ARE THERE COINCIDENCES IN MATHEMATICS?

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This, too, is probable, according to that saying of Agathon: "It is part of probability that many improbable things will happen."
—Aristotle: *Poetics*

The twentieth century may be partially characterized as far as mathematics is concerned as the century of intensive abstraction. The genesis and the program of abstraction was summed up in the first decade of the century by E. H. Moore of the University of Chicago, who wrote:

> The existence of analogies between the central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to their central features.

Putting it slightly differently, abstraction perceives the common features of diverse theories and fits them into a superstructure wherein they will be unified with respect to this common feature. There was clear apprehension on Moore's part as to what was happening and a prophetic vision and zeal as to what would happen.

As common examples of abstraction, one may cite the algebraic structures of all sorts: semigroups, groups, rings, fields, manifolds, and abstract spaces, such as Hilbert and Banach spaces. But there are many others. As I write, I have before me books on abstract optimality theory, abstract harmonic analysis, and so on. The whole program of abstraction has been so pervasive and so influential that it would not miss the mark by much to say that, in the view of many contemporary mathematicians, mathematics is precisely the study of abstract structures of all sorts.

Returning to Moore's observation that diverse theories are unified with respect to their *common features*, we may now raise the embarrassing question: What is a common feature of two diverse theories? How does one recognize such a feature? And having recognized such a feature, from whence does one derive the confidence that it is a worthwhile task to unify along such a feature? This is not a question of mathematics as such, but a question in the history, psychology, aesthetics, and applications of mathematics, and ultimately it is answerable only in terms of a mathematical culture and certain values that are operative in it. I know of no way of defining—before the act—what a "common feature" means; this is allied to the old philosophical question of universals. But the act of abstraction does it for me. This is precisely one of its functions.

Having raised the question of possible definitions, I should like to twist Moore's quotation still further and ask: Is it possible that a common feature is a coincidence of form which one has observed in several different places? And I mean here to stress the word *coincidence* almost in its surprise aspect. I think that one variety, at least, of a common feature may be identified with coincidence.

I cannot define coincidence. But I shall argue that coincidence can always be elevated or organized into a superstructure which performs a unification along the coincidental elements. The existence of a coincidence is strong evidence for the existence of a covering theory.

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I shall elucidate the experience of coincidence with a number of examples. The examples selected are formed, for the most part, from the simplest of mathematical ingredients.

Example 1. First, a personal experience. In the eighth grade of elementary school I learned that the diagonal of a rectangle was related to the sides by the Formula of Pythagoras,

$$d^2 = a^2 + b^2.$$

As an adjunct, I learned to carry out the laborious square root algorithm with pencil and paper. A week or so later, I learned that the diagonal of a three-dimensional box was related to the sides of the box by the formula

$$d^2 = a^2 + b^2 + c^2.$$

I recall distinctly that the feeling came over me that this was a great coincidence. Why was the formula not something like

$$d^2 = a^2 + b^2 + c^2 + (abc)^{2/3}?$$

There were plenty of difficult forms knocking around in my textbook. (The formula just given reduces to the two-dimensional case when a, b, or $c = 0$ and is dimensionally correct. But of course it does not reduce properly on all subspaces, which is precisely how the correct formula is obtained.) Moreover, I was grateful that on account of this coincidence I would not have to learn an additional laborious algorithm in order to compute the diagonals of a box. Not so if one wants the side of a cubic box, given its volume.

From the point of view of a mature mathematician, the use of the word *coincidence* in this connection might be regarded as demeaning. What we have here, he might say, is no *mere coincidence*. It is a general theory that extends all the way from the rectangle to the box, from the box to the hyper box, to the box in countably infinite dimensional Hilbert space, to the box in nonseparable spaces, to the ... It is part of a great, useful, expansible theory, part of a Grand Plan, part of the Decreed Order of the Universe. He might assert all this, forgetting for the moment that he has had to temper the Hilbert universe in such a way that the box makes sense:

$$a^2 + b^2 + c^2 + \cdots < \infty,$$

and that some authorities advocate regarding the theorem as a *definition*. (See [**9**, p. 36].)

