THE HISTORICAL DEVELOPMENT OF ALGEBRAIC GEOMETRY

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I. THEMES AND PERIODS

Modern algebraic geometry has deservedly been considered for a long time as an exceedingly complex part of mathematics, drawing practically on every other part to build up its concepts and methods and increasingly becoming an indispensable tool in many seemingly remote theories. It shares with number theory the distinction of having one of the longest and most intricate histories among all branches of our science, of having always attracted the efforts of the best mathematicians in each generation, and of still being one of the most active areas of research. Both are perhaps the best candidates for the perfect mathematical theory, according to Hilbert's ideas: if we agree with him that problems are the lifeblood of mathematics, then certainly we may say that algebraic geometry and number theory always have had more open problems than solved ones, and that each progress towards their solution has always brought with it a host of new and exciting methods.

Human minds being unable to grasp complex matters as a whole, I have thought it would be helpful to describe the history of algebraic geometry as a kind of two-dimensional pattern, where many varied trends of thought, belonging to a few big *themes*, weave their way as multicolored threads through the moving succession of years. It should, however, be emphasized from the start that such a presentation inevitably inflicts distortions on reality: these themes constantly react on one another, and any division of time into periods is bound to founder on the fact that periods almost always overlap.

With these reservations, we may first group the main ideas of algebraic geometry as follows:

(A) and (B) The twin themes of *classification* and *transformation*, hardly to be separated, since the general idea behind classification of algebraic varieties is to put together those which can be deduced from each other by some kind of "trans-

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It is hardly necessary to identify Prof. Dieudonné to our readers; still a few facts may prove interesting. Prof. Dieudonné studied at the Ecole Normale supérieure from 1924–27, was a Fellow at Princeton, Berlin, and Zürich, and received his Doctorate in 1931. He served on the faculties at Bordeaux, Rennes, Nancy, São-Paulo, Michigan, Northwestern, l'Institut des Hautes Etudes Scientifiques, and was the Dean of Faculty at Nice until his retirement. He held Visiting Professorships at Columbia, Johns Hopkins, Rio de Janeiro, Buenos Aires, Pisa, Maryland, Tata Institute Bombay, Notre Dame, and Washington. His honors include the Order of the Legion of Honor, the Order of the Academic Palms, and membership in the Academy of Sciences. He served as President of the Mathematical Society of France in 1964–65.

Prof. Dieudonné has published a number of books and about 135 research articles on analysis, topology, spectral theory, classical groups, formal Lie groups, and non-commutative rings. This article was prepared while the author was a Visiting Professor at the University of Maryland. *Editor*.

formation." Subordinate to these themes are the notion of *invariant*, both of algebraic type and of numerical type (such as dimension, begree, genus, etc.), and the concepts of *correspondence* and of *morphism*, which give precise meanings and extensions to the vague idea of "transformation."

(C) *Infinitely near points*: a thorny problem, which has plagued generations of mathematicians: the definition and classification of singularities, the correct definition of "multiplicity" of intersections, later the concept of "base points" of linear systems, and the recent introduction of rings with nilpotent elements, all belong to that theme.

(D) *Extending the scalars*: a giant step forward in the search for simplicity: the introduction of *complex* points and later of *generic* points were the forerunners of what we now consider as perhaps the most characteristic feature of algebraic geometry, the general idea of *change of basis*.

(E) *Extending the space*: another fruitful method for extracting understandable results from the bewildering chaos of particular cases: *projective geometry* and *n-dimensional geometry* paved the way for the modern concepts of "abstract" varieties and *schemes*.

(F) *Analysis and topology in algebraic geometry*. This theme beautifully exemplifies the cross-fertilization between various branches of mathematics. Out of a problem of integral calculus, the computation of elliptic integrals and of their generalizations, adelian integrals, Riemann developed the concept of Riemann surface (the first non-trivial example of "complex manifold"), invented algebraic topology, and he and his successors showed how these ideas completely renewed the theory of algebraic curves and surfaces. One hundred years later, history repeated itself when A. Weil transferred to algebraic geometry the notion of *fiber bundle*, and Serre the idea of using *sheaves* and their cohomology, which he and H. Cartan had shown to be so effective for complex manifolds.

(G) *Commutative algebra and algebraic geometry*. As we shall see, this has grown into the most important theme for modern algebraic geometry. Since Riemann introduced the field of rational functions on a curve, Kronecker, Dedekind and Weber the concepts of ideals and divisors, commutative algebra has become the workshop where the algebraic geometer goes for his main tools: local rings, valuations, normalization, field theory, and the most recent and most efficient of all, homological algebra.

Needless to say, within the scope of this article, it will be impossible to do more than deal with a few of the highlights of our history, leaving aside a large number of important developments which should be included in a reasonably complete survey.

II. FIRST PERIOD: "PREHISTORY" (CA. 400 B.C.–1630 A. D.)

If it is true that the Greeks invented geometry as a deductive science, they never (contrary to popular beliefs) made any attempt to divorce it from algebra. On the contrary, one of their main trends was to use geometry to solve algebraic problems, and this is best exemplified in the invention of the conics, the first curves which they thoroughly studied after straight lines and circles. The Greeks knew simple geometric constructions for the root of the equation $x^2 = ab$, a and b being given as *lengths* of segments, and the unknown x being considered as the side of a square; they usually wrote the equation as a "proportion" $a/x = x/b$. The "Delic problem" called for construction of a length x of given cube, $x^3 = a^2b$; this was transformed by Hippocrates of Chio (around 420 B.C.) into a "double proportion" $a/x = x/y = y/b$ for two unknown lengths x, y. Menechmus (ca. 350 B.C.) had the idea of considering the loci given by the two equations $ay = x^2$ and $xy = ab$, whose intersection has as coordinates x, y a solution of the problem. This may seem to involve knowledge of analytic geometry; actually the Greeks made extensive use of coordinates (in particular for the later theory of conics by Apollonius), without however reaching the general point of view of Descartes and Fermat (see below).

This method of solving equations by intersections of curves had in fact already been used in the 5th century B.C., and led to the invention of many curves, both algebraic and transcendental; of course, the distinction between the two kinds of curves could not be perceived during that period, and more generally, there was no attempt at classification, for which no rational basis existed. Besides planes and spheres, the Greeks also studied some surfaces of revolution, such as cones, cylinders, a few types of quadrics and even tori; after having discovered conics "analytically," Menechmus was also the first to recognize that they could be obtained as plane sections of a cone of revolution; and a bold construction of Archytas (late 5th century B.C.) gave a solution of the Delic problem by the intersection of a cone, a cylinder and a torus. Finally, in his astronomical work, Eudoxus was led to describe the intersection of a sphere and a cylinder as the trajectory of a movement conceived as the superposition of two rotations, which may be considered as the first example of a parametric representation of a curve.

III. SECOND PERIOD: "EXPLORATION" (1630–1795)

For once, this period has a very well-defined starting point, the independent invention by Fermat and Descartes of "analytic geometry," which certainly also marks the true birth of algebraic geometry. The main novelty compared to the way the Greeks used coordinates is that the *same* axes are used for *all* curves (fixed or variable) which are being considered in a problem, and above all the fact that the algebraic notation of Viète and Descartes opens the way to the consideration of arbitrary equations (where the Greeks could not go beyond the third or fourth degree). Within this frame, the distinction between algebraic and transcendental curves immediately emerges; the concept of dimension is already clear to Fermat, who explicitly states that a single equation defines a curve in 2 dimensions, a surface in 3 dimensions, and already hints at the possibility of generalization to higher dimensions. The degree of a plane curve is at once seen to be invariant with respect to a change of coordinates, and Newton knows that it is also invariant under a central projection (an operation which was familiar since the study of conic sections by the Greeks).

Themes A and B

The chief work of that period is one of exploration. Fermat shows that all curves of degree 2 are conics, and Newton classifies all plane cubics with respect to change of coordinates and projections; Euler classifies the quadrics, and the first skew curves, given as intersection of two surfaces, appear in the 18th century. The concept of parametric representation of a curve is fundamental in Newton's approach to calculus, and Euler knows how to get in certain cases a parametric representation from the cartesian equation. A beginning is made in the elucidation of the structure of singular points and inflexion points of algebraic plane curves, although limited to the most elementary cases, so that no general description is yet obtained.

Theme C

The problem of intersection of two algebraic plane curves is already tackled by Newton; he and Leibniz had a clear idea of "elimination" processes expressing the fact that two algebraic equations in one variable have a common root, and using such a process, Newton observed that the abscissas (for instance) of the intersection points of two curves of respective degrees m, n, are given by an equation of degree $\leqq mn$. This result was gradually improved during the 18th century, until Bézout, using a refined elimination process, was able to prove that, in general, the equation giving the intersections had exactly the degree mn; however, no general attempt was yet made during that period to attach to each intersection point an integer measuring the "multiplicity" of the intersection, in such a way that the sum of the multiplicities should always be mn. Bézout also generalized his elimination process to 3 dimensions, proving that the points of intersection of three algebraic surfaces of degrees m, n, p are in general given by an equation of degree mnp.

With the beginning of the consideration of algebraic families of algebraic curves a problem in a sense converse to the problem of intersections appeared, namely the determination of a curve of given degree n containing sufficiently many given points. It should be recalled here that this (linear) problem was the starting point for the theory of determinants, and the fact that $n(n+3)/2$ points in "general position" completely determine a curve of degree n, whereas two curves of degree n have in general n^2 common points, gave the first general example of the concept of rank for a system of linear equations ("Cramer's paradox").

We should finally stress the fact that a number of ideas fully developed during the next period may be traced back (in an embryonic form) to the 17th or 18th century, as we shall see below.

IV. THIRD PERIOD: "THE GOLDEN AGE OF PROJECTIVE GEOMETRY" (1795–1850)

Here again we have a rather sharp break with the past at the beginning of this period. In the space of a few years, with Monge and his school and especially with Poncelet, a new era begins with the simultaneous introduction of points at infinity and of imaginary points: "geometry" will now, for almost 100 years, exclusively mean geometry in the complex projective plane $\boldsymbol{P}_2(\boldsymbol{C})$ or the complex projective 3-dimensional space $\boldsymbol{P}_3(\boldsymbol{C})$. In fact, the fundamental idea of (real) projective geometry goes back to Desargues (17th century) who, trying to give mathematical foundations to the methods of "perspective" used by painters and architects, had introduced the concept of "point at infinity," and the use of central projections as a means of getting new theorems from classical results of Euclidean geometry; and although these ideas had inspired Pascal in his work on conics, they had very soon dropped into oblivion, due to the outlandish language of the author and the very limited diffusion of his book (which was for some time believed lost). Other mathematicians in the 18th century, in particular Euler and Stirling, had hinted at the existence of imaginary points, in order to state general theorems without distinction of various cases. This is precisely what is brilliantly accomplished by the new school: circles now intersect in 4 points as any two conics should, but two of the points are imaginary and at infinity; instead of several kinds of conics and quadrics, all nondegenerate conics (resp. quadrics) are now projectively equivalent; instead of the 72 kinds of cubics enumerated by Newton, only 3 remain projectively distinct; etc.

Themes D and E

The chief beneficiaries of these new ideas are at first the theory of conics, quadrics and of linear families of conics and quadrics; but curves and surfaces of degree 3 or 4 are also investigated in this way, revealing beautiful new theorems, such as the configurations of the 9 inflexion points of a plane cubic, the 27 lines on a cubic surface, the 28 bitangents to a plane quartic; the theorem of Salmon, proving the constancy of the cross ratio of the 4 tangents to a cubic issued from a point of the curve, was to gain even more significance later, as the first concrete example of a "module" in Riemann's sense for an algebraic curve.

Although, with Möbius, Plücker and Cayley, projective geometry received a sound algebraic basis by the use of homogeneous coordinates, a general tendency of the projective school was to minimize as much as possible algebraic computations, and to rely instead (beginning with Poncelet) on general heuristic "principles" which they did not bother to justify algebraically. The remarkable success they had in this direction was chiefly due to their skillful use of the idea of geometric *transformation*, which for the first time comes to the forefront in geometry, preparing the ground for Klein's famous "Program" linking geometry and the theory of groups. Most of the transformations they consider are linear: for instance, one of their favorite devices in the theory of conics is to consider a conic as the locus of two variable straight lines through two fixed points, one of them being derived from the other by a fixed linear transformation (an idea which, in some particular cases, goes back to Maclaurin). Similarly, in the study of the linear system of conics through 4 fixed points, they investigate the intersections of these conics with a fixed straight line D by considering the (linear) transformation which to a point M of D associates the second point of intersection with D of the conic of the system which contains M. Emboldened by the results obtained in this manner, they inaugurated what was to become the theory of *correspondences*, by considering what they called (α, β)-correspondences, i.e., relations between two points M, M' such that to each point M there exist α points M' related to M, and to each point M' there exist β points M related to M': when M and M' vary on the same projective line, Chasles' "correspondence principle" says that the number of points M (counted with multiplicities) coinciding with one of their transforms is $\alpha + \beta$ unless *every* point of the projective line has that property, a result which it is easy to justify algebraically. A beautiful application is the Poncelet "closure theorem" for polygons inscribed in a conic C and circumscribed to a conic C': for a given integer n, one defines on C a $(2,2)$-correspondence by assigning to $M \in C$ the nth point M_n in a sequence $M_0 = M, M_1, \cdots, M_n$, where each side M_iM_{i+1} is tangent to C' and the points M_i are on C. It is easily seen that for n even, one has $M = M_n$ if $M_{n/2}$ is a point common to C and C', and for n odd, $M = M_n$ if $M_{(n-1)/2} = M_{(n+1)/2}$, and the tangent to C at that point is also tangent to C'. There are thus at least 4 points M on C such that $M = M_n$, and by the correspondence principle, if there is still *one* more point having that property, then $M = M_n$ for *all* points on C (one uses of course the parametrization of a conic by the projective line).

Theme B

Later representatives of the projective school (notably Chasles in France, Steiner and von Staudt in Germany), somewhat intoxicated by the elegance of their methods, went so far as to insist that "pure" geometry should be entirely divorced from algebra and even (with von Staudt) from the concept of real number. As could be expected, such efforts did not lead very far, and probably hampered progress in the realization of the importance of linear algebra in classical geometry; it may be, however, that they paved the way for the later "abstract" algebraic geometry over a field different from $\boldsymbol{R}$ or $\boldsymbol{C}$.

In the general theory of algebraic curves (in $\boldsymbol{P}_2(\boldsymbol{C})$) and surfaces (in $\boldsymbol{P}_3(\boldsymbol{C})$), the main problems studied before Riemann are of an enumerative character: to give only one example of such problems, what is the number of conics tangent to 5 given conics in general position? (The correct answer is 3264.) Chasles, and later Schubert and Zeuthen proposed half-empirical formulas to solve these problems, based on an intuitive concept of "intersection multiplicity" which could only be justified much later. One of the main ideas of projective geometry, the concept of *duality*, led to the introduction of new "tangential" invariants for algebraic plane curves: the class (number of tangents through a point), the number of inflexion points and the number of double tangents, culminating in the famous "Plücker formulas"

Theme C

Theme A

$$m' = m(m-1) - 2d - 3s,$$

$$m = m'(m'-1) - 2d' - 3s',$$

$$s' - s = 3(m' - m),$$

where m is the degree of the curve, m' its class, d the number of double points, d' the number of double tangents, s the number of cusps, s' the number of inflexion points; no "higher singularities," either punctual or tangential, are supposed to occur.

