The Many Avatars of a Simple Algebra

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1. INTRODUCTION. Some mathematical structures show up in many different contexts, under many different guises. This is the case with the Weyl algebra. Born in the cradle of quantum theory, in the 1920s, it has come up in the representation theory of enveloping algebras and has played a key rôle in the creation of $\mathscr{D}$-module theory. It has recently returned to the parental home, under the auspices of deformation theory.

In this paper we survey the incarnations of the Weyl algebra associated to several formalisms of quantum mechanics. Beginning with the moment of conception in the 1920s, we work our way through matrix mechanics, Schrödinger's equation and Dirac's formalism. After a brief interlude where rings of differential operators are introduced, we return to quantum theory to look at *quantisation by deformation* and its version of the Weyl algebra.

2. QUANTUM MECHANICS. The story begins in May 1925, when W. Heisenberg fell ill with a bout of hay fever so vicious that he decided to ask for a fortnight leave to recover. He chose the island of Heligoland as a place to escape to. He must have been in a dreadful state indeed, because the landlady of the inn where he stopped for breakfast assumed, from his looks, that he had been involved in a fight the night before, [**21**, p. 248 ff].

In Heligoland, between walks and baths, Heisenberg carried on the work he had started in Göttingen. He was trying to develop a quantum mechanics, and his fundamental intuition was that it should deal only with observable quantities. Starting from that, Heisenberg developed a mathematical formulation of the theory. However it was not clear at first whether the mathematical scheme would be consistent or not.

Heisenberg felt that the real test of his scheme would be to check that it satisfied the law of conservation of energy. It took him a whole night to verify that energy was indeed conserved. Elated, he climbed a rock jutting out into the sea and watched the sun rise.

Let us see how Heisenberg arrived at his scheme of quantum mechanics. Consider an electron moving in an atom. If the system were classical, then we would have a function $x(t)$ describing the position of the electron as a function of time. We would also have Newton's equation

$$\ddot{x} + f(x) = 0.$$

Heisenberg decided that this equation ought to be retained, but that it would be necessary to find a new interpretation for $x(t)$. But the motion of the electron is periodic. Once again, if the system were classical, one could expand $x(t)$ as a Fourier series. In this case, the coefficients of the series would represent the amplitudes. In the quantum case these coefficients should depend on a quantum number. Developing the mathematical scheme along these lines, Heisenberg was led 'almost necessarily' to a very weird looking formula for the multiplication of amplitudes. In particular, as he explicitly stated in his original paper, these amplitudes do not commute; a fact that deeply troubled him.

At first Heisenberg hoped to remove the need for non-commutative amplitudes from his theory. Unable to 'improve' the paper, he decided to come out with it and handed it over to Max Born shortly before leaving for England, where he would speak at the Kapitsa Club in Cambridge.

3. MATRIX MECHANICS. Born did not look immediately at Heisenberg's manuscript. It was the end of term, he felt tired and 'afraid of hard thinking' [**22**, p. 8 ff]. However, when he read through it a few days later, he was fascinated. Born immediately began to work on Heisenberg's ideas. By simplifying Heisenberg's notation and re-writing the formulae for the multiplication of amplitudes he immediately realised that it was formally like the product of matrices. It is interesting to note that at the time matrices were not in the toolkit of every physicist. Luckily Born still remembered matrices from his student days in Breslau, twenty years back.

Soon Born began his own 'constructive work'. Denoting by $\mathbf{p}$ and $\mathbf{q}$ the momentum and position variables of Heisenberg's picture, Born realised that $\mathbf{pq}$ and $\mathbf{qp}$ were different because $\mathbf{p}$ and $\mathbf{q}$ were matrices. He also noted that Heisenberg's formulae gave only the diagonal entries of the commutator $[\mathbf{p}, \mathbf{q}] = \mathbf{pq} - \mathbf{qp}$, which had to be $i\hbar$. Here $\hbar$ denotes Planck's constant divided by 2π.

In Born's own words: 'repeating Heisenberg's calculation in matrix notation, I soon convinced myself that the only reasonable value of the non-diagonal elements should be zero' [**27**, p. 37]. Thus he arrived at the formula

$$\mathbf{pq} - \mathbf{qp} = i\hbar\mathbf{1}, \tag{3.1}$$

where $\mathbf{1}$ denotes the identity matrix. In his words, this formula was 'only a guess, and my attempts to prove it failed'.

A few days later, Born met Pauli, on the train between Göttingen and Hanover. Unable to resist his enthusiasm, he told Pauli about his matrices and his difficulties with the proof of (3.1). Instead of showing interest, as Born had expected, Pauli accused him of spoiling Heisenberg's idea with 'futile mathematics' [**27**, p. 37].

Having failed to engage Pauli's interest, Born turned to his former student P. Jordan. Working together, they developed Heisenberg's idea in the context of matrix calculus. This is the first time that (3.1) appears in print, with a 'proof' due to Jordan [**27**, p. 277].

