2} K_1(u, u_1) K_{n-1}(u_1, v)\, du_1 \tag{35}$$

$$= \int_0^{\pi/2} \cdots \int_0^{\pi/2} K_1(u, u_1) \cdot \prod_{i=1}^{n-2} K_1(u_i, u_{i+1}) \cdot K_1(u_{n-1}, v)\, du_1\, du_2 \cdots du_{n-1} \tag{36}$$

(the equivalence of these two formulas, and thus also of (33) with (34), is easily established by induction on n). We now interpret K_n and the integral (34) in terms of linear operators on $\mathrm{L}^2(0, \pi/2)$. Let T be the linear operator with kernel $K_1(\cdot, \cdot)$ on $\mathrm{L}^2(0, \pi/2)$:

$$(Tf)(v) = \int_0^{\pi/2} f(u) K_1(u, v)\, du = \int_0^{(\pi/2)-v} f(u)\, du. \tag{37}$$

Then we see from either (35) or (36) that $K_n(\cdot, \cdot)$ is the kernel of T^n:

$$(T^n f)(v) = \int_0^{\pi/2} f(u) K_n(u, v)\, du. \tag{38}$$

The next lemma gives the spectral decomposition of this operator T, and thus also of its powers T^n.

Lemma 3. *The transformation T is a compact, self-adjoint operator on* $\mathrm{L}^2(0, \pi/2)$. *Its eigenvalues, each of multiplicity one, are* $1/(4k + 1)$ $(k \in \mathbf{Z})$*; the corresponding orthogonal eigenfunctions are* $\cos((4k + 1)u)$.

Proof. T is self-adjoint because its kernel is symmetric: $K_1(u, v) = K_1(v, u)$. Compactness can be obtained from general principles (the functions in the image $\{Tf : \|f\| \le 1\}$ of the unit ball are a uniformly continuous family), or from the determination of T's eigenvalues. Let λ, then, be an eigenvalue of T, and f a corresponding eigenfunction, so

$$\int_0^{(\pi/2)-v} f(u)\, du = \lambda f(v) \tag{39}$$

for almost all v in $(0, \pi/2)$. Note that λ may not vanish, because then (39) would give $f = 0$ in $L^2(0, \pi/2)$. So we may divide (39) by λ, and use the left-hand side to realize f as a continuous function, and again to show that it is differentiable, with

$$f\left(\frac{\pi}{2} - v\right) = -\lambda f'(v) \tag{40}$$

for all v in $(0, \pi/2)$. Differentiating (40) once more, we find that

$$\lambda^2 f''(v) = -\lambda \frac{d}{dv} f\left(\frac{\pi}{2} - v\right) = \lambda f'\left(\frac{\pi}{2} - v\right) = -f(v), \tag{41}$$

whence

$$f(v) = A \cos\frac{v}{\lambda} + B \sin\frac{v}{\lambda} \tag{42}$$

for some constants A and B. But from (39) we see that $f(\pi/2) = 0$; substituting this into (40) we obtain $f'(0) = 0$, so $B = 0$. The condition $f(\pi/2) = 0$ then becomes $\cos(\pi/2\lambda) = 0$ and forces λ to be the reciprocal of an an odd integer, say $\lambda = 1/m$. Now λ and $-\lambda$ would both give rise to the same function $f(v) = \cos(v/\lambda)$, but only one of them may be its eigenvalue. To choose the sign, take $v = 0$ in (39), finding that

$$\lambda = \lambda f(0) = \int_0^{\pi/2} \cos(mu)\, du = \frac{1}{m} \sin\frac{m\pi}{2} = \frac{(-1)^{(m-1)/2}}{m}, \tag{43}$$

or equivalently that $\lambda = 1/m$ with $m \equiv 1 \bmod 4$. We then easily confirm that each of these $\lambda = 1, -1/3, 1/5, -1/7, \ldots$ and the corresponding $f(v) = \cos(v/\lambda)$ satisfy (39) for all v in $(0, \pi/2)$, completing the proof of the lemma. Alternatively, having obtained the eigenfunctions $\cos(u)$, $\cos(3u)$, $\cos(5u), \ldots,$ we need only invoke the theory of Fourier series to show that these form an orthogonal basis for $L^2(0, \pi/2)$ and then verify that they satisfy (39) with the appropriate λ. ∎