In this view, to talk about coincidences, *mere* coincidences, one would have to fasten upon a parallelism that is totally "accidental," "uncaused," something paltry, something almost to be disregarded and certainly not worthy of being honored with a proof.

Example 2. The numbers π and e are the outgrowths of substantial, but different, theories. Write down the decimal expansion of π and e:

Number of Digits

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
π	3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3
e	2.	7	1	8	2	8	1	8	2	8	4	5	9	0	4	5

Notice the coincidence in the 13th digit of π and the 13th digit of e (and no earlier digit). I call this a mere coincidence, because our first reaction upon seeing it may very well be "So what!" This, despite the number-mystical fact that the coincidence occurs at the 6th prime. I can call it a coincidence despite the fact that the statement

"The thirteenth decimal digits of both π and e are identical"

is, in fact, a mathematical theorem and can be proved rigorously by established methods of mathematical proof.

As we all know, it is possible in different ways to elevate this *mere* coincidence into something more substantial. Let us look for further coincidences:

Number of Digits

	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
π	2	3	8	4	6	2	6	4	3	3	8	3	2	7	9	5	0	2	8	8
e	2	3	5	3	6	0	2	8	7	4	7	1	3	5	2	6	6	2	4	9

In this way, we have discovered further coincidences at the 17th, 18th, 21st, and 34th digits. Our experience with this kind of problem now tells us that it is extremely unlikely that we can discover an elementary formula for the location of the nth coincidence; but exposure to probabilistic thinking, which is designed specifically to elevate chaos and mere coincidence to a Grand Plan, would lead us to say that we observed 5 coincidences in 36 observations. If, as seems likely, the digits of π and e are distributed at random and wholly independently from among $0,1,2,\dots,9$ (again, which seems likely psychologically, if not logically), then we would expect, on average, one coincidence every ten digits. Since the digits of π and of e are now available to hundreds of thousands of decimal places, a devotee of this problem might like to count the coincidences and subject them to a statistical analysis via the binomial distribution. (In a later example we refer to a similar investigation.) At any rate, the statement

"The decimal digits of π and e coincide, on average, once every ten digits"

is, to my knowledge, neither proved nor disproved. It is difficult of proof or disproof, but is credible and psychologically satisfying. Furthermore, the statement and the denial of the statement are both interesting; and if the denial turned out to be true, we would be inclined to decorate it with an exclamation point.

Example 3. Consider the formula

$$x = \sqrt{1141y^2 + 1}\,.$$

For $y = 1,2,3,\dots,100$, it is an easy job to show on a hand-held computer that the corresponding x is *not* an integer. We may be led to wonder whether x is ever an integer. The truth of the matter is that it is not an integer for $1 \leqslant y \leqslant 10^{25}$. The first value of y for which x is an integer is

$$y = 30{,}693{,}385{,}322{,}765{,}657{,}197{,}397{,}208.$$

Now we have no *direct* computational experience of the dearth of integer solutions in the range $1 \leqslant y \leqslant 10^{25}$. The anomaly is explained within the Theory of the Pell Equation, which informs us that if the (periodic) continued fraction expansion of $\sqrt{d}$ has a long period then the first solution of the Diophantine equation

$$x^2 - dy^2 = 1$$

will be exceedingly large. The solution just quoted has been arrived at backwards via a comprehensive theory whose existence could have been surmised from experiments in the more modest range $1 \leqslant y \leqslant 1000$.

(See [**19**].)

Example 4. Consider the two numbers

$$A = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$$

$$B = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}\,.$$

I ask you now to go to a computer and, using a language that does multiple precision arithmetic, compute the numbers A and B. If you do, you will find that A and B agree at the very least to 26 significant figures:

$$7.38117\ 59408\ 95657\ 97098\ 72669.$$

Now either (1) This is an incredible coincidence, or (2) $A = B$ is an incredible identity, inasmuch as A and B do not appear "to the naked eye" to lie in the same algebraic field.