The version of quantum mechanics that follows from the work of Heisenberg, Born, and Jordan is called *matrix mechanics*. In it the *momentum* and *position* are represented by matrices. Denoting these matrices by $\mathbf{p}$ and $\mathbf{q}$, respectively, the *equations of motion* for an electron moving in one dimension, under a potential, take the form

$$\begin{aligned} \partial\mathbf{q}/\partial t &= \partial H/\partial\mathbf{p} \\ \partial\mathbf{p}/\partial t &= -\partial H/\partial\mathbf{q} \end{aligned} \tag{3.2}$$

where H is a function of $\mathbf{p}$ and $\mathbf{q}$. These are Hamilton's equations of motion, to which we return in the next section. The point to note here is that the equations involve two kinds of differentiation: by a scalar (time) and by matrices ($\mathbf{p}$ and $\mathbf{q}$). The first poses no problem, but the same cannot be said of the second. We return to this question in §5.

4. HAMILTONIAN MECHANICS. Let us briefly review a few facts about hamiltonian mechanics that we will require. Consider a particle of mass 1 moving along a straight line. Let q and p denote the position and momentum of the system. Since we have a classical system, these are numbers: the coordinates of phase space. Suppose that the particle is subject to a force $F(q,t)$, which depends on position and time.

Since the system is one dimensional, F can be derived from a potential $V(q,t)$, given by

$$V(q,t) = -\int_{q_0}^{q} F(q,t)\,dq.$$

Hence the total energy of the system, which is the sum of the potential and kinetic energy of the particle is

$$H(q,p,t) = \frac{p^2}{2} + V(q,t).$$

This is called the *Hamiltonian* or *Hamiltonian function* of the system. By Newton's second law

$$\frac{\partial p}{\partial t} = F(q,t) = -\frac{\partial H}{\partial q}.$$

On the other hand, a direct calculation shows that

$$\frac{\partial q}{\partial t} = p = \frac{\partial H}{\partial p}.$$

The equations

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\frac{\partial H}{\partial q} \\ \frac{\partial q}{\partial t} &= \frac{\partial H}{\partial p}, \end{aligned} \tag{4.1}$$

are called *Hamilton's equations* of motion.

We have thus obtained Hamilton's equations for a system that consists of a particle of unit mass moving on a straight line under a force $F(q,t)$. In general, a *Hamiltonian system* of one degree of freedom is a second order system whose motion is determined by equations of the form (4.1).

The quantities of classical mechanics are described in terms of infinitely differentiable complex-valued functions of p and q. For the sake of simplicity we shall restrict ourselves to polynomial functions. Thus we shall be concerned with the space $\mathbf{C}[p,q]$ of polynomials in two commuting variables, which we denote by S. The system of equations (4.1) can be written in a very compact form using an operation called the *Poisson bracket* which is defined, for $f, g \in S$, by

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}.$$

This is clearly a polynomial in p and q. The vector space S is a Lie algebra with respect to the Poisson bracket; but this will not be needed here.

Returning to (4.1), an easy calculation shows that if we write x for the vector (p,q), the equations can be re-written in the form

$$\dot{x} = \{H, x\} \tag{4.2}$$

where the bracket with H is calculated coordinatewise. At this stage this may seem just a little trick. In fact, the Poisson bracket is the algebraic counterpart of the symplectic structure that gives phase space its peculiar geometry; see **[2]**. Moreover, this formalism guided Dirac in his formulation of quantum mechanics, as we shall now see. For more details about the Hamiltonian formalism see **[26]** or **[1]**.

5 DIRAC. In the meantime, in Cambridge, Heisenberg mentioned his ideas on matrix mechanics at the end of his talk to the Kapitsa Club. One of the physicists present at the lecture was R. H. Fowler. In September 1925 Heisenberg sent the proofs of his paper to Fowler, who promptly handed them to his student Paul Dirac. Dirac looked at the paper but 'at first could not make much of it'. Returning to it two weeks later, he realised that it 'provided a clue to the problem of quantum mechanics'. Unaware of the developments in Germany, Dirac proceeded to work out his own version of quantum mechanics.

Instead of interpreting the quantum variables as matrices, Dirac calculated with them formally. To use the Hamiltonian formalism he had to find an interpretation for the operation of differentiation with respect to a quantum variable, as we have already observed at the end of §3. Dirac's solution was to point out that the quantum analogue of differentiation by $\mathbf{q}$ say, is taking the commutator with $\mathbf{p}$. Thus, if $\mathbf{f}$ is a function of $\mathbf{p}$ and $\mathbf{q}$ in the quantum algebra, then $d\mathbf{f}/d\mathbf{q} = [\mathbf{p}, \mathbf{f}]$.

One of Dirac's great contributions was his identification of the classical analogue of the quantum commutator. According to Bohr's *correspondence principle* the results of quantum mechanics should converge to the analogous classical results when Planck's constant tends to zero. This weird 'constant tends to zero' really means that the numerical value of the constant should be small when it is expressed in the units of action characteristic of the class of systems under consideration.