Corollary 3.1. *The transformation T^n is a compact, self-adjoint operator on $L^2(0, \pi/2)$. Its eigenvalues, each of multiplicity one, are $1/(4k+1)^n$ ($k \in \mathbf{Z}$), with corresponding eigenfunctions $\cos((4k+1)u)$.*

In particular, once $n \geq 2$, the sum of the eigenvalues of T^n converges absolutely, so T^n is of *trace class* (see Dunford and Schwartz [**5**, XI.8.49, pp.1086–7]), and its *trace* is the sum

$$\sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} = S(n) \tag{44}$$

of these eigenvalues. But it is known that the trace of a trace-class operator is the integral of its kernel over the diagonal (a continuous analog of the fact that the trace of a matrix is the sum of its diagonal entries [**5**, XI.8.49(c), pp. 1086–7]). Thus the trace $S(n)$ of T^n is given by the integral (34), i.e., by the volume of Π_n. So we have an alternative proof of Theorem 1, in which the power sum (44) appears naturally, without separating the cases of even and odd n.

For future use, we give the orthogonal expansion of an arbitrary L^2 function and a consequence of Corollary 3.1:

Corollary 3.2. *For any f in $L^2(0, \pi/2)$ we have*

$$f = \sum_{k=-\infty}^{\infty} f_k \cos((4k+1)u), \tag{45}$$

with coefficients f_k given by

$$\frac{4}{\pi}\int_0^{\pi/2} f(u)\cos((4k+1)u)\,du. \tag{46}$$

For each $n \geq 0$ we have

$$\int_0^{\pi/2} f(u)\,(T^n f)(u)\,du = \frac{\pi}{4}\sum_{k=-\infty}^{\infty} \frac{f_k^2}{(4k+1)^n}. \tag{47}$$

Proof. Formulas (45) and (46) follow as usual from the orthogonality of the eigenfunctions $\cos((4k+1)u)$ and the fact that

$$\int_0^{\pi/2} \cos^2((4k+1)u)\,du = \frac{\pi}{4} \tag{48}$$

for each integer k. This, together with the eigenvalues of T^n given in Corollary 3.1, yields (47) as well. ■

4. ALTERNATING PERMUTATIONS, $S(n)$, AND Π_n. We shall see that $S(n)$ is closely related with the enumeration of alternating (also known as "up-down" or "zig-zag") permutations of n letters. A permutation σ of

$$[n] := \{1, 2, \ldots, n\} \tag{49}$$

is said to be *alternating* if

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > < \cdots \gtrless \sigma(n) \tag{50}$$

(the last inequality is $<$ or $>$ for n even and odd, respectively); such σ is *cyclically alternating* if, in addition, n is even and $\sigma(n) > \sigma(1)$ [**10**].[2] Let $A(n)$ be the number of alternating permutations of $[n]$, and $A_0(n)$ the number of cyclically alternating permutations when n is even. We tabulate these numbers for $n \leq 10$:

n	1	2	3	4	5	6	7	8	9	10
$A(n)$	1	1	2	5	16	61	272	1385	7936	50521
$A_0(n)$		1		4		48		1088		39680

The table suggests a relationship between $A(2m-1)$ and $A_0(2m)$, which we prove next.

[2]The term "permutation alternée" was introduced by André [**1**] (see also [**2**]) in the paper that first introduced alternating permutations and related their enumeration to the power series for $\sec x$ and $\tan x$. This notion should not be confused with the "alternating group" of even permutations of $[n]$.