V. FOURTH PERIOD: "RIEMANN AND BIRATIONAL GEOMETRY" (1850–1866)

The importance of Riemann in the history of algebraic geometry can hardly be overestimated, but in his two fundamental contributions, the "transcendental" approach *via* abelian integrals and the introduction of the field of rational functions on a curve, he built on basic ideas inherited from the previous period.

The origin of abelian integrals is the study of integrals of type

$$\int \frac{R(t)dt}{\sqrt{P(t)}}$$

where $P(t)$ is a polynomial of degree 3 or 4 and $R(t)$ a rational function; one of these integrals expresses the length of an arc of an ellipse (hence the name "elliptic integrals"). In the first half of the 18th century, Fagnano and Euler, looking for some substitute for the classical formula expressing the sum of two arcs of a circle, when the circle is replaced by an ellipse, found indeed that the sum

$$\int_a^x \frac{dt}{\sqrt{P(t)}} + \int_a^y \frac{dt}{\sqrt{P(t)}}$$

can be written

$$\int_a^z \frac{dt}{\sqrt{P(t)}} + V(x, y),$$

where z is an *algebraic* function of x and y, and V a rational or logarithmic function of x and y, and Euler had similar results for more general integrals.

At the beginning of his famous work of elliptic functions, Abel made a giant step forward by showing that the Fagnano-Euler relations were special cases of a very general theorem: he considers an arbitrary "algebraic function" y of x, defined as a solution of a polynomial equation $F(x, y) = 0$; an "abelian integral" $\int R(x, y)dx$ is an integral in which R is a rational function of x, y, in which y is replaced by the preceding algebraic function (for instance elliptic integrals correspond to $F(x, y) = y^2 - P(x)$). Then, if $G(x, y, a_1, \cdots, a_r) = 0$ is a second polynomial in x, y whose coefficients are rational functions of some parameters $a_1, \cdots, a_r$, and if (x_1, y_1), $(x_2, y_2), \cdots, (x_m, y_m)$ are the points of intersection of the two curves $F=0$, $G=0$, the sum

$$V = \int_{(a,b)}^{(x_1,y_1)} R(x, y)dx + \cdots + \int_{(a,b)}^{(x_m,y_m)} R(x, y)dx$$

is a rational or logarithmic function of the parameters a_j $(1 \leqq j \leqq r)$*; surprisingly enough, this is little more than an exercise in the theory of symmetric functions of the roots of a polynomial. But Abel does not stop there, and studies in detail the case, in which V is a constant; this leads him to the realization that in that case, *any* sum

$$\int_{(a,b)}^{(x_1,y_1)} R(x, y)dx + \cdots + \int_{(a,b)}^{(x_m,y_m)} R(x, y)dx$$

with *arbitrary* points (x_j, y_j) on the curve $F = 0$, can be expressed as the sum of a *fixed* number δ of values of the same integral, with upper limits algebraic functions of the (x_j, y_j); but, in contrast with the Fagnano-Euler formulas for elliptic integrals, he showed that the number δ may well be > 1, for instance when $F(x, y) = y^2 - P(x)$ with P of degree $\geqq 5$.

Abel, however, worked exclusively within the framework of analysis, and does not seem to have been acquainted with projective geometry. Furthermore, he obviously had no clear concept of integration in the complex plane (in 1826, Cauchy had hardly begun his work on that subject), and with the exception of a short and

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* Of course, the points x_j, y_j usually have complex coordinates; an integral

$$\int_{(a,b)}^{(x_j,y_j)} R(x, y)dx$$

is only properly defined when the path of integration in the complex plane C with extremities a and x_j has been fixed, and y_j is the value taken by y when x varies along the path, y is a continuous function of x and takes the value b at $x = a$. When the path is replaced by another one (with the same extremities), the value of the integral is modified by a "period."

By definition, a logarithmic function of the a_j has the form log S $(a_1, \ldots, a_r)$ where S is rational.

inconclusive note, he has no general discussion of the *periods* of his integrals. Thus, although Abel's theorem paved the way for Jacobi's breakthrough in the problem of inversion of hyperelliptic integrals*, Abel himself narrowly missed the concept of integral of the first kind and the definition of the genus of a curve (his failure to take into account the points at infinity has as a consequence the fact that the δ integrals he considers are not necessarily of the first kind).

When Riemann takes up the subject in 1851, the intervening years had seen the great development by Cauchy and his school of the theory of functions of a complex variable. Indeed, the starting point of Riemann has nothing to do with algebraic functions, but is the extension of Cauchy's theory to the "surfaces" he introduces in order better to deal with the so-called "multiform" functions of the most general (not necessarily algebraic) type. This was already far beyond the contemporary concepts, and during the 30 years following Riemann, it was the object of long and tedious explanations by the expositors of his theory. But the way Riemann uses this notion in order to attack the problem of abelian integrals is much more original still. Instead of starting (as would all his predecessors and most of his immediate successors) from an algebraic equation $F(s, z) = 0$ and the Riemann surface of the algebraic function s of z which it defines, his initial object is an n-sheeted Riemann surface without boundary and with a finite number of ramification points**, given *a priori* without any reference to an algebraic equation (Riemann

Theme F

* The natural idea of "inverting" the integral $\int_a^x (Q(t)\,dt)/\sqrt{P(t)} = u$ is to study x as a function of u, as Abel and Jacobi had done when P has degree 3 or 4; but Jacobi realized that, due to the existence of 4 periods, no meromorphic function of u could be a solution of the problem. Abel's theorem finally led him to the correct conception of the problem: one considers *two* equations

$$\int_a^x \frac{dt}{\sqrt{P(t)}} + \int_a^y \frac{dt}{\sqrt{P(t)}} = u, \quad \int_a^x \frac{t\,dt}{\sqrt{P(t)}} + \int_a^y \frac{t\,dt}{\sqrt{P(t)}} = v,$$

and one "inverts" them by expressing the symmetric functions $x + y$ and xy as functions of u and v; Abel's theorem yields an "addition formula" for these functions, from which one can show that they are meromorphic and quadruply periodic.

** The best way to define at least the part of the Riemann surface of a function $s(z)$ (defined by an algebraic relation $F(s, z) = 0$), containing no point at infinity, is to say that it is the subset of C^2 consisting of the pairs (s, z) satisfying the equation $F(s, z) = 0$; there is then no difficulty with the "crossing of sheets." Ramification points are those for which $\partial F/\partial s\,(s, z) = 0$; Puiseux proved in 1850 that if (s_0, z_0) is such a point, the surface decomposes at that point into a finite number of "branches" such that each branch can be represented by equations of type

$$z - z_0 = t^h, \; s - s_0 = a_1 t + a_2 t^2 \ldots,$$

where t (the "uniformizing parameter") is in a neighborhood of 0 in C and the series converges (the integer h depending on the branch).

This description is only correct, however, when at each ramification point (s_0, z_0) there is only one branch; if not, the point (s_0, z_0) must be replaced by as many points as there are branches; in other words the points of a Riemann surface are the *branches* at the various points of the curve.

takes care to complete each sheet with a point at infinity, and thus avoids Abel's difficulties with these points); then he attacks the problem in the most general manner possible: classify the integrals of *all meromorphic functions* on the surface. The work of Cauchy and Puiseux had brought to light the general idea of "periods" of such integrals, generally expressed (as in the example first given by Abel) as an integral taken along an arc joining two ramification points. Here again Riemann breaks entirely new ground: he realizes for the first time that topological concepts are closely related to the problem, and begins by essentially creating the topological study of compact orientable surfaces, attaching to such a surface S an invariantly defined integer $2g$, the minimal number of simple closed curves C_j on S needed to make the complement S' of their union simply connected. Then, instead of studying integrals of meromorphic functions, he *defines directly* integrals of the first and second kinds by their periodicity properties, as functions meromorphic on S', and tending on both sides of each C_j to limits which differ by a quantity k_j constant on C_j (a further reduction of the domain S' is needed to obtain similarly the integrals of the third kind, having logarithmic singularities)*; integrals of the first kind are those which have no pole on S. The existence of integrals of the three kinds is proved by Riemann as a consequence of what he calls the "Dirichlet principle," i.e., the existence of a harmonic function in S' taking prescribed values on the boundary (which allows him to prescribe at will the *real parts* of the k_j); and it is also by an ingenious use of the same principle that Riemann obtains the fundamental relation

$$g - 1 = w/2 - n$$

giving the genus in function of the number of sheets n, and the number w of ramification points (supposed to be of a "general" type).

The meromorphic functions on S are then the integrals of the first or second kind whose periods k_j all vanish, and Riemann shows that they may be expressed as rational functions of two of them, linked by an algebraic relation $F(s, z) = 0$, thus recovering the older point of view, but immeasurably enriched with new insights. The choice of these meromorphic functions s, z is in a large measure arbitrary, and this leads Riemann to his next big step forward, the general concept of *birational transformation* between two irreducible algebraic curves, corresponding to a biholomorphic mapping of their Riemann surfaces. Here again, Riemann was not without predecessors: already Newton and his followers had introduced quadratic transformations such as

Theme B

$$x' = 1/x,\ y' = y/x$$

in the plane, and observed that they thus transformed an algebraic curve into a

* One simply joins the singularity to one of the C_j by an arc, and deletes the arc from S'.

curve of different degree. "Inversion" in the plane and in 3 dimensional space had been intensively studied since the early 1820's, chiefly by "synthetic" geometers; finally, the passage from a plane curve to its transform by duality (exchanging punctual and tangential coordinates) was obviously a birational transformation between two algebraic curves, exchanging degree and class. But the startling novelty of Riemann's approach is of course the fact that to a class of "birationally equivalent" irreducible algebraic curves he was able to attach his topological invariant g, the *genus* of all the curves in the class. But he did not stop there, and by an evaluation (using two different methods) of the parameters on which a Riemann surface of genus g depended, he arrived at the conclusion that classes of isomorphic Riemann surfaces of genus $g \geqq 2$ were characterized by $3g - 3$ complex parameters varying continuously (for $g = 1$ there is only one parameter, and none for $g = 0$); the precise meaning of this result (the so-called theory of "moduli" of curves) was to remain until very recently among the least clarified concepts of the theory.*

Theme A

VI. FIFTH PERIOD: "DEVELOPMENT AND CHAOS" (1866–1920)

The extraordinary wealth of new ideas and methods introduced by Riemann provided inspiration for a steady development of algebraic geometry for over 80 years. But the grandiose synthesis he had envisioned and tried to materialize was almost immediately broken up by his successors. During that period there will be at least two or three schools of algebraic geometry, each using different methods, with little in common even in the fundamental concepts. Riemann's use of analysis, in particular in the "Dirichlet principle," exceeded the possibilities of his time, and he had obviously neglected all the difficulties bound to the existence of singular points on algebraic curves. The first task to which each school of algebraic geometry addressed itself was therefore the systematization of the birational theory of algebraic plane curves, incorporating most of Riemann's results with proofs in conformity with the principles of the school. Then, with varying success, they tried to extend their methods to the theory of algebraic surfaces and higher dimensional algebraic varieties.

VI a: The algebraic approach. Historically, this was the latest one, being initiated by two fundamental papers in 1882, one by Kronecker and one by Dedekind and Weber. But in the light of subsequent history, it is the trend which was to exert the deepest influence on the birth of our modern concepts; in particular, just as Riemann

* One should emphasize the fact that this only describes the first half of Riemann's paper on abelian integrals; the second part, which solves in a masterly way the inversion problem by the introduction of the general "thêta functions" has been, if anything, even more influential on the development of analysis.

had revealed the close relationship between algebraic varieties and the theory of complex manifolds, Kronecker and Dedekind-Weber brought to light for the first time the deep similarities between algebraic geometry and the burgeoning theory of algebraic numbers, which were to be some of the main driving forces during the next periods. Furthermore, this conception of algebraic geometry is for us the clearest and simplest one, due to our familiarity with abstract algebra; but it was precisely this "abstract" character which made it the least popular and least understood one in its time.

The work of Kronecker and of his immediate followers, Lasker and Macaulay, in the first two decades of the 20th century, was of a very general nature, and its importance only emerged in the later periods: it essentially consisted in setting up and consistently using an elimination method, far more flexible and powerful than the preceding ones, with the help of which it was for the first time possible to give a precise meaning to the concepts of *dimension* and of *irreducible variety** and to show that each variety (defined by an arbitrary system of algebraic equations) in projective n-space decomposed in a unique way into a union of irreducible varieties (in general of different dimensions).

Theme G

The goal of Dedekind and Weber in their fundamental paper was quite different and much more limited; namely, they gave purely algebraic proofs for all the algebraic results of Riemann. They start from the fact that, for Riemann, a class of isomorphic Riemann surfaces corresponds to a *field* K of rational functions, which is a finite extension of the field $\boldsymbol{C}(X)$ of rational fractions in one indeterminate over the complex field; what they set out to do, conversely, if a finite extension K of the field $\boldsymbol{C}(X)$ is given *abstractly*, is to reconstruct a Riemann surface S such that K will be isomorphic to the field of rational functions on S. Their very original and fruitful method may be presented in the following way: *if* the Riemann surface S was already known, at each point $z_0 \in S$, a rational function $f \neq 0$ would have an *order* $v_{z_0}(f)$, namely the integer (positive or negative) which is the degree of the smallest power in the Puiseux development $f(u) = \sum_k a_k u^k$ with respect to a "uniformizing parameter" u (equal to $z - z_0$ if z_0 is not a ramification point, to some power $(z - z_0)^{1/h}$ if z_0 is a ramification point). For a *fixed* $z_0 \in S$, the mapping $f \mapsto v_{z_0}(f)$ of K^* into $\boldsymbol{Z}$ is what is called a *discrete valuation* on K: we recall that this is by definition a mapping $w\colon K^* \to \boldsymbol{Z}$ such that $w(f+g) \geqq \inf(w(f), w(g))$ if $f+g \neq 0$, and $w(fg) = w(f) + w(g)$, which implies $w(1) = 0$ and $w(f^{-1}) = -w(f)$ (w is usually extended to K by taking $w(0) = +\infty$ by convention). What Dedekind and Weber do is to *reverse* this process, and *define* a "point of the Riemann surface S of K"

* An irreducible variety V in $\mathbf{P}_n(\mathbf{C})$ is characterized by the property that if the product PQ of two homogeneous polynomials is 0 in V, then one of the two polynomials P, Q must be 0 in V. The restrictions to V of the rational functions which are defined at one point of V at least then form a field whose transcendence degree over $\mathbf{C}$ is the dimension of V.

as a *nontrivial discrete valuation* on K (i.e., one which is not identically 0 on K^*: two proportional valuations are then identified).