Guided by this principle, Dirac discovered that the commutator divided by $i\hbar$ is the quantum analogue of the Poisson bracket of classical mechanics. The analogy allowed him to derive formula (3.1). Furthermore, defining H to be the Hamiltonian of the system, and assuming that 'the orders of the factors of the products occurring in quantum motion are unimportant' he wrote the fundamental quantum equation in the form

$$\dot{x} = [H, x],$$

in complete analogy with (4.1).

This analogy also helps to explain Dirac's formula for differentiation by $\mathbf{q}$. Indeed, if f is a polynomial in the (commutative) variables p and q, one immediately checks from the formula of the Poisson bracket that $\{p, f\} = \partial f/\partial q$. The corresponding quantum formula is obtained by replacing $\{,\}$ with $[,]$.

As his papers show, Dirac clearly understood that the quantum mechanical quantities defined a new sort of algebra, for which the multiplication was not commutative. He later called these quantities *q-numbers*—as opposed to *c-numbers*, which are the ordinary complex numbers. Dirac's papers can be found in **[27**, p. 307 and 417**]**.

6. QUANTUM ALGEBRA. Let us consider the algebraic background of the two interpretations of quantum mechanics that we have surveyed. Dirac assumes that he has 'quantities' that behave in a certain way. In other words, *symbols* that are subject to *relations*. In the one dimensional case two symbols $\mathbf{p}$ and $\mathbf{q}$ are required to represent momentum and position. They are related by $\mathbf{pq} - \mathbf{qp} = i\hbar \cdot \mathbf{1}$—where

$\mathbf{1}$ denotes the identity of the quantum algebra $\mathscr{A}$. To avoid unnecessary complication we *normalize* this relation to the form $\mathbf{pq} - \mathbf{qp} = \mathbf{1}$.

Algebraically, Dirac's quantum algebra is constructed beginning with the *complex free algebra* $\mathscr{F}$ in two generators x and y. The elements of the free algebra are linear combinations (with complex coefficients) of *words* in x and y. The product of two words is obtained by juxtaposition. The quantum algebra $\mathscr{A}$ is the quotient algebra of $\mathscr{F}$ by the two-sided ideal generated by $xy - yx - 1$. Thus $\mathbf{p}$ and $\mathbf{q}$ are the images of x and y in this quotient.

Every element of $\mathscr{A}$ is a linear combination of words in $\mathbf{p}$ and $\mathbf{q}$—a property that $\mathscr{A}$ inherits from its parent free algebra. Now, from $[\mathbf{p}, \mathbf{q}] = \mathbf{1}$ one deduces that

$$[\mathbf{p}, \mathbf{q}^k] = k\mathbf{q}^{k-1} \quad \text{and} \quad [\mathbf{p}^k, \mathbf{q}] = k\mathbf{p}^{k-1}$$

This agrees with Dirac's observation that commutation is analogous to differentiation. These commutation relations allow us to write every word in $\mathbf{p}$ and $\mathbf{q}$ as a linear combination of monomials $\mathbf{q}^k\mathbf{p}^m$. Thus every element of $\mathscr{A}$ is a linear combination of monomials of this form.

Working a little harder, we can show that the monomials $\mathbf{q}^k\mathbf{p}^m$, with $k, m \geq 0$, form a basis of $\mathscr{A}$ as a complex vector space [**6**, Proposition 1.2.1]. This can be used to define the degree of an element of $\mathscr{A}$. First define the *degree* of a monomial $\mathbf{q}^k\mathbf{p}^m$ to be $k + m$. Now write $\mathbf{d} \in \mathscr{A}$ as a linear combination of monomials of this form: the maximum of the degree of these monomials is called the degree of $\mathbf{d}$ and is denoted by $\deg(\mathbf{d})$.

The degree of $\mathscr{A}$ behaves in many ways like the degree of polynomials. For $\mathbf{d}_1, \mathbf{d}_2 \in \mathscr{A}$,

(1) $\deg(\mathbf{d}_1 + \mathbf{d}_2) \leq \max\{\deg(\mathbf{d}_1), \deg(\mathbf{d}_2)\}$,
(2) $\deg(\mathbf{d}_1\mathbf{d}_2) = \deg(\mathbf{d}_1) + \deg(\mathbf{d}_2)$, and
(3) $\deg[\mathbf{d}_1, \mathbf{d}_2] \leq \deg(\mathbf{d}_1) + \deg(\mathbf{d}_2) - 2$.

The proof of (1) is immediate, but the proof of (2) uses (3) and is somewhat convoluted. An immediate consequence of (2) is that $\mathscr{A}$ is an integral domain: it does not have any zero divisors. See [**6**, Ch. 2, §1].