If $A \neq B$, then we would still be left with the necessity of elevating the coincidence by placing it within a more comprehensive theory. After all, you just cannot throw integers and square roots around at random and expect to get 26-figure coincidence. There would have to be something behind the coincidence; and, whatever it is, that something would probably enable us to construct further incredibilities of a similar sort.

The reader may infer from the title of one of the references whether $A = B$ or $A \neq B$.

I first heard of this and similar coincidences through H. O. Pollak of the Bell Telephone Laboratories. His interest in the matter lay in an auxiliary question. Let A and B be irrational algebraic numbers. How can one prove or disprove the equation $A = B$ by systematic rational operations? (For example, the identity

$$1 + \sqrt{3} = \sqrt{3 + \sqrt{13 + 4\sqrt{3}}}$$

can be established by two squarings) and can this be carried out in polynomial time? At the time of writing, the last question had not yet been resolved.

(See **[15]**, **[18]**.)

Example 5. In elementary plane geometry, there are numerous theorems which contain a strong flavor of coincidence. Thus:

(a) the three medians in any triangle intersect in a common point.

Not only do we have this but

(b) the three angle bisectors intersect in a point, and
(c) the three altitudes intersect in a point.

Each of these circumstances represents a coincidence. Many observers view (a), (b), (c) as a further coincidence since their conclusions are identical: "They intersect in a point." This experience may be part of the reason for elevating the separate instances into a comprehensive theory:

CEVA'S THEOREM. *Let the sides of a triangle ABC be divided at L, M, N in the respective ratios* $\lambda : 1$, $\mu : 1$, $\nu : 1$. *Then the three lines AL, BM, CN are concurrent if and only if* $\lambda\mu\nu = 1$.

Now the ideas of (a), (b), (c) are easily extended to higher dimensional simplexes. What about the truth of the resulting statements? Focus attention on (c). Do the altitudes of a tetrahedron intersect? This is a problem about which the Harvard geometer J. L. Coolidge once wrote that most highly educated mathematicians do not know the answer. On the one hand, direct spatial intuition fails most of us. We simply do not have enough experience playing with wire tetrahedra and dropping perpendiculars. On the other hand, if one argues by verbal analogies, one is inclined to answer the question with "yes." *Se non è vero, è ben trovato.* If it is not true, it deserves to be true.

If, then, the statement is true, it is a coincidence of a more complex form than the simple triangle case ($n = 3$) and we should look for a fairly simple explanation. If the statement is false for $n = 4$, there are still, obviously, special instances of it which are true, e.g., the regular tetrahedra, and we would be inclined to deem interesting that class of tetrahedra for which it is true. In either case, then, natural curiosity and aesthetics would start from the sense of coincidence and force an appropriate theory to come into being.

(See [**2**].)

Example 6. A number of years ago, in the April issue of the *Scientific American*, Martin Gardner, who writes a regular column of mathematical recreation, ran a special "April Fool's" column. This was a collection of scientific hoaxes. One statement in particular caught the eyes of people who enjoy doing long computations:

"The number $N = e^{\pi\sqrt{163}}$ is an integer."

Now this statement is a shocker. If it is true, we would have a "fairly simple" real relationship between π and e. In view of all we know or have heard about such numbers, this does not seem likely. In view of the known complex relationship $e^{\pi\sqrt{-1}} = -1$, however, there might, perhaps, be a residue of doubt in our minds. N just might be an integer.

Well, let's go to the computer. But notice that a little hand-held computer is useless, for $e \approx 2.7$, $\pi \approx 3.1$, $\sqrt{163} \approx 12.8$, so that $N \approx e^{40} \approx 10^{17}$. Since hand computers carry 10 to 12 figures, we have got to go to a computer where we can work in multiple precision. Numbers such as $e, \pi, \sqrt{163}$ have been computed out to hundreds, if not thousands, of figures and have been published, but we had best write our own superaccurate programs for these numbers. At any rate, let us shoot for 20 figures.

