GENERALIZATIONS OF THEOREMS ABOUT TRIANGLES

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1. Introduction. Since one of the most powerful methods in mathematical research is the process of generalization, it is certainly desirable that young students be introduced to this process as early as possible. The purpose of this article is to call attention to the usually untapped possibilities for generalizing theorems on the triangle to theorems about the tetrahedron. Some of these, of course, do appear in our textbooks on solid geometry; but here I shall describe two situations where the appropriate generalizations seem to be generally unknown. The questions to be answered are: (1) What is the generalization to a tetrahedron of the angle-sum theorem for a triangle? (2) What is the corresponding generalization of the laws of sines and cosines for a triangle? Expressed in this form, the questions are certainly vague; for surely there are many generalizations. From these we are to select the ones which are most satisfying and which have a clear right to be called *the generalizations*. In attacking these problems we will need to reexamine the theorems as they are stated for a triangle, and perhaps to reformulate them so that the generalizations appear to be natural. Thus we have a bonus in that we learn additional ways of thinking about triangles.

2. The angle-sum theorem. Since this theorem is one of the most familiar in Euclidean geometry, it is strange that its three-dimensional generalization is not part of the classical literature on geometry. I ran across this generalization some years ago and have been putting the question to mathematicians wherever I find them. Only one of them, Professor Pólya, knew of it. He attributes it to Descartes [1].

The first question to be settled is that of the type of angles in a tetrahedron to be considered. It would be most natural to consider the inner solid angles and their sum. I remind you that the measure of a solid angle is the area of the region on the unit sphere which is the intersection of the sphere with the interior of the solid angle whose vertex is at the center. Thus the measure of the solid angle at a corner of a room is $4\pi/8=\pi/2$, and the measure of a "straight" solid angle is $4\pi/2=2\pi$. By considering a few cases, we conclude that the sum of the measures of the inner solid angles of a tetrahedron is not a constant. For example consider the situation in Figure 1, where all the points lie in a plane. If D is raised slightly, we have a tetrahedron the sum of whose interior solid angles is very near to 2π. On the other hand let us raise segment AB in the plane Figure 2 a small amount. Then we have a tetrahedron the sum of whose inner solid angles is very near to zero. Hence the obvious generalization is incorrect. As a matter of fact it has been proved [2] that the sum of the solid angles of a tetrahedron can take any value between 0 and 2π.

In order to make a fresh start, let us reformulate the triangle theorem in the statement: The sum of the outer angles of a triangle equals 2π. There are two possible definitions of an outer angle. The usual one is that it is the angle between a pair of successive directed sides (Fig. 3). This clearly does not general-

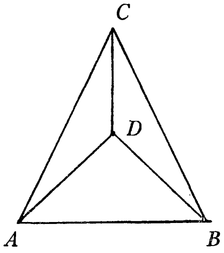


FIG. 1.

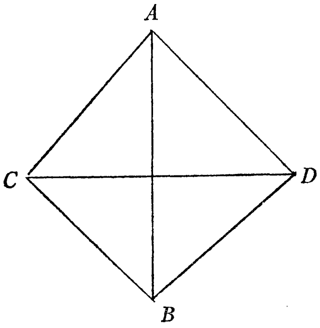
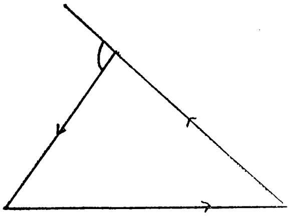


FIG. 2.