Now the nontrivial discrete valuations on the field $\mathbf{C}(X)$ are easily determined: one of them (the "point at infinity") w_∞ is such that $w_\infty(P) = -\deg(P)$ for any nonzero polynomial $P(X)$; the other ("finite points") correspond bijectively to the points $\zeta \in \mathbf{C}$, the corresponding valuation w_ζ being such that $w_\zeta(P)$ is the order of the zero ζ of $P(X)$ (equal to 0 if $P(\zeta) \neq 0$). It can easily be shown that for each discrete valuation w of $\mathbf{C}(X)$, there is a finite number of nonproportional valuations v_j *on* K such that for each j, v_j/e_j reduces to w on $\mathbf{C}(X)$, where e_j is an integer $\geqq 1$; one says that the v_j are the points of the Riemann surface S *above* w; the points above w_∞ are again called points at infinity, the other finite points.

The elements $f \in K$ for which $v(f) \geqq 0$ for all *finite* points v of S constitute exactly the elements of K which are *integral** over the ring of polynomials $\mathbf{C}[X]$; they form what we now call a *Dedekind ring* A, to which Dedekind's theory of *ideals* may be applied.** The maximal ideals $\mathfrak{P}_v$ of A correspond to the finite points $v \in S$: $\mathfrak{P}_v$ is the set of $f \in A$ for which $v(f) > 0$; the *fractionary ideals* of K are the A-modules $\mathfrak{a}$ contained in K and for which there is an element $c \neq 0$ in A such that $c\mathfrak{a} \subset A$; each of them can be written uniquely as a product $\mathfrak{P}_1^{\alpha_1}\mathfrak{P}_2^{\alpha_2}\cdots\mathfrak{P}_r^{\alpha_r}$, where the $\mathfrak{P}_j$ are maximal ideals of A and the α_j positive or negative integers. Another way of stating this result is to say that a fractionary ideal $\mathfrak{a}$ is the set of all $f \in K$ such that $v_j(f) \geqq \alpha_j$ for $1 \leqq j \leqq r$, where the valuations v_j correspond to the maximal ideals $\mathfrak{P}_j$, and $v(f) \geqq 0$ for the other finite valuations.

The consideration of the ideals of A, however, leaves the "points at infinity" out of the picture. This led Dedekind and Weber to generalize the concept of ideal and to introduce the notion of *divisor* on K. This is defined as a family $D = (\alpha_v)$ of integers $\alpha_v \in \mathbf{Z}$, where v runs through *all* points of S, and $\alpha_v = 0$ except for a finite number of points: writing $(\alpha_v) + (\beta_v) = (\alpha_v + \beta_v)$ defines the set $\mathscr{D}(K)$ of divisors of K as an *additive group* isomorphic to $\mathbf{Z}^{(S)}$, in which an *order relation* is naturally defined, $(\alpha_v) \leqq (\beta_v)$ meaning that $\alpha_v \leqq \beta_v$ for all $v \in S$; a divisor $D = (\alpha_v)$ such that $\alpha_v \geqq 0$ for all $v \in S$ is called *positive* or *effective*. The *degree* $\deg(D)$ of $D = (\alpha_v)$ is defined as $\sum_{v \in S} \alpha_v$ (positive or negative integer); the *support* of D is the set of the $v \in S$ for which $\alpha_v \neq 0$. One of the problems considered by Riemann was the determination of rational functions on a Riemann surface having poles of orders $\leqq \alpha_P$ for prescribed points P (in finite number) on S. Using his bold expression of functions as sums of abelian integrals, he found that there existed rational functions having that property for an *arbitrary* choice of the points P as long as $\sum_P \alpha_P \geqq g + 1$, whereas if $\sum_P \alpha_P \leqq g$, this was only possible for *special* positions of the points P. This result was completed by his student Roch, and put in its final form by Dedekind

* Recall that an element x of a ring R is *integral* over a subring S if it satisfies an equation of type $x^m + a_1 x^{m-1} + \ldots + a_m = 0$, with $a_j \in S$.

** Dedekind had developed this theory for algebraic number fields from 1870 on.

and Weber in the following way: the problem is a special case of the study of the set $L(D)$ of rational functions $f \in K$ satisfying the conditions

$$v(f) \geqq -\alpha_v \text{ for all } v \in S \tag{1}$$

for a given divisor $D = (\alpha_v)$; it follows from the axioms of valuations that $L(D)$ is a complex vector subspace of K, and it can be shown that this subspace has *finite* dimension $l(D)$.

A fractionary ideal may be described as the union of the increasing family of spaces $L(D_m)$, where $D_m = (\alpha_v)$ is such that the α_v coincide with the $-\alpha_j$ for the v_j, are equal to 0 for the other finite points, and to m for the points at infinity.

The relations (1) can be written in a different way. For each $f \in K^*$, there are only a finite number of valuations $v \in S$ such that $v(f) \neq 0$; let $(f)_0$ (resp. $(f)_\infty$) be the positive divisor $((v(f))^+)$ (resp. $((v(f))^-)$) (in the "transcendental" interpertation, $(f)_0$ is the "divisor of zeroes" and $(f)_\infty$ the "divisor of poles" of the rational function f), and let $(f) = (f)_0 - (f)_\infty$ in the group $\mathscr{D}(F)$; (f) is called the *principal divisor* defined by f. It can be shown that $\deg((f)) = 0$ by purely algebraic arguments (in the transcendental picture, this is merely the *residue theorem*)*; in particular, if $v(f) \geqq 0$ for *all* $v \in S$, then $f \in \boldsymbol{C}$ (only constants are everywhere holomorphic on a Riemann surface) and if in addition $v(f) > 0$ for some v, then $f = 0$. With these definitions, the relations (1) for $f \neq 0$ are equivalent to the inequality

$$(f) + D \geqq 0 \tag{2}$$

in the ordered group $\mathscr{D}(K)$.

Principal divisors form a subgroup $\mathscr{P}(K)$ of $\mathscr{D}(K)$ (isomorphic to the group $K^*/\boldsymbol{C}^*$, two elements of K^* which have the same principal divisor differing by a constant factor by the previous remarks). Divisors belonging to the same *class* in the quotient group $\mathscr{C}(K) = \mathscr{D}(K)/\mathscr{P}(K)$ are called (linearly) *equivalent*: to say that D and D' are equivalent means therefore that there exists $f \neq 0$ such that $D' - D = (f)$; it is clear that $\deg(D') = \deg(D)$ and $l(D') = l(D)$ for equivalent divisors; two elements f, g of $L(D)$ are such that $(f) + D = (g) + D$ if and only if f/g is a constant, in other words, the set $|D|$ of *positive* divisors equivalent to D is identified to the projective space $P(L(D))$ of dimension $l(D) - 1$.

The *Riemann-Roch theorem* is then written in the following way:

$$l(D) - l(\Delta - D) = \deg(D) + 1 - g, \tag{3}$$

where g is the genus, and Δ belongs to a well-determined divisor class, called the *canonical class* of K. To define it in the transcendental interpretation, one considers on the Riemann surface S a *meromorphic differential form* ω: at each point P of S,

* One integrates the differential df/f on the boundary of the simply connected part S' of the Riemann surface, taking into account that each arc of that boundary comes twice in the integral with opposite orientations.

the differential form ω may be written $F(u)du$, where u is the uniformizing parameter in a neighborhood of P and F is meromorphic at P; if δ_P is the order of F at the point P, (δ_P) is a *canonical divisor*, and it does not depend on the choice of the uniformizing parameters. Any other meromorphic differential form may be written $f\omega$ with $f \in K$, hence all canonical divisors belong to the same *class*. There is a purely algebraic definition of Δ (see section VII b), and one proves that $\deg(\Delta) = 2g - 2$ for $g \geqq 1$ and $l(\Delta)=g$. Relation (3) implies Riemann's result on the poles of rational functions; more generally, if $\deg(D) \geqq g + 1$, (3) implies $l(D) \geqq 2$; if $D \geqq 0$, $L(D)$ always contains the constant functions, and to say that $l(D) \geqq 2$ means that it contains a non constant rational function. From the definition of $L(D)$, it follows that $l(D) = 0$ if $\deg(D) < 0$, hence, by (3), $l(D) = \deg(D) + 1 - g$ if $\deg(D) > 2g - 2$; in particular, for any divisor D such that $\deg(D) > 0$, $l(mD) = m \cdot \deg(D) + 1 - g$ for m large enough (although one may have $l(D) = 0$).

VI b: The Brill-Noether theory of linear systems of points on a curve. An irreducible plane curve Γ without singularity is identified to its Riemann surface, and a positive divisor may therefore be identified with a system of points of Γ, each being counted with a certain "multiplicity" which is a positive integer. Riemann's determination of the "special" systems of at most g points of Γ, which may be the poles of a rational function, had led him (by an extension of some earlier computations of Abel) to define these sets as intersections with Γ of a family of "adjoint" curves of smaller degree, subject to linear conditions on the coefficients of their equations, so that such a family may be considered as given by an equation $\sum_{j=1}^{r} \lambda_j P_j(x, y) = 0$ in nonhomogeneous coordinates, where the P_j are polynomials and the λ_j variable complex parameters. A number of points of intersection of these curves with Γ may be fixed (i.e., independent of the λ_j); as the intersection multiplicity of a common point of Γ and of an arbitrary curve Γ' is immediately defined since Γ has no singular point (it is the same as the intersection multiplicity of Γ' and the tangent to Γ), we may consider for each adjoint curve Γ' of the family the positive divisor $D = \sum_P m_P P - \sum_Q m_Q^0 Q$, where P runs through *all* the intersection points of Γ and Γ', m_P is the corresponding intersection multiplicity, Q runs through the *fixed* intersection points and m_Q^0 is the minimum value of m_Q when the λ_j vary. It is immediate to see that if D_0 is one of these divisors, corresponding to the values λ_j^0 of the parameters, then $D = D_0 + (f)$, where $f = (\sum_j \lambda_j P_j)/(\sum_j \lambda_j^0 P_j)$.

Conversely, given a divisor D_0 (positive or not), if $l(D_0) = r > 0$, the functions $f \in L(D_0)$ may be written $(\sum_{j=1}^{r} \lambda_j P_j(x, y))/Q(x, y)$, where the P_j and Q are polynomials and the λ_j arbitrary complex numbers; the positive divisors $(f) + D_0$ where $f \in L(D)$, are obtained by adding a fixed divisor to the variable divisor obtained as above from the points of intersection of Γ and of the curve $\sum_j \lambda_j P_j(x, y) = 0$.

The study of the vector spaces $L(D)$ attached to divisors is thus essentially equivalent to the study of the systems of points of intersection (with multiplicities) of Γ with the curves Γ' of a system of curves $\sum_j \lambda_j P_j(x, y) = 0$. It is in fact by means of the study of such systems of points, called "*linear series*" or "*linear systems*" on Γ, that the geometric school of Clebsch, Gordan, Brill, and Max Noether described the birational theory of algebraic plane curves after 1866. But they wanted to deal in this way, not only with curves without singularities, but with arbitrary algebraic curves, and linear systems of points are only easy to handle when the curve Γ has no singularities, or at most "nice" singularities such as double points with distinct tangents. One of the first efforts of that school was therefore to establish the possibility of finding a birational transformation of an arbitrary irreducible algebraic curve Γ into a plane curve with only double points with distinct tangents; a result proved independently by M. Noether in 1871 and equivalent to a theorem of algebra obtained by Kronecker in 1862. In view of the extension of this result during the later periods, it is worthwhile to note that a slightly weaker theorem may be obtained by a succession of birational transformations of the whole projective plane $\boldsymbol{P}_2(\boldsymbol{C})$ onto itself of the type

Theme C

$$x'/yz = y'/zx = z'/xy$$

(for suitable *homogeneous* coordinates), the so-called *quadratic transformations.* Such a transformation is bijective outside the sides of the triangle having as vertices the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ but sends each point of one side (not a vertex) to the opposite vertex, and is indeterminate at a vertex: however, two points approaching a vertex along distinct lines have transforms which tend to distinct limits on the opposite side, so that the transformation may be said to "blow up" a vertex to the opposite side, and *separates* the branches of a curve having different tangents at a vertex by transforming them to branches through different points of the transformed curve. By repeating conveniently this process, one may show that there is a transformed curve whose singular points are such that each has a number of distinct tangents equal to its multiplicity. To get curves with only double points, one uses birational transformations which are only defined on the given curve (and not in the plane).

It is during the same period, and in the same school, that n-dimensional algebraic geometry comes into its own for any value of $n \geqq 1$ (all algebraic varieties being considered as subvarieties of some $\boldsymbol{P}_n(\boldsymbol{C})$). As we shall see below, the study of algebraic varieties of dimension $\geqq 2$ was to have important repercussions on the theory of algebraic curves, with the concept of algebraic correspondences as subvarieties of a product variety, and the study of abelian varieties. We only mention here another fruitful consequence, the relation between linear series of points and rational mappings of an irreducible curve Γ into a projective space $\boldsymbol{P}_r(\boldsymbol{C})$: such a mapping can be written

Theme E

$$\phi : \zeta \to (P_1(\zeta), P_2(\zeta), \cdots, P_{r+1}(\zeta)),$$

where the P_j are homogeneous polynomials in the homogeneous coordinates of ζ, all of the same degree: if Γ' is the image of Γ by ϕ, the points of intersection of Γ by the system of curves $\sum_j \lambda_j P_j = 0$ are the inverse images by ϕ of the points of intersection of Γ' by variable hyperplanes. This observation, in connection with the theory of linear series, enables one to choose the P_j in such a way that ϕ is a birational transformation and Γ' has *no singular points*. Furthermore, the curve Γ' having these properties is uniquely determined up to a birational and *bijective* transformation (one says it is *the* nonsingular *model* of the field of rational functions of Γ).