A 20-figure computation yields $N = 262{,}53741{,}26407{,}68743.99$. Hmm. What have we here? The fractional part .99 is tantalizing. Let's go for 25 figures. The computer grinds away (requiring a time which is probably proportional to the cube of the number of figures desired), but finally prints out

$$N = 262{,}53741{,}26407{,}68743.99999\ 99.$$

The mystery thickens. Is it really possible that N is an integer? Of course one can't prove it by this kind of numerical computation, for however long a string of 9's were produced there would be some residue of doubt. But we could *disprove it*; so let's push the button once again and give the computer a really good whirl.

$$N = 262{,}53741{,}26407{,}68743.99999\ 99999\ 99250.$$

Well, that blows it. N isn't an integer after all. But notice: it differs from an integer by around 10^{-12}.

Now of course anyone well versed in the theory of transcendentals could have told us that $e^{\pi\sqrt{163}}$ is not an integer. It is what we think it should be: a transcendental number. This can be deduced as a consequence, for example, of the Gelfond-Schneider Theorem (1932), which tells us that if a is an algebraic number and is neither 0 nor 1, and if b is an algebraic irrational, then a^b is a transcendental.

Now $N = e^{\pi\sqrt{163}} = e^{-\pi i\sqrt{-163}} = (-1)^{\sqrt{-163}}$, so that by Gelfond-Schneider, N is transcendental. But the equation $N = \text{integer} + O(10^{-12})$ is remarkable, and surely requires some sort of an explanation. You just don't slap e's and π's and square roots around and expect to get integers to within 10^{-12}. (Forget obvious identities such as $\pi/\pi + e/e = 2$.) On the other hand, a 12-figure coincidence doesn't seem excessively long (certainly not as good as the 26-figure coincidence in Example 3); so one would expect that the explanation would be fairly complicated. And complicated it is.

The writer was first shown this mathematical amusement by Morris Newman in 1952 when he joined the National Bureau of Standards. Newman was a student of Hans Rademacher and had just computed N to high precision on one of the first electronic computers in the world: the SEAC. It is not unreasonable to suppose that the explanation of the coincidence is to be found in the kind of mathematics that Rademacher delighted in and was most skillful in: analytic and algebraic number theory. But we shall have to make a very long story short.

Let

$$\sigma_3(n) = \sum_{d|n} d^3 \quad \text{and} \quad x = e^{2\pi i \tau} \qquad (\operatorname{Im}\tau > 0).$$

Set

$$J(\tau) = \left\{1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) x^n\right\} \Big/ \prod_{n=1}^{\infty} (1 - x^n)^{24}.$$

J, as a function of x, has the Laurent expansion

$$J(\tau) = \frac{1}{x} + 744 + 196884x + \cdots.$$

The function $J(\tau)$ is invariant under the modular group $\tau' = (a\tau + b)/(c\tau + d)$, $(ad - bc) = 1$. It is known as the elliptic modular function, or Felix Klein's absolute modular invariant. Now the function $J(\tau)$ is intimately related to the theory of quadratic forms in two variables with integer coefficients. Let $Q(x,y) = ax^2 + bxy + cy^2$. Assume that $a > 0$, $d = b^2 - 4ac < 0$, and that a,b,c, have no common factor. Let $h(d)$ be the number of equivalence classes of Q under unimodular substitutions of determinant 1. It is a theorem of Weber that the quantity $J((-b + \sqrt{d})/2a)$ is an algebraic integer of exact degree $h(d)$. Hence, if $h(d) = 1$, then $J((-b + \sqrt{d})/2a)$ is an (ordinary) integer. It has been proved that there are only nine negative values of d for which $h(d) = 1$; they are $d = -1, -2, -3, -7, -11, -19, -43, -67, -163$.