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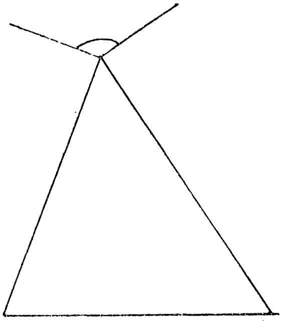

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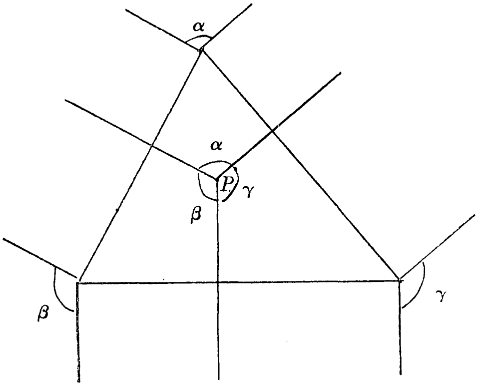

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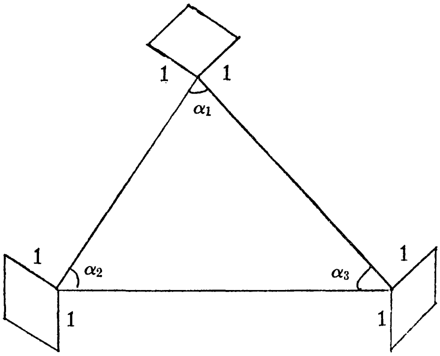

FIG. 6.

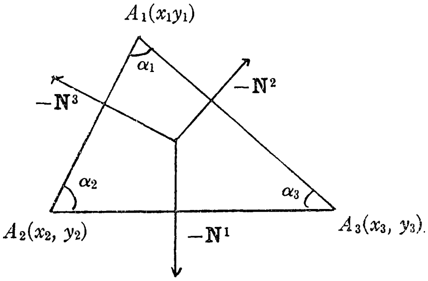

FIG. 7.

ize to three dimensions. Less familiar is the definition that an outer angle at a vertex is the angle between the two outward drawn normals to the two edges which meet at this vertex (Fig. 4).

Using this second definition, we can construct an elegant proof of the theorem. Choose any point P in the interior of the triangle and draw the perpendiculars from P to the three sides (Fig. 5). Then the outer angles α, β, and γ are equal to the three angles formed at P. Hence $\alpha+\beta+\gamma=2\pi$.

Now we can generalize at once. To find the corresponding theorem on the tetrahedron, first define the outer angle at a vertex as the trihedral angle formed by the three outer normals to the three faces meeting at this vertex. Choose an interior point P and draw the perpendiculars from P to the four faces. By the same argument that we used for the triangle, we find that

THEOREM 1. *The sum of the outer angles of a tetrahedron is* 4π.

By a straightforward generalization of the notion of an outer angle, we can similarly prove that

THEOREM 2. *The sum of the outer angles of any convex polyhedron is equal to* 4π.

There is also an immediate generalization to higher dimensions.

3. The Laws of Sines and Cosines. Before considering the generalization of these laws to a tetrahedron, let me give unfamiliar proofs of them which will suggest the proper generalization.

First, consider the Law of Sines. At each vertex (Fig. 6) draw the unit outer normals to the sides meeting at that vertex and complete the parallelograms determined by these pairs. By a familiar theorem of trigonometry the areas of these parallelograms are respectively $\sin(\pi-\alpha_1)=\sin\alpha_1$, $\sin(\pi-\alpha_2)=\sin\alpha_2$, and $\sin(\pi-\alpha_3)=\sin\alpha_3$. We shall proceed to compute these areas in terms of the coordinates of the vertices of the triangle (Fig. 7), choosing the notation appropriately so that $A_1A_2A_3$ are labeled in a counterclockwise fashion.

The equation of side A_2A_3 is

$$xN_x^1 + yN_y^1 + (x_2y_3 - x_3y_2) = 0$$

where N_x^1 and N_y^1 are respectively the cofactors of x_1 and y_1 in the determinant

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Thus the vector $\mathbf{N}^1$ with components (N_x^1, N_y^1) is normal to A_2A_3; $-\mathbf{N}^1$ is an outer normal; and $\mathbf{U}^1=-\mathbf{N}^1/a_1$ is the unit outer normal (where a_1 is the length of A_2A_3). More generally $\mathbf{U}^i=-\mathbf{N}^i/a_i$ $(i=1, 2, 3)$ are the three outer normals, where N_x^i and N_y^i are the cofactors of x_i and y_i respectively and a_i is the length of the side to which $\mathbf{U}^i$ is normal.