Lemma 4. *For all $m = 1, 2, 3, \ldots,$ the counts $A(2m-1)$ and $A_0(2m)$ are related by*

$$A_0(2m) = mA(2m-1). \tag{51}$$

Proof. We construct a 1-to-m correspondence between alternating permutations of $[2m-1]$ and cyclically alternating permutations of $[2m]$. Note first that, if σ is a cyclically alternating permutation of $[2m]$, then so is

$$\big(\sigma(2j+1), \sigma(2j+2), \ldots, \sigma(2j+2m-1)\big) = \sigma \circ \tau^{2j} \tag{52}$$

for each $j = 0, 1, 2, \ldots, m-1$, where τ is the $2m$-cycle sending each i to $i+1$ (and as usual $i+1$ and $2j+1, 2j+2, \ldots$ are taken mod $2m$). This partitions the cyclically alternating permutations into sets of m. Now each of these sets contains a unique permutation σ_0 taking $2m$ to $2m$. But such σ_0 correspond bijectively to the alternating permutations $(\sigma_0(1), (\sigma_0(2), \ldots, \sigma_0(2m-1))$ of $[2m-1]$. This establishes the identity (51). ■

The number $A_0(n)$ of cyclically alternating permutations of $[n]$ is given by the formula

$$A_0(n) = 2^{n-1}(2^n - 1)|B_n| \tag{53}$$

(see, for instance, Stanley [**10**]). By (9), this is equivalent to

$$A_0(n) = \left(\frac{2}{\pi}\right)^n n!\, S(n). \tag{54}$$

The identity (53) is usually obtained by identifying the exponential generating functions, not via $S(n)$ and the zeta function. But $A_0(n)$ can also be expressed directly in terms of the volume of the polytope Π_n (or rather in terms of the scaled polytope (29)), thus leading to (54) and showing that the appearance of $S(n)$ there is not merely accidental.

A general principle for enumerating permutations [**10**] shows that the number of cyclically alternating permutations of $[n]$ is $n!$ times the volume in the unit cube $(0, 1)^n$ of the region $P_0(n)$ determined by the inequalities

$$t_1 < t_2 > t_3 < t_4 > \cdots < t_n > t_1. \tag{55}$$

(Recall that "cyclically alternating permutations of $[n]$" can exist only if n is even.) This is because $P_0(n)$ is the *order polytope* associated to the partial order $\prec_0$ on $[n]$ in which

$$1 \prec_0 2 \succ_0 3 \prec_0 4 \succ_0 \cdots \prec_0 n \succ_0 1 \tag{56}$$

and all other pairs in $[n]$ are incomparable. In general, the (open) order polytope associated by Stanley to a partial order $\prec$ on $[n]$ is the set of all $(t_1, \ldots, t_n)$ in the unit cube such that $t_i < t_j$ whenever $i \prec j$; and the volume of this polytope is an integer multiple of $1/n!$. Specifically, we cite Corollary 4.2 from [**10**].

Lemma 5. *The volume of the order polytope associated to any partial order $\prec$ on $[n]$ is $1/n!$ times the number of permutations σ of $[n]$ such that $\sigma(i) < \sigma(j)$ whenever $i \prec j$.*

In other words, the volume is $1/n!$ times the number of extensions of $\prec$ to a linear order on $[n]$. To see that this is equivalent to the assertion of the lemma, consider that there are $n!$ linear orders on $[n]$, each determined by the permutation of $[n]$ that sends the the minimal element to 1, the next one to 2, and so on until the maximal element is sent to n. This order extends $\prec$ if and only if $\sigma(i) < \sigma(j)$ whenever $i \prec j$.

Lemma 5 appears in **[10]** as part of a corollary to a much more powerful theorem. For our purposes the following direct proof suffices.