If we now select $a = b = 1$, $c = 41$, then $d = -163$. Hence $J((-1 + \sqrt{-163})/2)$ is an integer. With $x = e^{2\pi i \tau}$ and $\tau = (-1 + \sqrt{-163})/2$, it follows that

$$x^{-1} = \exp\left(-2\pi i \frac{(-1 + \sqrt{-163})}{2}\right) = -e^{\pi\sqrt{163}}.$$

Therefore, by the Laurent expansion for $J(\tau)$,

$$J\left(\frac{-1 + \sqrt{-163}}{2}\right) = \text{integer} = -e^{\pi\sqrt{163}} + 744 + 196884e^{-\pi\sqrt{163}} + \cdots.$$

Hence,

$$e^{\pi\sqrt{163}} = \text{integer} + 744 + 196884e^{-\pi\sqrt{163}} + \cdots.$$

Since $e^{-\pi\sqrt{163}} \approx 10^{-17}$, we have

$$e^{\pi\sqrt{163}} = \text{integer} + O(10^{-12}),$$

as we have discovered by a brute calculation. The same device works for $e^{\pi\sqrt{n}}$ whenever $n = 1,2,\ldots,67,163$, but the value 163 naturally gives the sharpest agreement.

The reader needs scarcely to be informed that all of this was discovered by working forward, from the theory of modular invariants and quadratic forms, pushing toward the mysterious coincidence, rather than by starting from $e^{\pi\sqrt{163}}$ as a "shot in the dark" and working backward toward an explanation.

(See [**14**], [**21**], [**20**].)

Example 6′. Along lines that are similar to Examples 6 and 4, consider a problem posed by H. P. Robinson. Let $\zeta(x)$ be the zeta function and let

$$I = \int_0^{\infty} \{\zeta(x) - 1\} dx, \qquad L = \lim_{n\to\infty} \left\{ \sum_{k=2}^{n} \frac{1}{\ln k} - \int_0^{n} \frac{dx}{\ln x} \right\}.$$

Numerical agreement is reported to 43 places. Does $I = L$?

Query: Was numerical agreement obtained as a result of a computation or inferred from a proof of $I = L$?

(See [**16**].)

Example 7. The Riemann Hypothesis is currently the most notorious of unsolved problems of analysis, if not of the whole of mathematics. Riemann's zeta function is defined for $\mathrm{Re}\, z > 1$ by means of the Dirichlet Series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and is extended to the rest of the complex plane (where possible) by analytic continuation. The Riemann Hypothesis is that all the complex roots of the zeta function have real part equal to $\frac{1}{2}$. Many interesting consequences of this theorem are known, many equivalent formulations are known, but thus far a proof or a disproof has eluded investigators.

Many years ago, G. H. Hardy proved that infinitely many roots of the zeta function have real part equal to $\frac{1}{2}$. By laborious computations and laborious analyses of truncation errors, it has been shown that the first 70,000,000 roots all have the real parts equal to $\frac{1}{2}$. Coincidence? Or is this sufficient evidence to convince one of the truth of the Riemann Hypothesis?

Some authorities say no. They point out that in the theory of the zeta function, as well as in prime number theory, functions whose growth is as slow as that of $\log\log x$ occur often, and since $\log\log x = 10$ if x is around $10^{10{,}000}$, of what significance is a paltry 70,000,000? There is even a case in the literature of primes when a prevailing tendency was proved by Littlewood to cease prevailing when $n > 10^{10^{10^{34}}}$ (Skewes's number, since lowered.)

But Good and Churchhouse have latched onto another "coincidence." The Möbius function $\mu(n)$ is defined as follows:

$$\mu(n) = +1 \text{ if } n \text{ has an even number of distinct prime factors}$$

$$\mu(n) = -1 \text{ if } n \text{ has an odd number of distinct prime factors}$$

$$\mu(n) = 0 \quad \text{if } n \text{ has a repeated prime factor.}$$

The "visual" evidence shows that $\mu(n)$ behaves in an erratic fashion. One can show without too much difficulty that the probability

$$\text{that } \mu(n) = +1 \text{ is } 3/\pi^2$$

$$\text{that } \mu(n) = -1 \text{ is } 3/\pi^2$$

$$\text{that } \mu(n) = 0 \text{ is } 1 - 6/\pi^2.$$

Now, the Strong law of large numbers in probability theory tells us that if μ_n is a random variable selected N times with the probabilities just listed, then

$$\sum_{n=1}^{N} \mu_n < CN^{1/2+\epsilon}$$

with probability one. But it has been known for some time that the Riemann Hypothesis is equivalent to the inequality

$$\sum_{1}^{N} \mu(n) \leqslant CN^{1/2+\epsilon}.$$

Hence, if we could somehow identify N selections at random of μ from the distribution above with the first N deterministic values $\mu(1),\ldots,\mu(N)$, then the Riemann Hypothesis would be proved. To check numerically whether this identification is valid, Good and Churchhouse computed $\mu(n)$ for