The area of the outer parallelogram at A_1 of which two sides are $\mathbf{U}^2$ and $\mathbf{U}^3$ is

$$\sin \alpha_1 = \begin{vmatrix} U_x^2 & U_y^2 \\ U_x^3 & U_y^3 \end{vmatrix} = \frac{1}{a_2 a_3} \begin{vmatrix} N_x^2 & N_y^2 \\ N_x^3 & N_y^3 \end{vmatrix}.$$

By a classical theorem on determinants (Bôcher, Introduction to Higher Algebra, p. 31) it follows that

$$\begin{vmatrix} N_x^2 & N_y^2 \\ N_x^3 & N_y^3 \end{vmatrix} = \Delta \cdot 1.$$

Hence

$$\sin \alpha_1 = \frac{\Delta}{a_2 a_3} \quad \text{and} \quad \frac{\sin \alpha_1}{a_1} = \frac{\Delta}{a_1 a_2 a_3}.$$

In a similar fashion we prove that

$$\frac{\sin \alpha_1}{a_1} = \frac{\sin \alpha_2}{a_2} = \frac{\sin \alpha_3}{a_3} = \frac{\Delta}{a_1 a_2 a_3}$$

which is the familiar Law of Sines.

To arrive at the Law of Cosines, we begin with a theorem of Möbius.

THEOREM 3. $\mathbf{N}^1 + \mathbf{N}^2 + \mathbf{N}^3 = 0$.

This theorem follows from the facts that $N_x^1 + N_x^2 + N_x^3 = 0$ and $N_y^1 + N_y^2 + N_y^3 = 0$. These may be computed directly, or they may be proved by expanding the determinants

$$\begin{vmatrix} 1 & y_1 & 1 \\ 1 & y_2 & 1 \\ 1 & y_3 & 1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 1 \\ x_3 & 1 & 1 \end{vmatrix} = 0.$$

This theorem can be rewritten in the form:

$$\mathbf{N}^1 = -\mathbf{N}^2 - \mathbf{N}^3.$$

Now take the scalar product of each side of this equation with itself. The result is

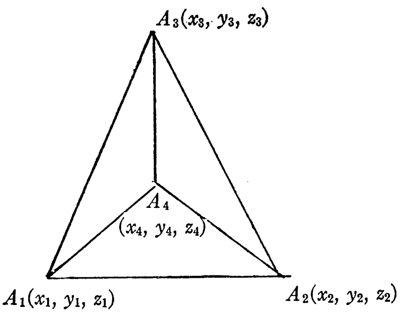

FIG. 8.

$$\mathbf{N}^1\cdot\mathbf{N}^1 = \mathbf{N}^2\cdot\mathbf{N}^2 + \mathbf{N}^3\cdot\mathbf{N}^3 + 2\mathbf{N}^2\cdot\mathbf{N}^3.$$

Since $\mathbf{N}^i\cdot\mathbf{N}^i = a_i^2$, and $\mathbf{N}^2\cdot\mathbf{N}^3 = -a_2a_3\cos\alpha_1$, this becomes $a_1^2 = a_2^2 + a_3^2 - 2a_2a_3\cos\alpha_1$.

4. The Generalized Laws of Sines and Cosines. These generalizations are due to Grassmann, but are relatively unfamiliar. Their proofs follow the lines just given in Section 3.