The degree can be used to prove several properties of $\mathscr{A}$. For example, in §2 of [**7**], Dirac characterizes all the derivations of $\mathscr{A}$. Recall that a *derivation* D of $\mathscr{A}$ is a $\mathbf{C}$-linear operator of $\mathscr{A}$ that satisfies $D(\mathbf{d}_1\mathbf{d}_2) = \mathbf{d}_1 D(\mathbf{d}_2) + D(\mathbf{d}_1)\mathbf{d}_2$, for every $\mathbf{d}_1, \mathbf{d}_2 \in \mathscr{A}$. With Dirac, we note that the order of $\mathbf{d}_1$ and $\mathbf{d}_2$ in the formula cannot be changed. An easy way to produce derivations of $\mathscr{A}$ is to use the commutator. Given $\mathbf{f} \in \mathscr{A}$, define $P(\mathbf{d}) = [\mathbf{d}, \mathbf{f}]$, for $\mathbf{d} \in \mathscr{A}$. As one checks easily, this is a derivation of $\mathscr{A}$. Derivations of this form are called *inner derivations* of $\mathscr{A}$.

Dirac showed that all the derivations of $\mathscr{A}$ are inner. Let us sketch the proof. Let D be a derivation of $\mathscr{A}$. Since commutation by $\mathbf{p}$ and $\mathbf{q}$ behaves like differentiation, we can find $\mathbf{f} \in \mathscr{A}$ such that $D(\mathbf{p}) = [\mathbf{f}, \mathbf{p}]$ and $D(\mathbf{q}) = [\mathbf{f}, \mathbf{q}]$. The actual calculation is reminiscent of the way one finds a potential function for a conservative polynomial vector field on the plane. Now, using induction on the degree of $\mathbf{d} \in \mathscr{A}$, one can check that $D(\mathbf{d}) = [\mathbf{f}, \mathbf{d}]$.

Another very important property of $\mathscr{A}$ is that it has no proper two-sided ideals, except zero. In other words, $\mathscr{A}$ is a *simple* algebra. However, it is not a division ring: $\mathbf{p}$ cannot have an inverse because, when multiplied by any element of $\mathscr{A}$ it gives rise to an element of degree at least 1. Actually, the only invertible elements of $\mathscr{A}$ are the constants.

The proof that $\mathscr{A}$ is simple goes as follows: suppose that J is a non-zero two-sided ideal of $\mathscr{A}$. Choose a non-zero element $\mathbf{d} \in \mathscr{A}$. Commuting with $\mathbf{p}$ is formally equivalent to differentiation by $\mathbf{q}$. Hence commuting $\mathbf{d}$ with $\mathbf{p}$ enough times we obtain an element $\mathbf{d}' \in \mathscr{A}$ that does not involve $\mathbf{q}$. But J is a two-sided ideal of $\mathscr{A}$. Thus every time we commute an element of J with $\mathbf{p}$, we get an element of J. Hence $\mathbf{d}' \in J$. Now repeat the process with $\mathbf{d}'$, this time commuting it as many times as necessary with $\mathbf{q}$, until we arrive at a non-zero constant. Thus J contains a non-zero constant, and so $J = \mathscr{A}$; this is what we wanted to prove. For details see [**6**, Theorem 2.2.1].

We can also describe in a mathematical way the relation that Dirac found between the quantum commutator and the Poisson bracket. Quantum mechanics is represented by $\mathscr{A}$, and classical mechanics is represented by the complex algebra S of polynomial functions on the variables p and q, which stand for momentum and position. Thus S is a commutative algebra.

Let B_k be the set of elements of $\mathscr{A}$ of degree $\leq k$ and let $S(k)$ be the set of homogeneous polynomials of degree k in S. We define a map $\sigma_k \colon B_k \to S(k)$ as follows: If $\mathbf{d} \in B_k$ has degree k, ignore the monomials of degree $< k$, and replace $\mathbf{p}$ by p and $\mathbf{q}$ by q in the monomials of degree k. This gives a homogeneous polynomial of degree k, which we denote by $\sigma_k(\mathbf{d})$. For example, if $\mathbf{d} = \mathbf{q}^4\mathbf{p}^5 + 7\mathbf{q}^3\mathbf{p}^6 + 6\mathbf{p}^5 + 3\mathbf{pq}$, then $\mathbf{d}$ has degree 9 and $\sigma_9(\mathbf{d}) = q^4p^5 + 7q^3p^6$. This is called the *symbol map* of degree k of $\mathscr{A}$; it is a linear map of vector spaces, no more. Note that if $\mathbf{d} \in B_k$ has degree $< k$ then its symbol of degree k is zero. This construction is well-known from partial differential equation theory.

Now to the relation with the Poisson bracket. Let $\mathbf{d}_1$, $\mathbf{d}_2$ be elements of $\mathscr{A}$ of degrees k_1 and k_2 respectively. By (2), the commutator $[\mathbf{d}_1, \mathbf{d}_2]$ has degree at most $k_1 + k_2 - 2$. One can now check that

$$\sigma_{k_1+k_2-2}([\mathbf{d}_1, \mathbf{d}_2]) = \{\sigma_{k_1}(\mathbf{d}_1), \sigma_{k_2}(\mathbf{d}_2)\}.$$

This is one way to express the relation discovered by Dirac. We will come across another way, more in the spirit of the correspondence principle, in §11.