$n \leqslant 33{,}000{,}000$. The number of n's for which $\mu(n) = 0$ was 12,938,407. Now

$$33{,}000{,}000(1 - 6/\pi^2) = 12{,}938{,}405.6.$$

Here we have 8-figure accuracy. Incredible!

Despite these coincidences, authorities have asserted that there is still but paltry evidence for believing in the truth of the Riemann Hypothesis. But it seems to me that this great coincidence shows either (1) the Riemann Hypothesis is true or (2) you have got a very difficult theorem on your hands to explain away (the 8-figure accuracy). And it is psychologically and almost morally incumbent upon you to do so.

(See [**5**], [**8**], [**11**], [**13**].)

Example 8. *Pappus's Theorem*; *Program Verification*. One of the principles that is drummed into every aspiring young mathematician is that one cannot prove a general theorem by merely proving special instances of it. One must present a proof that embraces all cases. Not infrequently does a student get his knuckles rapped for dealing only with a special case. (You assumed the triangle has a right angle; you assumed that α was real, but it may be complex; and so on.) Yet this principle contains an aspect which is palpably wrong. In the first place, the general theorem has been inferred, like as not, from several special cases. Second, if the special case is sufficiently random or "odd-ball," then its truth implies—with high probability—the general truth and, as we shall see, for "sufficiently odd-ball" cases, it implies the exact truth.

Consider, as an example, a classical theorem of fourth-century mathematics proved by Pappus. In Fig. 1, the lines l_1 and l_2 are selected at will and on these points $P_1, \ldots, P_6$ are selected at will. They are then cross-connected as indicated.

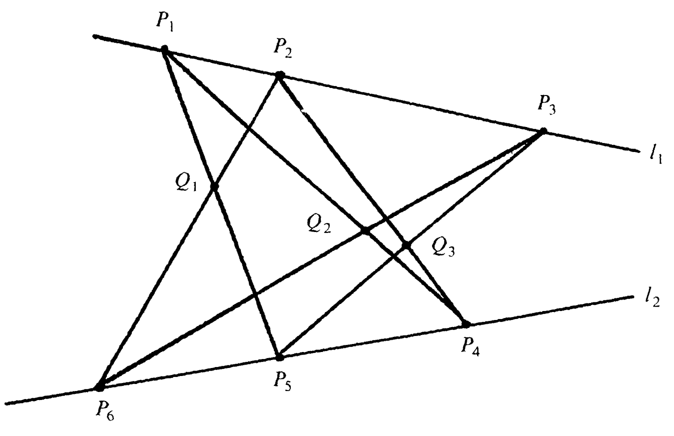

FIG. 1

The theorem now asserts that the points Q_1, Q_2, Q_3 will always be collinear. This theorem may very well have been found experimentally by fooling around with a ruler and pencil. Its ingredients are of the very simplest.

It should be observed that Pappus's proof and updated versions of it to be found in elementary projective geometry embody an element of great ingenuity. Also, this theorem lies at the heart of abstract projective geometry over arbitrary algebraic structures insofar as it is equivalent to the commutativity of the elements of the structure.

The verification of Pappus's theorem in any particular rational numerical case can be carried out completely in the field of rational numbers. Hence it can be carried out with exact precision on a digital computer. The assertion that Q_1, Q_2, Q_3 are collinear is equivalent to the statement that

$$D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

where $Q_i = (x_i, y_i)$. Now, beginning from integers of smallish size for the coordinates of $P_1, \ldots, P_6$, the arithmetic mixing process intrinsic in the geometric figure leads to very large integers for the numerator and denominator of the points Q_i. Therefore the discovery that $D = 0$ becomes an incredible arithmetic identity unless, of course, one believes that the theorem is true in general.