Proof. Decompose the closed unit n-cube into $n!$ simplices, one for each permutation σ of $[n]$, so that the simplex indexed by σ consists of all $t_1, \ldots, t_n$ in $[0, 1]$ with

$$t_{\sigma^{-1}(1)} \le t_{\sigma^{-1}(2)} \le \cdots \le t_{\sigma^{-1}(n)}. \tag{57}$$

Each of these simplices has the same volume; hence this common volume is $1/n!$. The union of those simplices indexed by σ satisfying $\sigma(i) < \sigma(j)$ whenever $i \prec j$ is the closure of the order polytope of $\prec$. Thus the volume of this polytope is $1/n!$ times the number of such σ. ∎

Equivalently, and perhaps more intuitively: the volume of the order polytope is the probability that n independent variables t_i drawn uniformly at random from $[0, 1]$ satisfy $t_i < t_j$ whenever $i \prec j$, i.e., that their linear order inherited from $[0, 1]$ extends the partial order $\prec$. This, however, is the same as the probability that a randomly chosen permutation of $[n]$ yields a partial order extending $\prec$, because (excepting the negligible case that some t_i coincide) the order of the t_i determines a permutation, and all permutations are equally likely.

We return now to the order polytope (55) associated to $\prec_0$. The affine change of variables

$$u_i = \begin{cases} t_i, & i \text{ odd;} \\ 1 - t_i, & i \text{ even,} \end{cases} \tag{58}$$

which has constant Jacobian determinant $(-1)^{n/2}$, transforms this region (55) to the familiar polytope $v_i > 0$, $v_i + v_{i+1} < 1$,[3] whose volume we have already identified with $(2/\pi)^n S(n)$ in two different ways. Thus

$$A_0(n) = \left(\frac{2}{\pi}\right)^n n!\, S(n) \tag{59}$$

for all even n.

By Lemma 4, we also recover a formula for $A(2m-1)$:

$$A(2m-1) = \frac{A_0(2m)}{m} = \frac{2^{2m-1}(2^{2m}-1)}{m}|B_{2m}| = \frac{2^{2m+1}}{\pi^{2m}}(2m-1)!\, S(2m). \tag{60}$$

[3]Richard Stanley notes that this polytope (29) is also a special case of a construction from his paper **[10]**: it is the *chain polytope* associated with the same partial order $\prec_0$. The result in **[10]** that contains our Lemma 4 asserts that the chain and order polytopes associated with any partial order have the same volume. In general this is proved by a piecewise linear bijection, but for partial orders of "rank 1" (i.e., for which there are no distinct a, b, c such that $a \succ b \succ c$), the polytopes are equivalent by a single affine chain of variables. The partial order $\prec_0$ has rank 1, as does the partial order we define next in (62) to deal with $A(n)$. Stanley's affine change of variables for these two partial orders is just our (58); thus this part of our argument is again a simple special case of his.

In other words,

$$A(n) = \frac{2^{n+2}}{\pi^{n+1}} n!\, S(n+1) \tag{61}$$

when n is odd. We next prove this formula directly for all n, whether even or odd.

Theorem 4. *The number $A(n)$ of alternating permutations of $[n]$ is given by (61) for every positive integer n.*

Proof. Let $\prec$ be the partial order on $[n]$ in which

$$1 \prec 2 \succ 3 \prec 4 \succ\prec \cdots \substack{\succ \\ \prec} n \tag{62}$$

and all other pairs in $[n]$ are incomparable. Then $A(n)$ is the number of permutations σ of $[n]$ such that $\sigma(i) < \sigma(j)$ whenever $i \prec j$. Accordingly, $A(n)$ is $n!$ times the volume of the associated order polytope

$$\{(t_1, t_2, \ldots, t_n) \in \mathbf{R}^n : 0 < t_i < 1,\ t_1 < t_2 > t_3 < t_4 > < \cdots \substack{> \\ <} t_n\}. \tag{63}$$

The change of variables (58) transforms (63) into the region

$$\{(v_1, v_2, \ldots, v_n) \in \mathbf{R}^n : v_i > 0,\ v_i + v_{i+1} < 1\ (1 \le i \le n-1)\}. \tag{64}$$