Consider a tetrahedron (Fig. 8) whose vertices are ordered so that

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} > 0.$$

Then the vector $\mathbf{N}^1$ whose components (N_x^1, N_y^1, N_z^1), are the cofactors of x_1, y_1, z_1 respectively in Δ, is normal to the face $A_2A_3A_4$. The length of $\mathbf{N}^1$, namely a_1, is equal to twice the area of this face. The vector $\mathbf{U}^1 = -\mathbf{N}^1/a_1$ is the unit outer normal to this face. Other normals $\mathbf{N}^i$ and $\mathbf{U}^i$ are defined in a similar fashion.

We now define the generalized sine ("G-sin") of the inner trihedral angle at A_1 to be the volume of the parallelopiped whose edges are $\mathbf{U}^2$, $\mathbf{U}^3$, and $\mathbf{U}^4$. Thus

$$G\text{-}\sin\alpha_1 = \begin{vmatrix} U_x^2 & U_y^2 & U_z^2 \\ U_x^3 & U_y^3 & U_z^3 \\ U_x^4 & U_y^4 & U_z^4 \end{vmatrix} = \frac{-1}{a_2a_3a_4}\begin{vmatrix} N_x^2 & N_y^2 & N_z^2 \\ N_x^3 & N_y^3 & N_z^3 \\ N_x^4 & N_y^4 & N_z^4 \end{vmatrix} = \frac{(-1)\Delta^2(-1)}{a_2a_3a_4} = \frac{\Delta^2}{a_2a_3a_4}.$$

By a continuation of this argument, we obtain the Generalized Law of Sines:

THEOREM 4.

$$\frac{G\text{-}\sin\alpha_1}{a_1} = \frac{G\text{-}\sin\alpha_2}{a_2} = \frac{G\text{-}\sin\alpha_3}{a_3} = \frac{G\text{-}\sin\alpha_4}{a_4} = \frac{\Delta^2}{a_1a_2a_3a_4}.$$

To establish the Generalized Law of Cosines, we observe that we can prove the following generalization of the Theorem of Möbius.

THEOREM 5. $\mathbf{N}^1+\mathbf{N}^2+\mathbf{N}^3+\mathbf{N}^4=0$.

Then writing

$$\mathbf{N}^1 = -\mathbf{N}^2 - \mathbf{N}^3 - \mathbf{N}^4$$

and $f_i = a_i/2 =$ area of the ith face, we prove as above the result:

THEOREM 6.

$$f_1^2 = f_2^2 + f_3^2 + f_4^2 - 2[f_2f_3\cos(f_2, f_3) + f_2f_4\cos(f_2, f_4) + f_3f_4\cos(f_3, f_4)],$$

where (f_i, f_j) is the inner dihedral angle of the tetrahedron between the faces whose areas are f_i and f_j respectively.

We also have another, rather novel, generalization if we start from $\mathbf{N}^1+\mathbf{N}^2 = -\mathbf{N}^3-\mathbf{N}^4$. The result is

THEOREM 7.

$$f_1^2 + f_2^2 - 2f_1f_2 \cos(f_1, f_2) = f_3^2 + f_4^2 - 2f_3f_4 \cos(f_3, f_4).$$

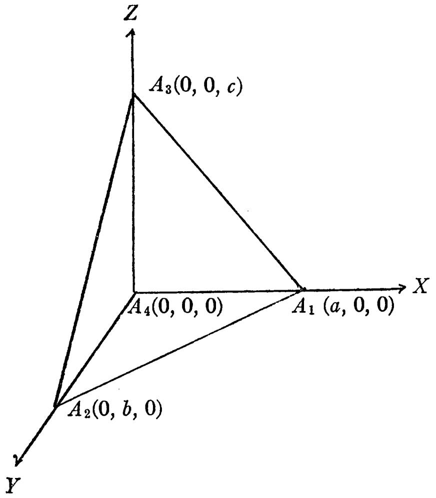

FIG. 9.