VI c: Integrals of differential forms on higher dimensional varieties. As soon as 1870, Cayley, Clebsch and M. Noether inaugurated the study of abelian integrals on irreducible algebraic surfaces, by considering, on a surface S in $\boldsymbol{P}_3(\boldsymbol{C})$ given by an equation $F(x,y,z) = 0$ in nonhomogeneous coordinates, double integrals of type $\iint R(x,y,z)dxdy$, where R is a rational function; after 1885, Picard began a thorough investigation of the properties of these integrals, as well as of simple integrals $\int P(x,y,z)dx + Q(x,y,z)\,dy$, where P, Q are rational and the differential is exact*. His method, which (conveniently generalized) is still very useful, consists in looking at the sections of the surface by the planes $y = \text{const.}$, applying Riemann's theory to abelian integrals on these curves (which in general are irreducible), and studying the way in which they depend on the parameter y; in particular, if p is the genus of the curve for general values of y, the $2p$ periods of the abelian integrals of the first kind satisfy a linear differential equation or order $2p$ (as functions of y), the so called Picard-Fuchs equation, which plays an important part in the theory. The algebraic surfaces considered by these mathematicians were usually supposed to be without singular points, or at most to have only "nice" singularities (double curves with distinct tangent planes except at finitely many points and no singular points except finitely many triple points); starting with M. Noether, many attempts were made to prove that any algebraic surface could be transformed into surfaces without singularities (not necessarily immersed in $\boldsymbol{P}_3(\boldsymbol{C})$,

Theme F

Theme C

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* The exact meaning of a simple integral $\int P(x,y,z)\,dx$ consists in assigning to each piecewise differentiable mapping $t \to (x(t), y(t), z(t))$ of an interval $[a,b] \subset \boldsymbol{R}$ into S (a "singular 1-simplex") the number $\int_a^b P(x(t), y(t), z(t))\, x'(t)\,dt$. Similarly, the double integral $\iint R(x,y,z)\,dxdy$ assigns to each piecewise differentiable mapping $(u,v) \to (x(u,v), y(u,v), z(u,v))$ of a triangle $T \subset \boldsymbol{R}^2$ into S (a "singular 2-simplex") the number

$$\iint_T R(x(u,v), y(u,v), z(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\,dudv.$$

One can then define in an obvious way the value of simple (resp. double) integrals over 1-chains (resp. 2-chains), i.e., formal linear combinations of 1-simplices (resp. 2-simplices) with coefficients in $\boldsymbol{Z}$ (or in $\boldsymbol{R}$, or in $\boldsymbol{C}$). Generalizations to higher dimensions are obvious, once one defines an n-simplex as a piecewise differentiable mapping of the "standard n-simplex" defined by the inequalities $x_j \geqq 0$ $(1 \leqq j \leqq n)$, $x_1 + x_2 + \dots + x_n \leqq 1$ in $\boldsymbol{R}^n$.

but in higher dimensional projective spaces), but no satisfactory proof was found until much later.

Very early it appeared that the theory of algebraic surfaces exhibited some features which had no counterpart in the theory of algebraic curves. Two irreducible surfaces without singularities may be birationally equivalent without being isomorphic. If p_g denotes the number of linearly independent double integrals of the first kind on an irreducible surface S (i.e., integrals which are finite over any 2-cell of S), the corresponding number for a surface S' birationally equivalent to S is not necessarily the same. The number p_g is the obvious counterpart of the genus of a curve; but very soon also, it was realized that the other definition of the genus of a curve, using the "adjoints" of Riemann, also generalized to surfaces, but might give a number p_a different from p_g (see in VIII-a its exact definition in modern terms); p_g was called the *geometric genus* and p_a the *arithmetic genus* of S, and the difference $q = p_g - p_a$ (which is always $\geqq 0$) the *irregularity* of the surface (for instance, Cayley found that for *ruled* surfaces $p_g = 0$ and $p_a < 0$ in general).

Theme A

It soon also became apparent that the properties of abelian integrals on a surface or a higher dimensional variety were to a large extent subordinate to the topological properties of the variety. H. Poincaré had particularly in mind the applications to algebraic geometry when, in 1895, he started to give mathematical substance to Riemann's intuition of higher dimensional "Betti numbers" by inventing the "simplicial" machinery which made rigorous proofs possible*; algebraic varieties (and more generally analytic varieties) are amenable to this technique due to the fact that they are *triangulable*, a fact for which Poincaré himself sketched a proof, which was later made entirely rigorous by van der Waerden. Using this machinery and the Picard technique of variable plane sections, Poincaré was able to bring to a satisfactory conclusion previous efforts by Picard and the Italian geometers and to prove that the irregularity q of an algebraic surface without singularity is equal to $R_1/2$, where R_1 is the first Betti number, and also equal to the number of independent simple abelian integrals of the first kind. Around 1920, Lefschetz considerably developed these techniques and generalized them to algebraic varieties of arbitrary dimension, concentrating in particular on the determination of the number of cycles on such a variety V which are homologous to cycles contained in algebraic subvarieties of V: for instance, if V is a projective variety of complex dimension n, and H a hyperplane section of V, the natural mappings

Theme F

* Let us recall that to an n-chain is attached a well determined $(n-1)$ - chain, its boundary; *n-cycles* are the n-chains whose boundary is 0, and the n-th *homology group* $H_n(M, \mathbf{Z})$ (resp. $H_n(M, \boldsymbol{R})$, resp. $H_n(M, \boldsymbol{C})$) of a manifold M, with coefficients in $\mathbf{Z}$ (resp. $\boldsymbol{R}$, $\boldsymbol{C}$) is the quotient of the group of n-cycles with coefficients in $\mathbf{Z}$ (resp. $\boldsymbol{R}, \boldsymbol{C}$) by the subgroup consisting of the boundaries of the $(n+1)$-chains. The Betti number R_p is the dimension of the real vector space $H_p(M, \boldsymbol{R})$.

$H_i(H,\mathbf{Z}) \to H_i(V,\mathbf{Z})$ of homology groups are bijective for $0 \leqq i \leqq n-2$ and surjective for $i = n-1$. He also showed that for an algebraic variety V, one had $R_{2p} > 0$, $R_p \geqq R_{p-2}$ for $p \leqq n$ (complex dimension of V) and that the Betti numbers R_{2p+1} of odd dimension were even.

VI d: Linear systems and the Italian school. The definition of divisors, given in VI-a, carries over to any field K finitely generated over $\mathbf{C}$; on a nonsingular model V having K as field of rational functions, the discrete nontrivial valuations of K now correspond to irreducible subvarieties of V of *codimension* 1. It is still true that $\deg((f)) = 0$ for principal divisors, and that $L(D)$ is a finite dimensional subspace of K for all divisors D. The concept of *linear system* of subvarieties of codimension 1 may therefore be associated to the notion of divisor as in VI-b. Around 1890, the Italian school of algebraic geometry, under the leadership of a trio of great geometers: Castelnuovo, Enriques and (slightly later) Severi, embarked upon a program of study of algebraic surfaces (and later higher dimensional varieties) generalizing the Brill-Noether approach *via* linear systems: they chiefly worked with purely geometric methods, such as projections or intersections of curves and surfaces in projective space, with as little use as possible of methods belonging either to analysis and topology, or to "abstract" algebra.

These limitations implied serious difficulties in the definition of the main concepts and the use of geometric methods. The chief trouble was that whereas on curves one can work almost exclusively with *positive* divisors, this is not the case any more for surfaces: for instance if $p_g = 0$, the canonical divisor (defined as in VI-a, but for meromorphic differential 2-forms) is not equivalent to a positive divisor, hence does not correspond to a linear system of curves. This compelled the Italians to introduce complicated "virtual" notions for linear systems, which obscured the significance of much of their results.

Working under such considerable handicaps, it is amazing to see how many new and deep results were discovered by the Italian geometers. It would be extremely long and intricate to describe these results in their own language (see for instance [**16**]) and we shall postpone the definition of the most important notions which they introduced until we can use the much simpler modern formulation.

Theme A

Let us only mention here a few of the beautiful theorems characterizing (up to birational equivalence) simple types of surfaces by the values of the arithmetical genus p_a and new invariants defined by, Enriques, the *plurigenera* P_k ($k \geqq 2$): a *rational* surface (i.e., birationally equivalent to a plane) is characterized by the relations $p_a = 0$, $P_2 = 0$, surfaces with $p_a < -1$ are ruled, whereas the surfaces such that $P_4 = P_6 = 0$ are either rational or ruled; finally, a surface for which $p_a = P_3 = 0$ and $P_2 = 1$ is birationally equivalent to the Enriques surface of degree 6 having the 6 edges of a tetrahedron as double lines (it is not a rational surface, although $p_g = 0$).

VII. SIXTH PERIOD: "NEW STRUCTURES IN ALGEBRAIC GEOMETRY" (1920–1950)

The general trend towards the unification of mathematics by the study of the *structures* underlying each theory, which started to get momentum in the 1920's, was particularly apparent in the development of algebraic geometry; the striking kinships between algebraic varieties and complex manifolds on the one hand, algebraic numbers on the other, which had been discovered in earlier periods, now became organic parts of the fundamental concepts of algebraic geometry. One of the effects of this broadened point of view was to loosen the exclusive grip held until then by projective and birational methods over algebraic geometry, and prepare the way for a far more flexible approach.

VII a: Kählerian varieties and the return to Riemann. Ever since Gauss's fundamental paper of 1826 on the theory of surfaces and Riemann's inaugural lecture of 1854 defining n-dimensional riemannian geometry, the concept of *differential manifold*, defined by "maps" and differentiable "transition functions" between maps*, had gradually become more and more precise as the fundamental topological concepts needed to express them were defined and studied in the last part of the 19th century and the beginning of the 20th. One of the most important developments in that direction was the introduction of the general concept of *exterior differential p-form* on a differential manifold (locally defined by expressions

$$\sum_{i_1 < i_2 < \cdots < i_p} A_{i_1 i_2 \cdots i_p}(x) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

in the local coordinates) and of their integrals on *p-chains* (generalizing the earlier notions of "curvilinear" and "surface" integrals), due to H. Poincaré and E. Cartan. At the very beginning of his papers on algebraic topology, Poincaré had pointed out the connection between the homology of a compact differential manifold V and the exterior differential forms on V (of which the classical Stokes' theorem is the simplest example). This was made precise by De Rham's famous theorems in 1931, starting from the duality between chains and forms given by the integral $\langle C, \omega\rangle = \int_C \omega$; due to the generalized Stokes' formula $\langle C, d\omega\rangle = \langle bC, \omega\rangle$ (where b is the boundary and d the exterior derivative), this yields a duality, pairing the real homology groups $H_i(V, \boldsymbol{R})$ of V and the cohomology groups $H^i(\Lambda)$**, where Λ is the "complex" of exterior differential forms

$$0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n \xrightarrow{d} 0 \qquad (n = \dim V), \tag{4}$$

(Λ^j is the $\boldsymbol{R}$-vector space of the j-forms).

* If M is a differential manifold of dimension $n, \phi: U \to \mathbf{R}^n$, $\psi: Y \to \mathbf{R}^n$ two maps of open sets U, V, of M onto $\mathbf{R}^n$, the "transition function" from U to V is the mapping (only defined when $U \cap V \neq \emptyset$) $x \mapsto \psi(\phi^{-1}(x))$ of $\phi(U \cap V)$ onto $\psi(U \cap V)$.

** $H^i(\Lambda)$ is the quotient of the kernel $d^{-1}(\Lambda^{i+1})$ by the image $d(\Lambda^{i-1})$.

A projective algebraic variety without singularity of (complex) dimension n has a natural underlying structure of differential manifold of dimension $2n$, but in fact it has a much richer structure. In the first place, it is a *complex* manifold, which means that for the "maps" which define the differential structure and which take their values in $\boldsymbol{C}^n$ ($= \boldsymbol{R}^{2n}$), the "transition functions" are *holomorphic*; it follows that the space Λ_C^p of (complex) differential p-forms for $1 \leqq p \leqq 2n$ decomposes naturally into a direct sum of vector spaces $\Lambda_C^{r,s}$ corresponding to the pairs of integers such that $r + s = p$; for $r \leqq n$ and $s \leqq n$, the forms in $\Lambda_C^{r,s}$ (called forms of *type* (r, s)) are those which for *complex* local coordinates $z^1, z^2, \cdots, z^n$, are written

Theme F

(5) $$\Sigma A_{j_1 \cdots j_r k_1 \cdots k_s}(x) dz^{j_1} \wedge \cdots \wedge dz^{j_r} \wedge d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_s},$$

where the $A_{j_1 \cdots k_s}$ are differentiable functions with complex values (not holomorphic in general). For $r > n$ or $s > n$, one takes $\Lambda_C^{r,s}$ as reduced to 0 by convention.

But this is not the end of the story. It is possible to define on a projective complex space, and by restriction on any complex compact submanifold of such a space (which is necessarily an algebraic variety by a theorem of Chow) a riemannian ds^2 which is *kählerian*, i.e., can be written locally as a hermitian form

$$ds^2 = \sum_{j,k} h_{jk} dz^j d\bar{z}^k \text{ with } h_{kj} = \bar{h}_{jk}$$

which has the property that the corresponding exterior 2-form (which is *real* valued)

(6) $$\Omega = (i/2) \sum_{h,j} h_{jk} d\bar{z}^k \wedge dz^j$$

is *exact* (i.e., $d\Omega = 0$).

Beginning around 1930, Hodge, in a series of remarkably original papers, showed how to use these facts to investigate the homology of compact kählerian varieties. On a riemannian manifold, Beltrami had shown that it is possible to define an operator which generalizes the usual laplacian, and therefore enables one to define *harmonic functions* on the manifold. By a very imaginative generalization, Hodge was able to define similarly, on any compact riemannian manifold, the notion of *harmonic exterior differential forms*, and to prove that there existed a unique such form in any cohomology class in any $H^j(\Lambda)$; from that result, he deduced the uniqueness and existence of a harmonic p-form having given periods on homologically independent p-cycles, thus obtaining a complete generalization of Riemann's fundamental result, and showing that Riemann's use of "Dirichlet's principle" was far more than a technical device (fortunately for Hodge, the theory of elliptic partial differential equations had advanced far enough to spare him the difficulties which had plagued Riemann's approach). Turning next to complex kählerian manifolds, the space H^p of harmonic p-forms with complex coefficients splits into a direct sum of $p + 1$ spaces $H^{r,s}$ consisting of (complex) harmonic forms of type (5), for $r + s = p$ (with $H^{r,s} = 0$ if $r > n$ or $s > n$); it can be shown that $H^{p,0}$ consists exactly of the *holomorphic* p-forms (or "differential forms of the first kind"), i.e., those for which in (5), $s = 0$ and the $A_{j_1 \dots j_p}$ are holomorphic. As complex conjugation transforms $H^{r,s}$ into $H^{s,r}$, they have the same dimension, and this shows that the dimension of H^p, i.e., the Betti number R_p, is even when p is odd. On the other hand, one easily verifies that the (real) 2-form Ω defined in (6) is harmonic, as well as all its exterior powers, which proves that $R_{2k} \geqq 1$ for every integer k. Finally, $\phi \mapsto \Omega \wedge \phi$ is shown to be an injective mapping of H^p into H^{p+2} for $p \geqq n - 2$, from which the inequality $R_{p+2} - R_p \geqq 0$ follows; all the Lefschetz's theorems on Betti numbers of algebraic varieties are thus "explained" and shown to belong in fact to the theory of kählerian manifolds (there are compact kählerian manifolds which are not isomorphic to projective algebraic varieties). We shall return to the Hodge's theory when in the next period it merges into sheaf cohomology.