(N.B. This looks like the familiar $(2/\pi)\Pi_n$, but in fact strictly contains $(2/\pi)\Pi_n$, because we do not impose the condition $v_n + v_1 < 1$.) On the other hand, under the further linear change of variable $v_i = (2/\pi)u_i$, the region (64) maps to

$$\{(u_1, u_2, \ldots, u_n) \in \mathbf{R}^n : u_i > 0,\ u_i + u_{i+1} < \pi/2\ (1 \le i \le n-1)\}, \tag{65}$$

a region whose volume is $(\pi/2)^n$ times larger than that of (64). Thus the volume of (64) is

$$\begin{aligned} &\left(\frac{2}{\pi}\right)^n \int_0^{\pi/2} \cdots \int_0^{\pi/2} \prod_{i=1}^{n-1} K_1(u_i, u_{i+1})\, du_1 \cdots du_n \\ &= \left(\frac{2}{\pi}\right)^n \int_0^{\pi/2} \cdots \int_0^{\pi/2} K_{n-1}(u_1, u_n)\, du_1 \cdots du_n. \end{aligned} \tag{66}$$

Now by (38) the function

$$u_n \mapsto \int_0^{\pi/2} K_{n-1}(u_1, u_n)\, du_1 \tag{67}$$

in $\mathrm{L}^2(0, \pi/2)$ is the image under T^{n-1} of the constant function 1. Thus the integral (66) is $(2/\pi)^n$ times the inner product of 1 and $T^{n-1}1$ in $\mathrm{L}^2(0, \pi/2)$. By (47) in Corollary 3.2, this inner product is

$$\frac{\pi}{4} \sum_{k=-\infty}^{\infty} \frac{c_k^2}{(4k+1)^{n-1}}, \tag{68}$$

where c_k is the coefficient of $\cos((4k+1)u)$ in the orthogonal expansion of 1. Using (46) of the same corollary, we calculate

$$c_k = \frac{4}{\pi}\int_0^{\pi/2} \cos((4k+1)u)\, du = \frac{4}{\pi}\left.\frac{\sin((4k+1)u)}{4k+1}\right|_0^{\pi/2} = \frac{4}{\pi}\frac{1}{4k+1}. \tag{69}$$

Therefore,

$$\frac{A_n}{n!} = \left(\frac{2}{\pi}\right)^n \left(\frac{4}{\pi}\right) S(n+1), \tag{70}$$

from which (61) follows. ∎

Remark. Some time before publishing [**10**], Stanley proposed the computation of the volume of the region in (65) as a MONTHLY problem [**9**]. The published solution [**8**] used generating functions to express the volume in terms of E_n or B_{n+1}. But the "Editor's Comments" at the end of the solution include the note: "Several solvers observed that [$n!$ times the volume of the polytope] is the number of zig-zag permutations of $1, 2, \dots, n, \dots$." This suggests that, even if the elementary proof of Lemma 4 is not yet in the literature, it is obvious enough that these solvers at least implicitly recognized it, and applied it together with the change of variables (58) to relate the polytope volume to A_n.

Theorem 2, together with Lemma 4, yields the following amusing corollary: as $m \to \infty$,

$$\frac{A_0(2m)}{A(2m)} = m\frac{A(2m-1)}{A(2m)} = \frac{\pi}{4}\frac{S(2m)}{S(2m+1)} \to \frac{\pi}{4}. \tag{71}$$

That is, a randomly chosen alternating permutation σ of $[2m]$ is cyclically alternating with probability approaching $\pi/4$! The convergence is quite rapid, with error falling as a multiple of 3^{-2m}; for instance, for $m = 5$ we already find $39680/50521 \approx 0.785416$, while $\pi/4 \approx 0.785398$.

When n is even, say $n = 2m$, the formula (61) simplifies to

$$A(2m) = (-1)^m E_{2m} = |E_{2m}| \tag{72}$$

by (10). We have thus given a combinatorial interpretation of the positive integer $(-1)^m E_{2m}$, as promised in the introduction: it is the number of alternating permutations of $[2m]$.

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