7. MATRIX REPRESENTATIONS. Let us now turn to the Heisenberg-Born-Jordan version of quantum mechanics. In it $\mathbf{p}$ and $\mathbf{q}$ are matrices. First of all notice that there cannot be two finite matrices whose commutator is $\mathbf{1}$. The easiest way to see this is to observe that the trace of a commutator is always zero, while the trace of the identity matrix is always non-zero. Therefore any such matrices must be infinite.

Thus we are led to a representation of $\mathscr{A}$ into the algebra $M_\infty(\mathbf{C})$ of infinite matrices with complex coefficients. In other words, we must construct a homomorphism of algebras of $\mathscr{A}$ into $M_\infty(\mathbf{C})$. This is easy, given that we have defined $\mathscr{A}$ as a quotient of a free algebra. It is enough to find two matrices P and Q in $M_\infty(\mathbf{C})$ such that $PQ - QP = \mathbf{1}$; for example

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

First define a map $\theta \colon \mathscr{F} \to M_\infty(\mathbf{C})$ by $\theta(x) = P$ and $\theta(y) = Q$. Since $PQ - QP = \mathbf{1}$, it follows that $xy - yx - 1$ belongs to the kernel of θ. Thus θ induces a map $\bar{\theta}\colon \mathscr{A} \to M_\infty(\mathbf{C})$. But we have already seen that $\mathscr{A}$ is simple. In particular, the image of $\ker(\theta)$ in $\mathscr{A}$ must be zero. Hence $\bar{\theta}$ is injective. In other words, the subalgebra of $M_\infty(\mathbf{C})$ generated by P and Q is isomorphic to $\mathscr{A}$.

Thus we have two ways of describing the quantum algebra $\mathscr{A}$: as a quotient of a free algebra (Dirac's way) or as a subalgebra of a matrix algebra (the Heisenberg-Born-Jordan way). A third scheme for doing quantum mechanics leads into yet one more description, perhaps the most fruitful, in terms of differential operators.

8. WAVE MECHANICS. It has long been known that light shows phenomena that are better explained in terms of waves, and others that make better sense if it is thought of as a stream of small particles. Quantum theory reached a compromise, affirming a dual nature for light, both wave and particle. In 1924, Louis de Broglie, then a student working towards his doctorate in Paris, understood that the wave-particle dualism ought to be truly universal. If that were so, then a 'particle' such as an electron should also present the same dual nature of wave and particle. Langevin sent a copy of de Broglie's thesis to Einstein, who wrote in reply: 'he has lifted a corner of the great veil'.

De Broglie's work was the starting point of a third version of quantum mechanics, developed by the Austrian physicist Erwin Schrödinger in 1925. Schrödinger's starting point can be best summed up in the aphorism *where there is a wave, there must also be a wave equation.* Actually this was reportedly said by P. Debye at the end of a colloquium in Zurich, in which Schrödinger explained de Broglie's work to his department; see [**23**, Ch. 6]. Using de Broglie's formulae and a reasonable heuristic argument, Schrödinger arrived at a very neat partial differential equation. For an electron of mass m moving in one dimension under a potential V, the equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi. \tag{8.1}$$

At first this equation was hailed as a return to the good old days, with some physicists hoping that it would drive out those strange matrices and non-commutative quantities. Moreover, it was deterministic. However, what did the function ψ represent? It takes complex values, for a start. Max Born once again came to the rescue, and proposed that the *wave function*, as ψ came to be called, did not represent any physical quantity whatsoever. Only the square of its modulus $|\psi(x,t)|^2$ had a physical interpretation. It represented the probability of finding an electron at x at the moment t. Despite much initial dispute, this became the accepted interpretation.

This also rescued the *uncertainty principle*, which affirms that one cannot measure at the same time and with arbitrary precision, the position and momentum of a particle. In its mathematical form, it is a consequence of the quantum commutation relation (3.1). Since ψ is a solution of a differential equation, it behaves deterministically. But ψ cannot be measured. What one can measure is $|\psi|^2$, a mere probability.

Let us spell out this scheme in more detail. The wave functions live in the space $\mathscr{L}^2$ of square integrable functions defined on the real line and taking complex values. This is a Hilbert space. In particular, it is endowed with an inner product; if $\psi_1, \psi_2 \in \mathscr{L}^2$ then

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathbf{R}} \psi_1 \overline{\psi_2}\, dx.$$

A wave function ψ must be square integrable because $\langle \psi, \psi \rangle$ is equal to the probability of finding the particle somewhere in the real line, which must be 1. However not every element of $\mathscr{L}^2$ is a wave function: wave functions must be differentiable, if they are going to satisfy Schrödinger's equation. The *observables* correspond to Hermitian operators on $\mathscr{L}^2$. Despite the name, these cannot be observed directly. The magnitudes that are observed correspond to the eigenvalues of these operators. Since the observables are Hermitian operators, the eigenvalues are real numbers, which is what one would expect of physical quantities.