VII b: Abstract algebraic geometry. It is well known that, from 1900 to 1930, the general concepts of algebra (mostly confined until then to real or complex numbers) were developed in a completely abstract setting, the notion of algebraic *structure* (such as group, ring, field, module, etc.) becoming the fundamental one and relegating to second place the nature of the mathematical *objects* on which the structure was defined. It was therefore quite natural to think of an "abstract" extension of algebraic geometry, in which the coefficients of the equations and the coordinates of the points would belong to an arbitrary field. Already Dedekind and Weber, in their 1882 paper, had observed that all their arguments only used the fact that the basic field was algebraically closed (and of characteristic 0, a notion which had not yet been defined then). Even notions which seem linked to analysis, such as derivatives and differentials, had algebraic counterparts: a *derivation* in a commutative ring A is an additive mapping $x \mapsto Dx$ of A into itself such that $D(xy) = x \cdot Dy + (Dx) \cdot y$, and a *differential* is an A-linear mapping $\omega\colon \mathfrak{D} \to A$ of the A-module of all derivations into A; for each $x \in A$, dx is the linear form $D \mapsto Dx$ on $\mathfrak{D}$, and p-forms are defined by the usual methods of exterior algebra. Theme G

The motivation for the development of abstract algebraic geometry was therefore a natural outcome of the progress of algebra; after 1930, a more powerful impulse was to come from number theory, as we shall see below.

As it was apparent that a large part of the foundations of classical algebraic geometry came from geometric intuition, more or less justified by appeals to analysis or topology, a thorough examination of the basic concepts, from the exclusive viewpoint of algebra, was necessary in order to carry out an ambitious program of algebraic geometry over an arbitrary field. This groundwork, which at the same time created most of modern commutative algebra, was chiefly due to E. Noether, W. Krull, van der Waerden, and F. K. Schmidt in the period 1920–1940, and to Zariski and A. Weil from 1940 on.

The first two of these mathematicians use the geometric language very sparsely; their results are almost always expressed in the language of rings and ideals, and it was only after 1940 that the importance of their work was properly appreciated: the decomposition into primary ideals in noetherian rings, the properties of integrally closed rings, the extensive use of valuations, the notion of localization and the fundamental properties of local rings are all due to them. (A local ring is a commutative ring A in which there is only one maximal ideal. The typical example consists of the rational functions (elements of the field $\mathbf{C}(X)$) for which a given point $\zeta \in \mathbf{C}$ is not a pole: they form the local ring of $\mathbf{C}(X)$ at the point ζ). A similar remark may be made on the foundational work of Zariski, probably the deepest one in that period; although it is usually expressed in the language of projective geometry, it mostly belongs to local algebra and its central position in algebraic geometry was only recognized in the next period. The contribution of F. K. Schmidt (in connection with his work on number theory which we describe below) essentially consisted in extending the Dedekind-Weber theory to curves defined over an algebraically closed field of *any characteristic*.

The most conspicuous progress realized during that period is the successful definition, in algebraic geometry over an arbitrary field, of the concepts of *generic point* and of *intersection multiplicity*, due to the combined efforts of van der Waerden and A. Weil. The Italians (not to speak of their predecessors) used these notions with a freedom which, to their critics of the orthodox algebraic school, bordered on recklessness. As long as the underlying field was $\mathbf{C}$, the notion of "elements in general position" could be easily justified by an appeal to continuity (although the Italians seldom bothered to prove that these elements formed *open sets* in the spaces they considered). On the other hand, Lefschetz had made the elementary but fundamental observation that when two subvarieties U, V of $\mathbf{P}_n(\mathbf{C})$, of complementary dimensions r and $n - r$, intersect transversally in simple points, the number of these points is equal (for convenient orientations) to the *intersection number* $(U \cdot V)$ of the *cycles* U, V, in the sense of algebraic topology; as this number is known to be invariant under homology, it was quite natural to take it as the number of intersections of U and V (counted with multiplicities) in the most general cases. This justified the extensive use of intersection multiplicity by the Italian geometers, in particular the "self-intersection" number $(C \cdot C)$ of a curve on an algebraic surface. (Unfortunately, the complexity of the Italian definitions was such that it was often impossible to be sure that the same words meant the same things in two different papers; hence the numerous controversies between geometers of that school, such as the one which occurred as late as 1943 between Enriques and Severi, see [**4**] and [**10**].)

These foundations of course disappeared in algebraic geometry over an arbitrary field, and this was one of the reasons why no algebraic proofs valid over any field (even of characteristic 0) had been found for the results obtained in the theory of algebraic surfaces by transcendental or geometric methods. In 1926, van der Waerden saw that to gain the freedom which Analysis gave for classical geometry over the complex field, one had only to return to the process which had allowed the passage from real to complex geometry, namely *enlarge* the field k to which the coefficients of the equations of a variety and the coordinates of its points are supposed to belong: if K is any extension of k, these equations are still meaningful when the coordinates are taken in K. Giving a general form to ideas which went back at least to Gauss, he introduced the idea of *specialization* over k of any set of elements $x_1, \cdots, x_m$ in an arbitrary extension K of k: it is a mapping which to each x_j assigns an element x'_j of an extension K' of k (which may be equal to K), in such a way that for every homogeneous polynomial $P \in k[X_1, \cdots, X_m]$ for which $P(x_1, \cdots, x_m) = 0$, one also has $P(x'_1, \cdots, x'_m) = 0$ (van der Waerden always works in projective spaces, or finite products of such spaces). Suppose then that V is an irreducible algebraic variety in $\boldsymbol{P}_n(k)$, and let K be the field of rational functions on V; one may assume that V is not contained in a hyperplane of $\boldsymbol{P}_n(k)$; for $1 \leqq j \leqq n$, the restriction ξ_j to V of the rational function $x \mapsto x^j/x^0$ (where $x^0, x^1, \cdots, x^n$ are homogeneous coordinates of a point, $x \in \boldsymbol{P}_n(k)$) is an element of K; if V_K is the variety in $\boldsymbol{P}_n(K)$ defined by the same equations as V, the point $(1, \xi_1, \cdots, \xi_n)$ belongs to V_K. Van der Waerden calls this point a *generic point* of V, for it is immediate to check that for *any* extension K' of k, *any* point of $V_{K'}$ is a specialization of $(1, \xi_1, \cdots, \xi_n)$. Such points can then be used in the same way as the "general points" of the Italians, despite their apparently tautological character: any theorem proved for generic points (and of course expressible by algebraic *equations* (not inequalities!) between their coordinates) is valid for *arbitrary* points of corresponding varieties. Van der Waerden then proceeded to apply this new tool with great virtuosity to many problems of algebraic geometry, and in particular to the definition of multiplicity of intersection of two varieties in abstract algebraic geometry, which had not yet been given a meaning except in the case of the intersection of two curves on a surface without singularity. However, Poncelet, as a consequence of his general vague "principle of continuity," had already proposed to define the intersection multiplicity at one point of two subvarieties U, V of complementary dimensions by having V (for instance) *vary* continuously in such a way that for some position V' all the intersection points with U should be *simple*, and counting the number of these points which collapsed to the given point when V' tended to V; in such a way, the total number of intersections (counted with multiplicities) would remain constant ("principle of the conservation of number"), and it is thus that Poncelet proved Bézout's theorem, by observing that a curve C in the plane belonged to the continuous family of all curves of the same degree m, and that in that family there existed curves which degenerated into a system of straight lines, each meeting a fixed curve Γ of degree n in n distinct points. Many mathematicians in the 19th century had extensively used such arguments, and in 1912, Severi had convincingly

Theme D

Theme C

argued for their essential correctness. The concept introduced by van der Waerden was based on similar ideas: under suitable conditions, the multiplicity of a solution $y = (y_0, \cdots, y_n) \in \boldsymbol{P}_n(k)$ of a system of equations $P_\mu(x, y) = 0$, where x is a point of an irreducible variety V, is the number of the solutions η of the system $P_\mu(\xi, \eta) = 0$, where ξ is the generic point of V, which specialize to y when ξ specializes to x. Using this definition, he was finally able to attach to every irreducible component C of the intersection of two irreducible varieties V, W of an "ambient" nonsingular variety U, an integer $i(C, V \cdot W;\ U) \geqq 0$, the multiplicity of C in $V \cap W$, provided *all* irreducible components of $V \cap W$ were "proper," i.e., had a dimension equal to $\dim V + \dim W - \dim U$.

Unfortunately, this restriction considerably reduced the usefulness of the notion of multiplicity. Using more powerful algebraic devices, A. Weil could define an intersection multiplicity $i(C, V \cdot W;\ U)$ when it is *only* supposed that C is proper (the other components of $V \cap W$ can have larger dimensions); furthermore, he showed that this number did not depend on the method used to define it (other, quite different methods, were later given by Chevalley and Samuel), once it possessed the "natural" properties similar to those of the intersection number in algebraic topology; this he showed to be the case for his definition, and it enabled him to develop in abstract algebraic geometry a calculus of "cycles" patterned on the calculus of chains introduced by Poincaré (irreducible subvarieties replacing simplices). In this context, divisors on an irreducible variety of dimension n were the cycles of dimension $n - 1$ (one also says that they have *codimension* 1).

Weil then went on to break away, for the first time, from projective algebraic geometry: for his purposes (see below) he needed constructions of algebraic varieties similar to the "gluing together" constructions of manifolds in algebraic topology or differential geometry, which had been familiar since the beginning of the century; he showed that this could be done by using as "transition functions" biregular mappings of complements of subvarieties in affine varieties (the Zariski topology was not yet in use at that time), and he could also define in this context the notion of "complete variety" which is the counterpart of the concept of compact space in "abstract" algebraic geometry (in classical projective geometry, all algebraic subvarieties are complete).

Theme E

VII c: Zeta functions and correspondences. A. Weil's work was chiefly motivated by problems which had arisen in the early 1920's in number theory. In his thesis of 1923, E. Artin had observed that algebraic congruences modulo a prime p, in 2 variables, i.e., of the form $F(x, y) \equiv 0 \pmod{p}$, where F is a polynomial with integral coefficients, could be interpreted as algebraic equations over the prime field $\boldsymbol{F}_p = \boldsymbol{Z}/p\boldsymbol{Z}$ (and similarly the "higher congruences" in the sense of Dedekind were algebraic equations over an arbitrary *finite* field $\boldsymbol{F}_q$ $(q = p^d)$). He further noticed that the analogy, already exploited by Dedekind and Weber, of finite extensions of the field $C(X)$ with algebraic number fields, was here much closer, since the residual fields of the valuations of a finite extension K of $F_q(X)$ are *finite* fields (extensions of F_q) just as for number fields (whereas they are equal to C in classical algebraic geometry). This enabled him to define, in complete analogy with the Riemann-Dedekind zeta function of an algebraic number field, the *zeta function of* K, and to extend to it the classical theory: functional equation and the location of the poles. However, his treatment was entirely algebraic, without any kind of geometric interpretation; a little later, F. K. Schmidt observed that a much simpler and more natural treatment was achieved if one completely modeled the theory after Dedekind and Weber, by introducing *divisors* (or "points of the abstract Riemann surface") instead of ideals; it can then easily be shown that the zeta function can be defined by the equation (for $u = q^s$)

$$\frac{d}{du}(\log Z(u)) = \sum_{m=1}^{\infty} N_m u^{m-1}, \qquad Z(0) = 1,$$

where N_m is the number of points of the curve whose coordinates belong to the extension F_{q^m} of F_q of degree m. It turns out that this function is much simpler than in the classical case; in fact it is a rational function

$$Z(u) = P_{2g}(u)/(1-u)(1-qu),$$

where P_{2g} is a polynomial of degree $2g$ (g being the genus of K). F. K. Schmidt further discovered the remarkable fact that the functional equation

$$Z(1/qu) = q^{1-g}u^{2-2g}Z(u)$$

was nothing else but the analytic expression of the Riemann-Roch theorem!

At the same time, arithmeticians had been endeavoring to obtain an evaluation of N_1, the number of points of the nonsingular curve Γ corresponding to K with coordinates in F_q, and had obtained estimates of the form $|N_1 - (q+1)| \leqq Cq^\alpha$, with C independent of q and $1/2 < \alpha < 1$; they had observed that $\alpha = 1/2$ would be the best possible result. Hasse became interested in the problem and remarked that the result was a consequence of the so-called "Riemann hypothesis for curves over finite fields," namely the fact that all the zeroes of the polynomial P_{2g} lay on the circle $|u| = q^{1/2}$, this fact implying the inequality

$$|N_1 - (q+1)| \leqq 2g \cdot q^{1/2} \tag{7}$$

in an elementary way. In 1934, he succeeded in proving this result for $g = 1$, by adapting to the case of finite fields ideas from the theory of complex multiplication of elliptic functions. He and Deuring observed furthermore that an extension to values $g \geqq 2$ would have to be based on the theory of correspondences.

This is what A. Weil proceeded to do. An irreducible correspondence between two irreducible curves Γ_1, Γ_2 is an irreducible curve on the surface $\Gamma_1 \times \Gamma_2$, and in general a *correspondence* between Γ_1 and Γ_2 is a *divisor* on $\Gamma_1 \times \Gamma_2$; degenerate correspondences are those of the form $\{x_1\} \times \Gamma_2$ or $\Gamma_1 \times \{x_2\}$ $(x_i \in \Gamma_i)$ and linear combinations of such with integral coefficients; correspondences are called *equivalent* if they differ by the sum of a principal divisor and a degenerate correspondence. For $\Gamma_1 = \Gamma_2 = \Gamma$, one defines as in set theory the *composition* $X \circ Y$ of two correspondences; it can be proved that, together with the addition of divisors, this defines on the set of equivalence classes $\mathfrak{A}(\Gamma)$ a structure of *ring* with unit element (the class of the diagonal Δ of $\Gamma \times \Gamma$). The degrees $d(X)$ and $d'(X)$ of a correspondence are defined as the integers, such that the first (resp. second) projection of X is the cycle $d(X) \cdot \Gamma$ (resp. $d'(X) \cdot \Gamma$); on the other hand, for two correspondences X, Y which intersect properly, $I(X \cdot Y)$ is the degree of the cycle $X \cdot Y$. One can then show that the integer

Theme B

$$S(X) = d(X) + d'(X) - I(X \cdot \Delta)$$

only depends on the equivalence class ξ of X, and has the property of a *trace*, i.e., $S(\xi \cdot \eta) = S(\eta \cdot \xi)$ for two elements of $\mathfrak{A}$. Furthermore, to each correspondence X is associated another one X', deduced from X by the symmetry automorphism of $\Gamma \times \Gamma$; if ξ, ξ' are the classes of X and X', one has $S(\xi \cdot \xi') \geqq 0$, equality being only possible for $\xi = 0$ in $\mathfrak{A}$. This theory was first developed in 1885 by Hurwitz, using Riemann's theory of abelian integrals, and the inequality for the trace was obtained by Castelnuovo (of course for the classical case); using his theory of intersection multiplicities, A. Weil was able to extend all these results to curves over arbitrary fields. He then observed that in the Hasse problem, the number N_m was exactly $I(F^m \cdot \Delta)$, where F is the "Frobenius correspondence" which to each point of Γ associates its transform by the automorphism of Γ corresponding to the automorphism $t \mapsto t^q$ of the algebraic closure of $\boldsymbol{F}_q$; from which it follows by definition that $S(F^m) = 1 + q^m - N_m$, and expressing the inequality $S(\xi \cdot \xi') \geqq 0$ where ξ is the class of $a \cdot \Delta + b \cdot F^m$, for arbitrary integers a, b, one gets $|N_m - q^m - 1| \leqq 2g \cdot q^{m/2}$, which generalizes (7) and implies the "Riemann hypothesis."