If now one allows the points $P_1, \ldots, P_6$ to have irrational coordinates, then the numerical verification of the identity $D = 0$ is, in the strict sense, impossible on a computer. However, it can be done by setting up the irrational quantity as an abstract variable, to be processed as a variable and not numerically, and this is tantamount to a rigorous proof.

In a similar way, consider a program P residing in a digital computer. Let us suppose it is such that given an input value x, the computer prints out the value $y = P(x)$. We have a certain ideal program $\tilde{P}$ in mind and we should like to inquire whether $P \equiv \tilde{P}$. In most instances, it is simply out of the question to test whether $P(x) = \tilde{P}(x)$ for all the possible input values x.

To limit the problem, suppose we know a priori that both P and $\tilde{P}$ are selected from a limited family of possible programs. Say they are both polynomials in a single variable x of degree $\leqslant n$. Then, from our theory of polynomials, we can assert that if $x_1, \ldots, x_{n+1}$ are n distinct inputs and if $P(x_i) = \tilde{P}(x_i)$ $i = 1, \ldots, n+1$, then $P \equiv \tilde{P}$. Thus a limited testing implies complete identity. We can assert more.

If we test out the program with a transcendental element x^* (which, in the numerical sense, is inaccessible to the computer), then $P(x^*) = \tilde{P}(x^*)$ implies $P \equiv \tilde{P}$. Therefore, one special (but sufficiently odd-ball) case implies complete identity.

But we can assert more. Suppose that P and $\tilde{P}$ are extracted from the class of *computable functions*. Then a simple countability argument provides us with a value x^* such that $P(x^*) = \tilde{P}(x^*)$ implies $P \equiv \tilde{P}$. Thus we have in this statement a theory of coincidences wherein a single coincidence of sufficient intensity implies complete identity.

(See [**1**], [3], [4], [**6**], [7], [**12**], [**17**].)

Example 9. This example is of a different order. The Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

plays a role in the theory of gravitational attraction, in electro- and magneto-statics, in fluid flow, in the theory of elasticity, etc. Coincidence, or part of a Grand Order? Again, to inject a personal note, when I first learned this, it had the strong flavor of a coincidence, and my teacher and my textbook certainly emphasized this aspect as well as the fact that one wouldn't have to learn totally different theorems. The solution of problems of fluid flow by means of electrostatic analog devices seemed to confirm the miracle.

Today the books are more sophisticated and attempt to unify the whole by pointing out that all those physical phenomena are subject to the $1/r$ force law, to similar laws of invariance, of conservation, and of the operation of the maximum principle.

To restore a sense of coincidence here, consider a different aspect of the equations of mathematical physics. The relationship between real, two-dimensional potential theory and the theory of analytic functions of a complex variable is one of the great glories of mathematics. Many novices anticipate that there must be a corresponding relationship between higher dimensional potential theory and a theory of functions of a hypercomplex variable.

This is not true in the simplistic sense. But the drive toward preserving the coincidence has brought forth theories, e.g., application of quaternions or Bergman's theory of integral operators of several complex variables, which in some degree can be regarded as substitutes. (See [**10**].)

But enough of examples: It is time now that we drew some inferences from them.

From time to time mathematicians perceive certain similarities of form which elicit an element of surprise. Such a similarity may be called a coincidence. The surprise calls for an explanation. The explanation, if it is forthcoming, serves partially to kill the surprise.

The existence of the coincidence implies the existence of an explanation. If the coincidence is of a high degree of improbability, then there is more to explain and the explanation will be easier in the sense that it involves a more easily accessible theory. If the coincidence is only of a medium order of improbability, the explanation will be more difficult.

A Platonic philosophy of mathematics might say that there are no coincidences in mathematics because all is ordained. In the words of Alexander Pope (*Essay on Man*):

All nature is but art unknown to thee
All chance, direction which thou can'st not see.

But for the working mathematician, coincidence exists. He feels it, he identifies it, he uses it as an inductive and constructive element. He pursues its implication along certain lines. To some extent, he even brings it about.

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