5. Supplementary matters. Another approach to the Generalized Law of Sines is to begin with a right tetrahedron (Fig. 9). Then it would be reasonable to define

$$G\text{-}\sin \alpha_1 = \frac{\text{Area } A_2A_3A_4}{\text{Area } A_1A_2A_3} = \frac{bc}{\{b^2c^2 + a^2c^2 + a^2b^2\}^{1/2}}.$$

Let us show that this agrees with our previous definition of G-sin α_1. We have:

$$\Delta = \begin{vmatrix} a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Then

$$G\text{-}\sin \alpha_1 = \frac{\Delta^2}{a_2a_3a_4} = \frac{a^2b^2c^2}{(ac)(ab)\{b^2c^2 + a^2c^2 + a^2b^2\}^{1/2}} = \frac{bc}{\{b^2c^2 + a^2c^2 + a^2b^2\}^{1/2}}.$$

Also we have the reassuring result that for our right tetrahedron:

$$(G\text{-}\sin \alpha_1)^2 + (G\text{-}\sin \alpha_2)^2 + (G\text{-}\sin \alpha_3)^2 = 1.$$

It is natural to ask whether G-$\sin \alpha_1$ is actually the sine of the measure of the inner or the outer solid angle at A_1; the answer is "no". To give an elementary counter-example we consider the right tetrahedron with $a=b=c=1$. Then G-$\sin \alpha_1 = 1/\sqrt{3}$; sine (measure of inner solid angle at A_1) $=1/3$; and sine (measure of outer solid angle at A_1) $=\sin 7\pi/6 = -1/2$.

As a matter of fact, G-$\sin \alpha$ is not even a functon of either the inner or the outer solid angles at the given vertex. Rather it depends directly on the face angles of the outer trihedral angle. If these angles are λ, μ, ν and $s=(\lambda+\mu+\nu)/2$, then

$$G\text{-}\sin \alpha = \{\sin s \sin (s-\lambda) \sin (s-\mu) \sin (s-\nu)\}^{1/2}.$$

References

1. R. Descartes, Oeuvres, vol. 10, pp. 257–276.

2. J. W. Gaddum, The sums of the dihedral and trihedral angles of a tetrahedron, Amer. Math. Monthly, 59 (1952) 370–375.

ON SUMS OF INVERSES OF PRIMES

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1. Let P be the set of prime numbers and let $p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, etc. It is a well known and useful fact that the infinite series $\sum_{i=1} p_i^{-1}$ diverges. It is further known that the approximate rate at which $\sum_{i=1} p_i^{-1}$ diverges, is

$$\sum_{p_i \leq x} p_i^{-1} = \ln \ln x + K + O(1/\ln x),$$

where K is a constant independent of x [1].

It seems natural to ask the following question: If $S \subseteq P$ does $\sum_{p_i \in S} p_i^{-1}$ converge or diverge?

The answer, of course, depends on S. For example, if $P-S$ is finite then surely $\sum_{p_i \in S} p_i^{-1}$ diverges; on the other hand, if S is finite $\sum_{p_i \in S} p_i^{-1}$ converges. The only case which has interest is when $P-S$ and S are both infinite.

For integers k and t, let $S(k, t) = \{kh+t\}_{h=1}^{\infty} \cap P$. Dirichlet proved that $S(k, t)$ is infinite if $(k, t)=1$, [1]. If $k>2$, then $P-S(k, t)$ is also infinite. It is known that $\sum_{p_i \in S(k,t)} p_i^{-1}$ diverges; in fact

$$\sum_{\substack{p_i \in S(k,t) \\ p_i \leq x}} p_i^{-1} = (\phi(k))^{-1} \ln \ln x + K_1 + O(1/\ln x)$$

where K_1 is a constant and ϕ is the Euler phi function [2].

It may occur that S may be so defined that one is not able to say if S is finite or not. The convergence of $\sum_{p \in S} p^{-1}$ still allows that S could be finite but adds little credence to it being so. To illustrate this point consider Brun's Theorem; that is if S^* is the set of all twin primes then $\sum_{p \in S^*} p^{-1}$ converges [3].