In the end, it turned out that Schrödinger's wave mechanics is equivalent to the matrix formulation. Note that in both versions one has operators: matrices, in matrix mechanics; differential operators, in wave mechanics. But in matrix mechanics, the matrices themselves change with time. The fundamental equations (3.2) relate the time derivative of a matrix with the quantum equivalent of the Hamiltonian. In wave mechanics, it is the wave function that changes with time. The differentiable operators act on the space of functions in the usual way.

The connection between the two pictures comes through an operator $U(t)$ that takes the wave function at t_0 into the wave function at t, namely $\psi(x,t) = U(t)\psi(x,t_0)$. It can be deduced from physical considerations that $U(t)$ is a unitary operator. Let X_0 be an observable in wave mechanics. Mathematically we are talking about a Hermitian operator in $\mathscr{L}^2$. Write $X = X(t) = U(t)^t X_0 U(t)$; this is the 'matrix' that corresponds to X_0 in matrix mechanics. Differentiating this formula with respect to t and using Schrödinger's equation, we arrive at

$$\dot{X}(t) = [H, X(t)],$$

which is the fundamental quantum equation in Dirac's form. For more details see [**8**, Ch. V, §28].

We can recreate the quantum algebra $\mathscr{A}$ in the language of wave mechanics. This time we will be handling differential operators in $\mathscr{L}^2$. The operators we want to consider are $\partial/\partial x$ and multiplication by x. For the sake of simplifying the formulae, let us denote these operators by ∂ and x, respectively. If ψ is a wave function, then

$$[\partial, x](\psi) = \partial(x\psi) - x\partial(\psi) = \psi.$$

Since this holds for every ψ, we conclude that $[\partial, x] = 1$, the identity operator in $\mathscr{L}^2$. Thus, proceeding as in §7, we can show that $\mathscr{A}$ is isomorphic to the complex subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathscr{L}^2)$ generated by the operators ∂ and x. In wave mechanics, these operators correspond to momentum and position, as was to be expected.

9. $\mathscr{D}$-MODULES. We can represent the algebra $\mathscr{A}$ more economically as an algebra of differential operators if we use polynomial functions. Let us start in a little more generality. Let R be a commutative algebra over $\mathbf{C}$. We define the *ring of differential operators* $\mathscr{D}(R)$ inductively as a subalgebra of $\mathrm{End}_{\mathbf{C}}(R)$. Since an element of R gives rise to a linear operator of R by multiplication, the inductive definition begins with $\mathscr{D}^0(R) = R$, the operators of order zero. The operators of order k are

$$\mathscr{D}^k(R) = \{d \in \mathrm{End}_{\mathbf{C}}(R) : [d, a] \in \mathscr{D}^{k-1}(R) \text{ for all } a \in R\}.$$

Let $\mathscr{D}(R)$ be the union of all $\mathscr{D}^k(R)$ for $k \geq 0$. This turns out to be a subalgebra of $\mathrm{End}_{\mathbf{C}}(R)$, though the proof is not quite obvious, see [**6**, Ch. 3, §1].

It is easy to calculate $\mathscr{D}^1(R)$ explicitly. It is generated, as an R-module, by 1 and the $\mathbf{C}$-derivations of R. In particular, if $R = \mathbf{C}[x]$, the polynomial ring in one variable, then $\mathscr{D}^1(R) = R + R\partial$, where ∂ denotes the operator differentiation by x. Thus the quantum algebra $\mathscr{A}$ is contained in $\mathscr{D}(R)$ as the algebra generated by x and ∂. Working a little harder, we can prove that $\mathscr{A} = \mathscr{D}(R)$; for details see [**6**, Ch. 3, §2]. Hence the quantum algebra is the algebra of differential operators of the ring of polynomials in one variable.

The preceding definition of rings of differential operators appears in Grothendieck's *Eléments de geométrie algébrique* [**14**, proposition 16.8.8]. The notoriety of rings of differential operators nowadays is mainly due to $\mathscr{D}$-module theory. A $\mathscr{D}$-module is a finitely generated module over the algebra of differential operators of the coordinate ring of a smooth affine algebraic variety. To handle general varieties one must introduce sheaves [**3**, Ch. VI].

The importance of the theory lies in its numerous applications, which extend from mathematical physics to number theory. One of the most famous is to the representation theory of algebraic groups, where $\mathscr{D}$-modules were used to settle the Kazhdan-Lusztig conjecture in 1981. A very important $\mathscr{D}$-module theoretic theorem used in the solution of the conjecture is the *Riemann-Hilbert correspondence*. This is a result of the noblest parentage. Its genealogical tree includes Riemann's memoir on the hypergeometric function, Hilbert's 21st problem, and the work of Deligne on regular connections.