VII d: Equivalence of divisors and abelian varieties. The introduction of varieties of arbitrary dimension had been particularly useful because it allowed to consider as points in a projective space of sufficiently high dimension geometric objects such as lines, conics, etc. In 1937, Chow and van der Waerden showed quite generally that it is possible to consider the irreducible algebraic subvarieties of given dimension and degree in a given $\boldsymbol{P}_n(k)$ as the points of some algebraic variety in a suitable $\boldsymbol{P}_N(k)$. From this result it follows that it is possible to give a precise meaning (for an arbitrary field k) to the concepts of "specialization of cycles" and of "algebraic family of cycles" which had been used in the classical case by the Italian school.

In particular, one can define the concept of *algebraic equivalence* of two divisors D_1, D_2 on a nonsingular variety V as meaning that they belong to a common irreducible algebraic family of divisors. Another concept of equivalence is *numerical equivalence*, meaning that for any curve C on V, the intersection numbers $(D_1 \cdot C)$ and $(D_2 \cdot C)$ are equal. If one denotes by G, G_n, G_a, G_l the group of divisors on V and its subgroups formed of divisors equivalent to 0 for numerical, algebraic and linear equivalence, one has $G \supset G_n \supset G_a \supset G_l$. Severi for the classical case, and Matsusaka for arbitrary characteristic proved that the group G_n/G_a is finite. A deeper result, proved by Severi for complex algebraic surfaces, following earlier results of Picard, is that the group G/G_n is a free finitely generated commutative group $\boldsymbol{Z}^\rho$; this result was extended by Néron for arbitrary fields and in any dimension. Finally, it was known since Riemann that for an irreducible algebraic curve over $\boldsymbol{C}$, the group G_a/G_l was naturally endowed with a structure of g-dimensional algebraic nonsingular variety (g being the genus of the curve) which, as a topological group, is isomorphic to a *complex torus* $\boldsymbol{C}^g/\Gamma$, where Γ is a lattice in $\boldsymbol{C}^g$ (discrete group isomorphic to $\boldsymbol{Z}^{2g}$); this variety is called the *Jacobian* of the curve, and it had been used since Clebsch to study the geometry on an algebraic curve. In general, a complex torus $\boldsymbol{C}^n/\Gamma$, where Γ is a lattice in $\boldsymbol{C}^n$ (isomorphic to $\boldsymbol{Z}^{2n}$) can only be given the structure of an algebraic variety if the lattice Γ satisfies certain bilinear relations which had been already found by Riemann; it is then called an *abelian variety*. The work of Picard and his successors proved that for an arbitrary nonsingular algebraic variety V over $\boldsymbol{C}$, the group G_a/G_l was again equipped with a structure of abelian variety, called the *Picard variety* of V. Following his work on the Riemann hypothesis, A. Weil developed the general theory of abelian varieties over an arbitrary field (as "abstract" varieties), and was able to define the Jacobian of a curve. Later work of Chow and Matsusaka proved that abelian varieties can still be imbedded in projective space in the general case, and extended to any field the definition of the Picard variety.

VIII. SEVENTH PERIOD: "SHEAVES AND SCHEMES" (1950–)

After 1945, the considerable progress brought in algebraic topology, differential topology and the theory of complex manifolds by the introduction of sheaves and spectral sequences (both due to J. Leray) completely renewed the concepts and methods of algebraic geometry, both "classical" and "abstract," simplifying old definitions and results and opening new ways leading to the solution of old problems.

VIII a: The Riemann-Roch theorem for higher dimensional varieties and sheaf cohomology. The Riemann-Roch problem for an irreducible algebraic variety V is the computation of the dimension $l(D)$ of the vector space $L(D)$ for an arbitrary divisor D on V by some formula similar to the Riemann-Roch theorem for curves (3).

The Italian geometers had attacked the problem for surfaces, but succeeded only in getting a *lower bound* for $l(D)$, expressed in terms of $\deg(D)$ and birational invariants of the surface S, of D and of $\Delta - D$ (where Δ is a canonical divisor).

Theme F

In the 1930's, study of differential geometry and in particular of E. Cartan's method of moving frames had finally led to the definition of *vector bundles* over a differential manifold M: such a bundle is a differential manifold $\mathbf{E}$ with a projection $p\colon \mathbf{E} \to M$ such that the fibers $p^{-1}(x)$ for any $x \in M$ are real (resp. complex) vector spaces of fixed dimension r (the *rank* of $\mathbf{E}$), and locally on M, $\mathbf{E}$ looks like the product of M and $\boldsymbol{R}^r$ (resp. $\boldsymbol{C}^r$); in other words each point of M has an open neighborhood U for which there is a diffeomorphism ϕ transforming $p^{-1}(U)$ onto $U \times \boldsymbol{R}^r$ (resp. $U \times \boldsymbol{C}^r$) in such a way that ϕ transforms *linearly* each fiber $p^{-1}(x)$ into $\{x\} \times \boldsymbol{R}^r$ (resp. $\{x\} \times \boldsymbol{C}^r$). A *section* of $\mathbf{E}$ is a differentiable mapping $s\colon x \mapsto s(x)$ of M into $\mathbf{E}$ such that $s(x) \in p^{-1}(x)$ for every $x \in M$. Over a complex manifold M, one can similarly define holomorphic vector bundles by taking $\mathbf{E}$ as a complex manifold, the projection p being holomorphic, the fibers $p^{-1}(x)$ complex vector spaces, and ϕ (in the above definition) being also holomorphic. Important examples of vector bundles are the *tangent bundle* $\mathbf{T}(M)$, where the fiber $p^{-1}(x)$ consists of the tangent vectors to M at x (so that the rank is $\dim(M)$), and the *bundle of p-covectors* on M, whose sections are the exterior differential p-forms on M (see VII a).

The concept of *divisor* can be generalized to arbitrary complex manifolds M: if (U_α) is an open covering of M, one considers in each U_α a meromorphic function h_α, such that in $U_\alpha \cap U_\beta$, h_β/h_α is holomorphic and $\neq 0$ everywhere; two such systems (h_α), (h'_λ) corresponding to coverings (U_α), (U'_λ) are identified if h_α/h'_λ is holomorphic and $\neq 0$ in $U_\alpha \cap U'_\lambda$ for any pair (α, λ) of indices, and these classes of systems (h_α) are called divisors on M. One sees that for projective algebraic varieties over $\boldsymbol{C}$, this notion coincides with the old one: for instance, if $M = \boldsymbol{P}_n(\boldsymbol{C})$, and $D = \sum_k m_k S_k$ is a divisor on M, where each S_k is an irreducible hypersurface defined by an equation $F_k(x_0, x_1, \cdots, x_n) = 0$, F_k being an irreducible homogeneous polynomial of degree d_k, one covers $\boldsymbol{P}_n(\boldsymbol{C})$ with the $n+1$ open sets U_j $(0 \leqq j \leqq n)$, U_j being defined by the relation $x_j \neq 0$; one can then take as meromorphic function h_j in U_j the function

$$x \mapsto x_j^{-d} \prod_k (F_k(x_0, \cdots, x_n))^{m_k}$$

with $d = \sum_k m_k d_k$. In 1950, A. Weil observed that to a divisor D on a complex manifold M was naturally attached a complex vector bundle of rank 1 (what one calls a *line bundle*) $\mathbf{B}(D)$: with the previous notations, one "glues together" the complex manifolds $U_\alpha \times \boldsymbol{C}$ by taking as "transition function" from U_α to U_β the function $(x, z) \mapsto (x, (h_\beta(x)/h_\alpha(x))z)$, holomorphic in $(U_\alpha \cap U_\beta) \times \boldsymbol{C}$. Furthermore, if s is a holomorphic section of $\mathbf{B}(D)$, the restrictions s_α of s to U_α are such that in $U_\alpha \cap U_\beta$ one has $s_\beta = (h_\beta/h_\alpha)s_\alpha$, hence there is a meromorphic function f on M such that the restriction of f to U_α is s_α/h_α for each α; for an algebraic variety M this is equivalent to $(f) + D \geqq 0$, and therefore $L(D)$ can be interpreted as the vector space $\Gamma(\mathbf{B}(D))$ of all *holomorphic sections* of the line bundle $\mathbf{B}(D)$. For instance, if $M = P_n(C)$, and $D = H$, a hyperplane in $P_n(C)$, the transition functions for $\mathbf{B}(H)$ are

$$(x, z) \mapsto \left(x, \frac{x_k}{x_j} z\right)$$

in $U_j \cap U_k$ (with the notations introduced above), and $\Gamma(\mathbf{B}(H))$ is the vector space of all linear forms $(x_0, \cdots, x_n) \mapsto \lambda_0 x_0 + \cdots + \lambda_n x_n$ in C^{n+1}.

Now to each complex vector bundle $\mathbf{E}$ over a differential manifold M of dimension n are attached, for each even integer $2j \leqq n$, well determined elements $c_j(\mathbf{E})$ of the cohomology group $H^{2j}(M, Z)$ called the *Chern classes* of $\mathbf{E}$*; when M is a complex manifold of real dimension $2n$, the Chern classes of $\mathbf{T}(M)$ are simply written c_j $(1 \leqq j \leqq n)$ and called the *Chern classes of* M; the number $\langle c_n, M \rangle$ (where M is considered as $2n$-cycle) is the Euler-Poincaré characteristic

$$\chi(M) = \sum_{j=0}^{2n} (-1)^j R_j.$$

Using the interpretation of divisors by line bundles and Hodge's theory of harmonic forms, Kodaira was able in 1951 to obtain, for compact kählerian manifolds of complex dimension 2, a "Riemann-Roch formula" in which the missing terms from the formula found by the Italian geometers were expressed by means of Chern classes; in 1952 he found a similar formula for kählerian manifolds of dimension 3.

Meanwhile, H. Cartan and Serre had discovered that Leray's concept of *sheaf* led to a remarkably simple and suggestive expression of the main results of the theory of complex manifolds. The holomorphic functions in open sets of such a manifold M satisfy Leray's axioms: if $\mathcal{O}(U)$ is the set of the complex functions holomorphic in the open set $U \subset M$, then, for every open covering (V_α) of U, a function $f \in \mathcal{O}(U)$ is entirely determined by its restrictions $f | V_\alpha \in \mathcal{O}(V_\alpha)$, and conversely, given for each α an $f_\alpha \in \mathcal{O}(V_\alpha)$ such that f_α and f_β have the same restriction to $V_\alpha \cap V_\beta$ for all pairs (α, β), there exists an $f \in \mathcal{O}(U)$ such that $f | V_\alpha = f_\alpha$ for all α.

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* One can define the concept of *direct sum* of vector bundles over M by defining it locally in an obvious way; for any differentiable map $f: M' \to M$, one defines the "pullback" $f^*(\mathbf{E})$ of a vector bundle $\mathbf{E}$ over M as the submanifold of the product $M' \times \mathbf{E}$ consisting of the pairs (x', z) such that $f(x') = p(z)$. The Chern classes of $\mathbf{E}$ can then be characterized by the following conditions, where one writes $c(\mathbf{E})$ for the sum $\sum_{j=0}^{\infty} c_j(\mathbf{E})$ (the sum is finite since the groups $H^{2j}(M)$ are 0 for $2j > \dim M$; one writes by convention $c_0(\mathbf{E}) = 1$): (i) $c(f^*(\mathbf{E})) = f^*(c(\mathbf{E}))$, where on the right hand side f^*: $H^*(M, Z) \to H^*(M', Z)$ is the natural mapping deduced from $f: M' \to M$.

(ii) $c(\mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \cdots \oplus \mathbf{E}_m) = c(\mathbf{E}_1) c(\mathbf{E}_2) \ldots c(\mathbf{E}_m)$ for any direct sum of vector bundles $\mathbf{E}_j$ over M^1 (product taken in the cohomology ring $H^*(M, Z)$).

(iii) $c(\mathbf{B}(H)) = 1 + h_n$ for a hyperplane $H \subset P_n(C)$, $h_n \in H^2(P_n(C), Z)$ being the cohomology class orresponding to the homology class of the $(2n-2)$-cycle H by Poincaré duality.

The sheaf thus defined is called the structural sheaf of M and written $\mathcal{O}_M$; one writes $H^0(U, \mathcal{O}_M)$ instead of $\mathcal{O}(U)$. More generally, for any complex vector bundle $\mathbf{E}$ over M, one defines the sheaf $\mathcal{O}(\mathbf{E})$ by replacing $\mathcal{O}(U)$ by the set of sections $\Gamma(U, \mathbf{E})$ of $\mathbf{E}$ above U, written $H^0(U, \mathcal{O}(\mathbf{E}))$; in particular one writes Ω_X^p the sheaf corresponding to the complex bundle of p-covectors on M, so that $H^0(U, \Omega_X^p)$ is the set of holomorphic exterior differential p-forms on U; for a divisor D on M, one writes $\mathcal{O}_X(D)$ instead of $\mathcal{O}(\mathbf{B}(D))$.