10. THE WEYL ALGEBRA. It did not take long for algebraists to notice that the quantum algebra $\mathscr{A}$ was an interesting object of study. In 1933, D. E. Littlewood wrote a paper [**18**] in which he proves most of the properties of $\mathscr{A}$ that we considered in §6. He also gives several examples of infinite matrices satisfying (3.1), among them the one of §7.

Littlewood's language is rather antiquated. But in 1937, K. A. Hirsch published a paper [**16**] in which he proves that a class of rings that includes $\mathscr{A}$ is a simple algebra. This is a thoroughly modern paper, written in the language of van der Warden's *Moderne Algebra*. His approach is essentially the one presented in §6. See also [**5**].

A great boost to the study of $\mathscr{A}$ came with the realization that it appears as a quotient of enveloping algebras of nilpotent Lie algebras by primitive ideals. This brought them into the fold of the representation theory of Lie algebras.

In fact $\mathscr{A}$ is the first member of a family of complex algebras. It corresponds to a quantum system with one degree of freedom. The equations for systems with n degrees of freedom were found by Heisenberg himself, as early as September 1925. They also give rise to complex algebras that are simple integral domains. J. Dixmier studied these algebras in a series of papers in the 1960s, and was the first to call them *Weyl algebras*, after a suggestion of I. Segal; see [**10**]. He also introduced the notation $A_n(\mathbf{C})$ for the algebra corresponding to a system of n degrees of freedom; see [**9**]. Both the name and notation have become standard.

The importance of the Weyl algebra has grown steadily in the last 30 years; see [**3**], [**6**], [**20**]. The work on non-commutative Noetherian rings that followed A. Goldie's famous theorems on quotient rings of Noetherian rings [**12**], [**13**] and the fact that the Weyl algebra is the simplest (but quite typical) ring of differential operators has only added to its importance.

11. DEFORMATIONS. It is time to return to quantum mechanics. The three schemes that we studied in §§6–8 give rise to the method known as *canonical quantisation.* First of all, by *quantisation* we mean the process of turning a classical system into its corresponding quantum system. This is not a well-defined process. In canonical quantisation one starts with the Hamiltonian H of the classical system and systematically replaces the classical variables position and momentum by the operators x and d/dx of wave mechanics. One may now write the corresponding Schrödinger equation and solve it.

Several other methods of quantisation have been proposed: geometrical quantisation, asymptotic quantisation, deformation quantisation. It is the last of these that we want to study here. It leads us into another way of describing the Weyl algebra: as a deformation of a polynomial ring.

Let S be a commutative $\mathbf{C}$-algebra. Denote by $S[[t]]$ the space of power series in one variable with coefficients on S. Note that we are considering $S[[t]]$ as a *vector space* only, and not as a ring. This is because what we really want to do is to define a new multiplication in $S[[t]]$. To do that, we start with a family of bilinear maps $B_j\colon S \times S \to S$, for $j \geq 0$. If $a, b \in S$, then their *$\star$-product* in $S[[t]]$ is

$$a \star b = \sum_{j=0}^{\infty} B_j(a, b)t^j.$$

Extending this linearly to the whole of $S[[t]]$, we obtain a multiplication in this space. The multiplication is associative if the B_j satisfy

$$\sum_{i+j=k} B_i\big(a, B_j(b, c)\big) = \sum_{i+j=k} B_i\big(B_j(a, b), c\big)$$

for $k \geq 0$ and all $a, b, c \in S$. This is not very easy to check for a given family of bilinear maps. Doing it recursively, one is led to consider Hochschild homology, as shown by M. Gerstenhaber in **[11]**. We do not pursue this line here; our aims are more modest.

Two further assumptions are usually made. Since we want the $\star$-product to be a *deformation* of S, we must have $B_0(a, b) = ab$, the original product in S. If the identity of S is to be the identity of $S[[t]]$ with the product $\star$, then we must also have $B_j(a, b) = 0$ for $j > 0$ if either a or b is a scalar.

Let us return to quantum theory. We have seen that one of the key features of Dirac's approach to quantum mechanics was the relation between the classical Poisson bracket and the quantum commutator. He arrived at this relation using the *correspondence principle*, which states that a quantum system should tend to its classical analogue when Planck's constant tends to zero. This is also the starting point of the deformation theoretic approach to quantum mechanics.

In this approach we begin with the classical phase space. Since we are considering only a particle moving in a straight line, phase space is a two dimensional space. The classical dynamical variables are functions on phase space, and we are assuming that they are polynomial functions, to keep the going easy. The same is true in the deformation theoretic scheme. So far, so good. What we have to define anew is the multiplication of these observables. Furthermore, it must somehow depend on Planck's constant.