There are many types of sheaves other than those derived from vector bundles, and the usefulness of sheaves derives from this versatility and from the many operations one can do with sheaves. In the first place, to a sheaf of groups $\mathscr{F}$ over M and to each point $x \in M$ is associated a group, the *stalk* $\mathscr{F}_x$ of $\mathscr{F}$ at x: for $\mathcal{O}(\mathbf{E}), \mathcal{O}(\mathbf{E})_x$ consists of the equivalence classes of sections of $\mathbf{E}$ over neighborhoods of x for the following relation: two sections are equivalent if they coincide on a neighborhood of x ("germs of sections"); the general definition of $\mathscr{F}_x$ is similar. For a sheaf of abelian groups $\mathscr{G}$ and a sheaf $\mathscr{N} \subset \mathscr{G}$ such that $\mathscr{N}_x$ is a subgroup of $\mathscr{G}_x$ for each x, one can then define a quotient sheaf $\mathscr{G}/\mathscr{N}$ such that $(\mathscr{G}/\mathscr{N})_x = \mathscr{G}_x/\mathscr{N}_x$. Each stalk $(\mathcal{O}_X)_x$ (written $\mathcal{O}_x$) is a local ring, and if $\mathscr{F}$, $\mathscr{G}$ are two sheaves such that $\mathscr{F}_x$ and $\mathscr{G}_x$ are $\mathcal{O}_x$-modules, then one can define a sheaf $\mathscr{F} \otimes \mathscr{G}$ such that $(\mathscr{F} \otimes \mathscr{G})_x = \mathscr{F}_x \oplus_{\mathcal{O}_x} \mathscr{G}_x$; one has $\mathcal{O}_x(D + D') = \mathcal{O}_x(D) \otimes \mathcal{O}_x(D')$ for divisors D, D'. The chief interest of sheaf theory is that sheaves of groups may be used to replace the *coefficients* in cohomology groups by "local coefficients" varying with $x \in M$. The cohomology groups $H^j(M, \mathscr{F})$ which one thus defines for each integer $j \geqq 1$ (one also writes $H^j(\mathscr{F})$) have the fundamental property that for any exact sequence of sheaves of abelian groups $0 \to \mathscr{N} \to \mathscr{G} \to \mathscr{G}/\mathscr{N} \to 0$, one has a "long exact sequence"

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$$(8)\ 0 \to H^0(\mathscr{N}) \to H^0(\mathscr{G}) \to H^0(\mathscr{G}/\mathscr{N}) \to H^1(\mathscr{N}) \to H^1(\mathscr{G}) \to H^1(\mathscr{G}/\mathscr{N}) \to H^2(\mathscr{N}) \to \cdots$$

Once these new tools were introduced in analysis it was soon recognized that the invariants introduced by the Italian school and by Hodge were easily expressed by sheaf cohomology. In the first place, if M is a compact connected kählerian variety of dimension n, Dolbeault and Serre proved that the corresponding space $H^{r,s}$ of harmonic forms of type (r, s) (see VII-a) is isomorphic to $H^s(\Omega_M^r)$; furthermore, for any divisor D on M, Serre discovered that there is a natural duality pairing the spaces

$$H^j(\mathcal{O}_M(D)) \text{ and } H^{n-j}(\Omega_M^n \otimes \mathcal{O}_M(-D)) = H^{n-j}(\mathcal{O}_M(\Delta - D))$$

"explaining" the intervention of the canonical divisor Δ in Riemann-Roch's theorem (3) (one has written $\Omega_M^n = \mathcal{O}_M(\Delta)$). By definition, the *geometric genus* of M can be written

$$(9)\qquad p_g = \dim(H^0(\Omega_M^n)) \text{ and also } p_g = \dim(H^n(\mathcal{O}_M))$$

by the isomorphism of $H^{r,s}$ and $H^{s,r}$; one has similar invariants for holomorphic exterior forms of all degrees $< n$. The *arithmetic genus* turns out to be the number

$$(10) \qquad p_a = \dim H^n(\mathcal{O}_M) - \dim H^{n-1}(\mathcal{O}_M) + \cdots + (-1)^{n-1}\dim H^1(\mathcal{O}_M)$$

and the plurigenera are given by

$$(11) \qquad p_k = \dim H^0(\mathcal{O}_M(k\Delta)).$$

In 1937, Eger and Todd introduced, on an algebraic nonsingular projective variety M of complex dimension n, "canonical" equivalence classes of algebraic cycles of dimension $n-j$, which later were recognized to correspond exactly *via* Poincaré duality, to the Chern classes c_j of M; furthermore, Todd discovered that the arithmetic genus of M could be computed by the formula

$$(12) \qquad (-1)^n p_a + 1 = \langle T_n(c_1, \cdots, c_n), M \rangle,$$

where T_n is a polynomial with rational coefficients in the Chern classes, defined by the following device: in the power series

$$\prod_{j=1}^{n} \frac{\gamma_j z}{1 - \exp(\gamma_j z)}$$

one considers the coefficient of z^n, which is a symmetric polynomial in the variables γ_j, and one expresses it in terms of the elementary symmetric functions of the γ_j; then one replaces each elementary symmetric function σ_j by c_j. For instance, the first three Todd polynomials are

$$T_1(c_1) = c_1/2, \qquad T_2(c_1, c_2) = (c_2 + c_1^2)/12,$$
$$T_3(c_1, c_2, c_3) = c_2 c_1/24.$$

In 1954, Hirzebruch generalized both Todd's result and the Riemann-Roch formulas of Kodaira by proving that for any divisor D on M, the expression

$$\dim H^0(\mathcal{O}_M(D)) - \dim H^1(\mathcal{O}_M(D)) + \cdots + (-1)^n \dim H^n(\mathcal{O}_M(D))$$

could be expressed as $\langle P(f, c_1, \cdots, c_n), M \rangle$, where f is the first Chern class of the bundle $\mathbf{B}(D)$, and P a polynomial which is obtained by the same device as above, starting from the power series

$$e^{fz} \prod_j \frac{\gamma_j z}{1 - \exp(\gamma_j z)}.$$

It was later recognized that in fact, Hirzebruch's formula was a particular case of a much more general theorem valid for all differential manifolds, the Atiyah-Singer index formula.

The Hirzebruch formula enables one to solve the Riemann-Roch problem when all cohomology groups $H^j(\mathcal{O}_M(D))$ are reduced to 0 for $j \geqq 1$. Kodaira found sufficient conditions for this fact to hold; for instance, it is true when one replaces D by $D + mH$ where H is the intersection of M and a hyperplane (in the projective space where M is imbedded) and $m > 0$ is large enough. He has also obtained a fundamental criterion for a compact kählerian manifold M to be isomorphic to a projective algebraic variety: there must exist on M a kählerian metric such that the cohomology class of the form Ω (equation (6)) in $H^2(M, \boldsymbol{R})$ belongs to $H^2(M, \boldsymbol{Q})$.

VIII b: The Serre varieties. In 1942, Zariski began a deep study of singularities of projective algebraic varieties over any field, in view of proving a desingularization theorem (which he succeeded to do for dimension $\leqq 3$ and over a field of characteristic 0); for that purpose, he used for the first time the general theory of valuations*, developed 10 years earlier by Krull. In the course of this work, he introduced the generalization of the "abstract Riemann surface" of Dedekind-Weber for an arbitrary field K of algebraic functions over a field k, defining it to be the set V of all valuations of K which vanish on k^*; but in addition, using ideas introduced a few years earlier by M. Stone, he defined on V (by purely algebraic considerations) a *topology* for which V became quasi-compact, although that topology is not Hausdorff in general: for instance, in the case of dimension 1, considered by Dedekind-Weber, the closed sets are V and all the finite subsets of V.

Theme C

By 1950 A. Weil observed that this "Zariski topology" could be defined on his "abstract varieties" (see VII-b); not only did it appreciably improve the exposition of the theory by allowing one to use a "geometric" language, but it also made possible a definition of *vector bundles* modeled on the classical one, and to extend to abstract varieties the relations between divisors and line bundles (see VIII-a). Going one step further, Serre, in 1955, had the idea to transfer in the same way the theory of sheaves to abstract varieties, using the Zariski topology instead of the usual one in Leray's definition. At the same time, he observed that the concept of sheaf made possible a much simpler definition of "abstract varieties," using the general idea of "ringed space" of H. Cartan, i.e., a topological space X on which is given a sheaf of rings $\mathcal{O}_X$; the advantage of this kind of structure is that it lends itself very easily to "gluing" ringed spaces along open subsets, the verification of the conditions of compatibility being usually trivial. In Serre's case the "pieces" which are glued together are *affine varieties* over an algebraically closed field k of

Theme G

* The only difference between the definition of a general valuation and the definition of a discrete valuation (see VI–a) is that the valuation may take its value in an *arbitrary* totally ordered group. For instance, the group $\boldsymbol{Z} \times \boldsymbol{Z}$ may be totally ordered by writing $(m, n) < (m', n')$ if either $m < m'$, or $m = m'$, and $n < n'$ ("lexicographic ordering"); one may then define on $\boldsymbol{C}(X, Y)$ a valuation with value in that totally ordered group by taking for $w(P)$, where P is a polynomial $\neq 0$, the smallest (m, n) in $\boldsymbol{Z} \times \boldsymbol{Z}$ for which the term in $X^m Y^n$ in P has a nonzero coefficient.

arbitrary characteristic: such a variety X is a (Zariski) closed set of some k^n (i.e., defined by polynomial equations), and $\mathcal{O}_X$ is the sheaf of rings such that for each open set $U \subset X$, $\mathcal{O}(U) = H^0(U, \mathcal{O}_X)$ consists of the restrictions to U of the rational functions $P(X)/Q(X)$ on k^n which are defined (i.e., $Q(x) \neq 0$) at every $x \in U$. Of course cohomology groups $H^j(\mathcal{F})$ can still be defined when $\mathcal{F}$ is a sheaf of modules over the rings $\mathcal{O}_x$; they are vector spaces over k and Serre computed the groups $H^j(\mathcal{O}_M(mH))$ for $M = \boldsymbol{P}_n(k)$ and H a hyperplane ($m \in \boldsymbol{Z}$); he also extended to arbitrary fields and to projective varieties his duality theorem; but when k has characteristic $p > 0$, most of the results obtained in the classical case by the methods of Lefschetz and Hodge fail to generalize: for instance, the dimension of $H^r(\Omega_X^s)$ and of $H^s(\Omega_X^r)'$ for a projective variety X are not necessarily equal. Nevertheless, Grothendieck and Washnitzer were able independently to extend Hirzebruch's formula to fields k of arbitrary characteristic, and Grothendieck, by the introduction of his "K-theory," gave a far reaching generalization of that formula. Finally, when k is the complex field, Serre showed that the cohomology groups obtained by using the Zariski topology coincided with the classical ones.

Being chiefly interested in cohomology, Serre did not dwell at length on the general properties of his varieties; these were investigated in detail by Chevalley almost simultaneously (in a different language, which we do not reproduce here). One of the points which should be emphasized is that with Serre and still more with Chevalley, birational geometry fades out of the picture and the concept of *morphism* comes to the fore. Until then, the center of interest was the theory of *complete* varieties, and it is only seldom that a correspondence between two such varieties X, Y, even if it assigns only one point of Y to a point of X (a $(1, n)$-correspondence in classical language), is defined at *every* point of X. A morphism $f: X \to Y$, where X and Y are Serre varieties, is on the contrary a *mapping of* X into Y, which is continuous for the Zariski topologies and such that for every point $x \in X$ and every affine neighborhood V of $y = f(x)$, there is an affine neighborhood U of x such that $f(U) \subset V$ and, for every function $s \in H^0(V, \mathcal{O}_Y)$, the function $x \mapsto s(f(x))$ defined in U, belongs to $H^0(U, \mathcal{O}_X)$. The main results of Chevalley are general theorems on morphisms and studies of special types of morphisms using results of commutative algebra going back to E. Noether and Krull. It had been known for a long time that the image $f(X)$ of X by a morphism $f: X \to Y$ was not even locally closed in Y in general; Chevalley showed however that when X is irreducible, $f(X)$ always contains a set which is open and dense in the subspace $\overline{f(X)}$ of Y. Another of Chevalley's results is that if X and Y are irreducible, and for each $x \in X$ one writes $e(x)$ the maximum of the dimensions of the irreducible components of $f^{-1}(f(x))$ which contain x, then the mapping $x \mapsto e(x)$ is upper semi-continuous in X (in other words, when x' is close enough to x, $e(x')$ is never $< e(x)$).

Theme B

Chevalley also showed how important concepts introduced by Zariski in the 1940's, and which A. Weil had already used in his theory of abstract varieties, led to very suggestive theorems on morphisms. For projective varieties, Zariski had observed that the "regularity" properties of a point $x \in X$ were linked very closely to the structure of the *local ring* $\mathcal{O}_x$ of the variety X at that point: x only belongs to one irreducible component if $\mathcal{O}_x$ has no zero divisors, and x is *simple* if $\mathcal{O}_x$ is a *regular* local ring (i.e., $\mathcal{O}_x$ is an integral domain whose field of fractions has a transcendence degree over the base field k (always assumed to be algebraically closed) equal to the dimension over k of the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$). A property, of which Zariski was the first to grasp the geometric significance, is the fact for $\mathcal{O}_x$ to be *integrally closed* in its field of fractions, in which case x is said to be *normal*. Zariski showed that simple (or normal) points of an irreducible variety formed an open dense set, and that the complement of the set of normal points has codimension *at least* 2. Furthermore, Zariski defined for each projective irreducible variety X its "normalization;" this can easily be extended to Serre varieties: for any finite extension L of the field of rational functions K of X, there is a variety X' and a morphism p: $X' \to X$ such that for each affine open set U of X, $p^{-1}(U)$ is an affine open set of X' and the ring $H^0(p^{-1}(U), \mathcal{O}_{X'})$ is the integral closure in L of the ring $H^0(U, \mathcal{O}_X)$; X' is called the normalization of X *in* L, and simply the normalization of X if $L = K$. The normalization of X is of course birationally equivalent to X, and its singular points form a subvariety of codimension $\geqq 2$; in particular, if X is a curve, X' has no singular points, and this is the simplest "desingularization" of a curve (valid in every characteristic).

The climax of Zariski's investigations on normal varieties had been his "main theorem" expressed in the language of birational correspondences; Chevalley showed that it implies a far more intuitive result about morphisms: suppose X and Y are irreducible and *normal* varieties, f: $X \to Y$ is a morphism such that $f(X)$ is dense in Y and each set $f^{-1}(y)$ is *finite* for $y \in Y$. Then f factorizes in $X \xrightarrow{g} Y' \xrightarrow{p} Y$ where Y' is the normalization of Y in the field of rational functions of X, and g is an *isomorphism* of X onto an *open* subvariety of Y'.

Finally, Chevalley defined the notion of *complete* variety in a much simpler way than before: X is complete if, for every variety Y, the second projection $X \times Y \to Y$ is a *closed* mapping.

The interest of Chevalley in such theorems was spurred by the theory of *algebraic groups*, which he and A. Borel brought to a high level of development during the 1950's; in that theory, both affine and complete varieties play an important part and the preceding theorems are powerful tools.

VIII c: Schemes and topologies. Until the 1950's, no one seems to have tried to give an *intrinsic* definition of an affine variety over an *algebraically closed field* k, independent of any imbedding of the variety in some "affine space" k^n, although the tools to do so were available since the 1890's. In his work on invariant theory, Hilbert had proved his famous "*Nullstellensatz*," one of the forms of which is that the maximal ideals of the algebra of polynomials $k[X_1, \cdots, X_n]$ are in one-to-one correspondence with the elements $z = (\zeta_1, \cdots, \zeta_n) \in k^n$, such an element corresponding to the ideal generated by the polynomials $X_1 - \zeta_1, \cdots, X_n - \zeta_n$. Just as Riemann attached to a projective curve the field of rational functions on that curve, so one may attach to an affine variety $V \subset k^n$ the ring $R(V)$ of the restrictions to V of all *polynomial* functions on k^n; this ring is a finitely generated algebra over k, which has no nilpotent elements (one says it is *reduced*); and by Hilbert's *Nullstellensatz*, the points of V are in one-to-one correspondence with the maximal ideals of $R(V)$. Conversely, it is readily seen that *any* reduced and finitely generated k-algebra has the form $R(V)$ for an affine variety determined up to isomorphism. Furthermore, when V is irreducible, it is even possible to define the sheaf $\mathcal{O}_V$ directly from the ring $R(V)$: for any open (Zariski) subset U of V which is defined as the set of points x such that $f(x) \neq 0$ for some $f \in R(V)$, one defines $\mathcal{O}(U)$ as the ring of rational functions of type g/f^m for $g \in R(V)$ and m a positive integer, and it is easy to see that this defines completely $\mathcal{O}_X$. Finally, if V, W are two affine varieties over k, we have seen above that to a morphism f: $V \to W$ corresponds a k-algebra homomorphism $R(f)$: $R(W) \to R(V)$; but the converse is also true, for Hilbert's *Nullstellensatz* implies that for any such homomorphism ϕ: $R(W) \to R(V)$, the inverse image $\phi^{-1}(\mathfrak{m})$ of a maximal ideal of $R(V)$ is again a maximal ideal in $R(W)$, and $\mathfrak{m} \mapsto \phi^{-1}(\mathfrak{m})$ is the morphism corresponding to ϕ. In the language of categories, which was beginning to be used in the late 1950's, the category of affine varieties over k was *equivalent* to the *dual* of the category of reduced finitely generated (commutative) k-algebras.