So let $S = \mathbf{C}[p, q]$ be the polynomial ring. We define a new product in $S[[\hbar]]$, the space of formal power series in $\hbar$, using the deformation theoretic approach just described. But what does it mean to say that the 'commutator corresponds to the Poisson bracket'? Let $f, g \in S$. Suppose we have constructed a deformation of $S[[\hbar]]$ given by a family of bilinear forms B_j, for $j \geq 0$. Forming the commutator of f and g as elements in the ring $S[[\hbar]]$ with this product, we get

$$f \star g - g \star f = \sum_{j \geq 0} \big(B_j(f, g) - B_j(g, f)\big)\hbar^j. \tag{11.1}$$

Since $B_0(f, g) = fg$, the first non-zero term of the power series in (11.1) is $(B_1(f, g) - B_1(g, f))\hbar$. But, according to Dirac, the commutator $f \star g - g \star f$ divided by $i\hbar$ ought to be equal to the Poisson bracket when $\hbar$ goes to zero. Thus $B_1(f, g) - B_1(g, f) = i\{f, g\}$. An easy way to achieve this is to require that $B_1(f, g) = i\{f, g\}/2$, since the Poisson bracket is skew symmetric.

As we saw in §4, the Poisson bracket is really a bidifferential operator in the arguments f and g. Thus we may boldly propose to extend this assumption to all the B_js. The question, of course, is: can one define a $\star$-product in $S[[\hbar]]$ satisfying all these conditions?

The answer is yes. This $\star$-product is called the *Moyal-Weyl product*. It was used by Moyal in [**24**] to study quantum statistical mechanics from the point of view of classical phase space. This product can be described in a very compact way if we use tensor products. First define the differential operator Π: $S \otimes_{\mathbf{C}} S \to S \otimes_{\mathbf{C}} S$ by $\Pi(f \otimes g) = \partial f/\partial p \otimes \partial g/\partial q - \partial g/\partial p \otimes \partial f/\partial q$. Now let Δ: $S \otimes_{\mathbf{C}} S \to S$ be the multiplication map $\Delta(f \otimes g) = fg$. One checks easily that the Poisson bracket can be written using Π and Δ as $\{f, g\} = \Delta\Pi(f \otimes g)$. More generally, the Moyal-Weyl $\star$-product of f and g is $f \star g = \Delta(\exp(i\hbar\Pi)(f \otimes g))$. As an example, let us calculate the coefficient of the term in $\hbar^2$ of $f \star g$. By definition it is $-\Delta(\Pi^2)(f \otimes g)$. An easy calculation shows that this is equal to

$$-\left(\frac{\partial^2 f}{\partial p^2}\frac{\partial^2 g}{\partial q^2} - 2\frac{\partial^2 f}{\partial p \partial q}\frac{\partial^2 g}{\partial p \partial q} + \frac{\partial^2 f}{\partial q^2}\frac{\partial^2 g}{\partial p^2}\right).$$

In particular, if either f or g has degree ≤ 1 then this term is zero.

More generally, if either f or g has degree $\leq k$, then $\Delta(\Pi^{k+1}(f \otimes g)) = 0$. Thus $f \star g$ is indeed a polynomial. Moreover $p \star q - q \star p = i\hbar\{p, q\} = i\hbar$. Normalizing $i\hbar$ to 1 we see that the algebra S with the Moyal-Weyl $\star$-product is isomorphic to $\mathscr{A}$. This is not quite a proof that these two algebras are isomorphic, because we have not verified that the Moyal-Weyl product is associative. We shall not check this here, but leave it, instead, to the conscientious reader as an exercise. For details see [**25**] and [**28**].

The deformation theoretic approach to the Weyl algebra is also interesting from an algorithmic point of view. Calculations with elements of the Weyl algebra are not exactly easy. The multiplication of two monomials of relatively small degree may give rise to a long string of terms. This is awkward to implement in a computer. The $\star$-product approach bypasses all this and gives a closed formula in terms of differentiation of polynomials, a calculation that computers can handle.

12. CONCLUDING REMARKS. G. H. Hardy says in *A Mathematician's Apology* that 'a mathematical idea is 'significant' if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas' [**15**, §11]. Having examined the evidence collected in the preceding sections, we can safely say that, by Hardy's criterion, the Weyl algebra is a 'significant idea'. This explains why it has been studied so intensely. A lot is known about the one-dimensional Weyl algebra $\mathscr{A}$. Its right ideals have been classified in [**4**] and [**17**], and its representation theory has been studied very thoroughly [**19**]. The same cannot be said of the many-dimensional Weyl algebras mentioned in §10.

But even $\mathscr{A}$ still hides some secrets. For example, it is not known whether all endomorphisms of $\mathscr{A}$ are surjective. This first appeared in print as 'Problème 11.1' in [**10**]; and it is closely related to the famous *Jacobian conjecture*; see [**6**, Ch. 4, §4].

ACKNOWLEDGMENTS. I wish to thank P. M. Cohn, M. P. Holland, and M. B. Alves for reading a preliminary version of this paper and for their many suggestions and comments. The work on this paper was partially supported by a grant from CNPq (Brazil).

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