Following a suggestion of Cartier, A. Grothendieck undertook around 1957 a gigantic program aiming at a vast generalization of algebraic geometry, absorbing all previous developments and starting from the category of *all* commutative rings (with unit) instead of reduced finitely generated algebras over an algebraically closed field. If one wanted to define a category which would be equivalent to the dual of the category of all commutative rings, a nontrivial modification was needed from the start, since if ϕ: $A \to B$ is a homomorphism of rings (sending unit element on unit element), the inverse image $\phi^{-1}(\mathfrak{m})$ of a maximal ideal of B is not in general a maximal ideal of A, whereas the inverse image $\phi^{-1}(\mathfrak{P})$ of a *prime* ideal of B is always a prime ideal of A. It was thus necessary to take as the set replacing the affine variety the *spectrum* of A, i.e., the set Spec(A) of all *prime* ideals of A; closed sets in Spec(A) are defined as sets of prime ideals containing a given (arbitrary) ideal of A, hence a "Zariski topology" for which, however, finite sets are no longer closed in general; finally, using work of Chevalley and Uzkov on localization dating from the 1940's, it is possible to give a meaning to g/f^m even when f is a zero-divisor of A, hence to define the sheaf $\mathcal{O}_X$ on $X = \text{Spec}(A)$ in the same way as for affine varieties. The ringed spaces thus obtained are called *affine schemes* and they form a category equivalent to the dual of the category of all commutative rings; finally, the usual "gluing process" for ringed spaces yields the category of *schemes* by replacing affine varieties by affine schemes.

The experience of the last 10 years has convinced the specialists that, in spite of the much greater amount of commutative algebra techniques which it requires, the theory of schemes is the context in which the problems of algebraic geometry are best understood and attacked. Among the features which distinguish it from previous conceptual frames for algebraic geometry, let us mention only the few following ones:

(1) The notion of *generic point*, which had disappeared from the Serre-Chevalley theory, is now reintroduced in a natural way: for instance, if A is an integral domain, its (unique) generic point is the prime ideal (0) in Spec(A); its "generic" property is expressed by the fact that its *closure* is the *whole* space Spec(A), and thus continuity arguments in the Italian style (but in the Zariski topology!) are now again available.

(2) The predominance of "relative" versus "absolute" notions, or, put in a different way, the fact that most of the times what is studied is not a scheme but a *morphism* of schemes $f: X \to S$, where S is often quite arbitrary (one also says that the study of such morphisms, for fixed S, is the study of "S-schemes"). This is particularly apparent when it comes to imposing *finiteness conditions* (without *any* such condition, there is very little likelihood of ever getting any deep result): Grothendieck has shown that, except for cohomological notions, one may usually allow the "base scheme" S to be free from finiteness assumptions (such as being noetherian, or of finite dimension, etc.), and the results only depend on finiteness conditions for the morphism f; this allows considerable freedom in the "change of bases" (see below).

(3) Given two "S-schemes" $f: X \to S$, $g: Y \to S$, there is an essentially unique triplet consisting in an S-scheme $X \times_S Y$ and two morphisms $p_1: X \times_S Y \to X$, $p_2: X \times_S Y \to Y$ such that $f \circ p_1 = g \circ p_2$, which is the "categorical" *product* of X and Y over S: this means that, given two morphisms $u: Z \to X$, $v: Z \to Y$ such that $f \circ u = g \circ v$, there is a unique morphism $w: Z \to X \times_S Y$ such that $u = p_1 \circ w$ and $v = p_2 \circ w$ (there is no similar result for Serre varieties; it easily follows from the existence of the tensor product $B \otimes_A C$ of arbitrary A-algebras, where A is any ring).

Most of the time this fundamental process is applied to study the morphism $f: X \to S$ by replacing the "base" S by another one Y, in such a way that the new morphism p_2, which is now written $f_{(Y)}: X_{(Y)} \to Y$ (the notation $X_{(Y)}$ replacing $X \times_S Y$) can be more easily handled. This "change of base" is probably the most powerful tool in the theory of schemes, generalizing in a bewildering variety of ways the old idea of "extending the scalars." To give only one example, consider at any point $s \in S$ the residual field $k(s) = \mathcal{O}_s/\mathfrak{m}_s$ of the local ring $\mathcal{O}_s$ at that point; then $X_s = X \times_S \text{Spec}(k(s))$ has as underlying space the "fiber" $f^{-1}(s)$ in X and (provided f satisfies finiteness conditions) it can be considered as a "variety" over the field $k(s)$ (in a slightly more general sense than with Serre). In this way, an S-scheme X may be considered as a "family of varieties" X_s parametrized by S

(generalizing the old Picard method (see VI-c)) and many properties of S-schemes may be obtained by a study of the fibers X_s.

(4) It may seem strange at first that one should consider affine schemes $\mathrm{Spec}(A)$ even when A has *nilpotent elements* other than 0; but in fact, this also corresponds to geometric facts which were not taken into account by older theories. For instance, consider the parabola $y^2 - x = 0$ in $\boldsymbol{C}^2$ and the mapping which projects it on the x-axis; in the language of schemes, we consider the affine schemes $U = \mathrm{Spec}(\boldsymbol{C}[X, Y]/(Y^2 - X))$, $V = \mathrm{Spec}(\boldsymbol{C}[X])$ and the morphism $p\colon U \to V$ which corresponds to the natural injection $\boldsymbol{C}[X] \to \boldsymbol{C}[X, Y]/(Y^2 - X)$ which sends X onto the class of X. A maximal ideal $(X - \zeta)$ in $\boldsymbol{C}[X]$ is identified with the point $\zeta \in \boldsymbol{C}$, and the fiber $V_\zeta = p^{-1}(\zeta)$ is the affine scheme $\mathrm{Spec}(\boldsymbol{C}[Y]/(Y^2 - \zeta))$; now, if $\zeta \neq 0$, the ring $\boldsymbol{C}[Y]/(Y^2 - \zeta)$ is isomorphic to the direct sum of two fields isomorphic to $\boldsymbol{C}$, corresponding to the fact that the fiber has two distinct points; but if $\zeta = 0$, $\boldsymbol{C}[Y]/(Y^2)$ has nilpotent elements: the two points have become "infinitely near" one another. It turns out that this is a general phenomenon: nilpotent elements in the local rings of a scheme are the algebraic counterpart of "infinitesimal" properties, and their presence allows a much more natural and flexible treatment of these properties than in classical algebraic geometry (see e.g. [**8**]).

(5) If we return to the concept of affine Serre variety, corresponding to a reduced finitely generated algebra A over an algebraically closed field k, the points of the variety are not *all* points of $\mathrm{Spec}(A)$, but only the *closed* ones, corresponding to all homomorphisms $A \to k$ which are k-homomorphisms, i.e., such that the composition with the natural mapping $k \to A$ gives the identity on k; similarly, if one wants to consider the points of variety "with coordinates in a field K extension of k" (see VII-b), one has to consider homomorphisms $A \to K$ which by composition $k \to A \to K$ give the homomorphism defining the extension K of k. This idea has been greatly generalized by Grothendieck: for an S-scheme $X \to S$ the "points of X in an arbitrary S-scheme T" (or more briefly the "T-points" of X) are by definition the morphisms $T \to X$ which, composed with $X \to S$, give the structural morphism $T \to S$; if we denote by $\mathrm{Mor}_S(T, X)$ the set of these "S-morphisms," it can easily be shown that $T \mapsto \mathrm{Mor}_S(T, X)$ is a *functor* from the category of S-schemes to the category of sets, and that the knowledge of that functor entirely determines the S-scheme X, which is said to "represent" the functor. This idea has become a very fruitful principle allowing the definition of schemes by the functor which they "represent," which is generally much easier (provided one has general theorems establishing the "representability" of functors); in particular, one transfers in that way to the theory of schemes many classical constructions such as projective spaces, Grassmannians, Chow varieties, Picard varieties, and one is able to give a general meaning to the concept of "moduli" introduced by Riemann for curves.

(6) It was early recognized that the Zariski topology on schemes had some unpleasant features regarding "vector bundles:" natural definitions of S-schemes $X \to S$, which in classical geometry gave vector bundles X over S, did not have in

general the property of being "locally" products of a (Zariski) neighborhood and a "typical fiber" (one says that they are not "locally trivial" for the Zariski topology). However, Serre observed that in important cases, a mild "extension of the base" $T \to S$, where T is an "etale covering" of S (which corresponds in classical geometry to an unramified covering with finitely many sheets) was enough to restore "local triviality." Starting from this remark, Grothendieck conceived the idea of replacing the Zariski topology on S by a new structure, called "*etale topology*," which is not any more a topology in the usual sense; essentially it consists in replacing the usual open subsets of S (or rather their natural injections $U \to S$) by etale coverings of S (one may say that the open sets are now "out of the space" instead of being parts of it). The important fact is that he was able to transfer to this new concept the definition of sheaves and of sheaf cohomology, and to show that this "etale cohomology" can partly remedy to the defects of the usual (Zariski) sheaf cohomology for varieties over a field of characteristic $p > 0$.

IX. OPEN PROBLEMS

To have some idea of the dozens of problems on which algebraic geometers are now working, one may consult for instance the various reports in [**18**], [**19**], or [**20**]. We will conclude by mentioning very briefly some of the most conspicuous ones.

(1) The famous problem of "desingularization" of algebraic varieties over a field k has been solved by Hironaka in all dimensions, when k has characteristic 0, and this result has become a very powerful tool in many problems of algebraic geometry, both classical and "abstract." For fields of characteristic $p > 0$, the problem is still open in dimensions $\geqq 3$; for dimension 2, the desingularization theorem has been proved by Abhyankar in all characteristics.

(2) The problem of Riemann's "moduli" has attracted much attention during the last 20 years, both in classical and in abstract geometry: the general idea is to prove the existence of a variety (or scheme) whose points would correspond to isomorphism classes of curves of a given genus over a given field; the most comprehensive results to date are those of Mumford, who has proved the existence of such a scheme; but much remains to be done regarding the properties of that scheme. One has similar results when curves of given genus are replaced by abelian varieties of given dimension; but already for algebraic surfaces, very little progress has been made on similar problems. Even when one considers "local" problems, i.e., how algebraic structures depending on parameters may "deform" in the neighborhood of a point in the parameter space, the results are far from final.

(3) In spite of the progresses brought by "etale cohomology" (and other similar theories based on other types of "Grothendieck topologies"), the cohomological properties of varieties over a field of characteristic $p > 0$ are not yet well understood, and nothing has yet satisfactorily replaced the abelian integrals in that case. Central in these problems are the "Weil conjectures" which he formulated as extensions to algebraic varieties of arbitrary dimension of his work on the zeta function of algebraic curves over finite fields; some of them have been proved by Grothendieck

and M. Artin, using etale cohomology, but the extension of the "Riemann hypothesis" has up to now resisted all efforts.

(4) In classical algebraic geometry, the theory of integrals of "second" of "third" kinds on projective algebraic varieties of arbitrary dimension is still incomplete, although much advanced recently by the work of Leray, Hodge-Atiyah and Griffiths on the concept of "residue." Generalizations of the Hodge theory to non compact algebraic varieties (over C) with singularities have recently been started by Deligne and others.

(5) One would expect that the precise knowledge of divisors under various "equivalence" concepts (see VII-d) should extend to "cycles" of arbitrary condimension, but even in the classical case that theory is still in an embryonic stage.

(6) Finally, the beautiful results of Castelnuovo and Enriques on the characterization of classes of surfaces by properties of their invariants have been greatly extended by Kodaira and Shafarevich [**11**], and generalized by Mumford to surfaces over an algebraically closed field of characteristic $p > 0$ [**19**], but much remains to be done, and practically no comparable results have been obtained in higher dimensions.

References

1. M. Baldassari, Algebraic varieties, Ergeb. der Math., Heft 12, Springer, Berlin-Göttingen-Heidelberg, 1956.

2. J. Dieudonné, Algebraic geometry, Advances in Math., 3 (1969) 233–321.

3. ———, Fondements de la géométrie algébrique moderne, Advances in Math., 3 (1969) 322–413.

4. F. Enriques, Sui sistemi continui di curve appartenenti ad una superficie algèbrica, Comm. Math. Helv., 15 (1943) 227–237.

5. F. Hirzebruch, Topological methods in algebraic geometry, Springer, Berlin-Heidelberg-New York, 3rd ed., 1966.

6. S. Lefschetz, L'Analysis Situs et la Géométrie algébrique, Gauthier-Villars, Paris, 1924.

7. D. Mumford, Geometric invariant theory, Erg. der Math. Heft 34, Springer, Berlin-Heidelberg-New York, 1965.

8. ———, Lectures on curves on an algebraic surface, Princeton Univ. Press, Princeton, 1966.

9. ———, Abelian varieties, Oxford Univ. Press, Oxford, 1970.

10. F. Severi, Intorno ai sistemi continui di curve sopra una superficie algèbrica, Comm. Math. Helv., 15 (1943) 238–248.

11. I. Shafarevich et al. Algebraic surfaces, Proc. Steklov Inst. of Math., Amer. Math. Soc., 1967.

12. B. L. van der Waerden, Einführung in die algebraische Geometrie, Springer, Berlin, 1939.

13. A. Weil, Foundations of algebraic geometry, Amer. Math. Soc. Coll. Publ., 29 (1946).

14. ———, Sur les courbes algébriques et les variétés qui s'en déduisent, Hermann, Paris, 1948.

15. ———, Introduction à l'étude des variétés kählériennes, Hermann, Paris, 1958.

16. O. Zariski, Algebraic surfaces, Erg. der Math. 2nd ed., Springer, Berlin-Heidelberg-New York, 1971.

17. ———, An introduction to the theory of algebraic surfaces, Lecture Notes in Math., 83, Springer, Berlin-Heidelberg-New York, 1969.

18. *Dix exposés sur la cohomologie des schémas*, North-Holland, Amsterdam-London, 1968.

19. *Global Analysis* (*Papers in honor of K. Kodaira*), Princeton Univ. Press, 1969.

20. *Actes du Congrès international des mathématiciens*, Nice, 1970, vol. I et II, Gauthier-Villars, Paris, 1971.