A Case Study in Mathematical Research: The Golay-Rudin-Shapiro Sequence

John Brillhart and Patrick Morton

1. INTRODUCTION. The case study we are presenting here is a re-creation of our original investigation into the Golay[1]-Rudin-Shapiro sequence [3]. We are particularly fond of this investigation, because of its unexpected simplicity and elegance. It contains a nice balance between reasonable and thought-provoking questions that lead us through the development of the subject and answers that arise from examining pertinent data as we go along. These answers lead in turn to more questions, etc. Our main purpose in re-creating this investigation is to show the evolution of questions and ideas that originally led us to our results. Thus, we are especially interested in highlighting some of the stances that mathematicians take in the middle of their work.

The standard and time-honored practice in mathematics—to erase all hint of the development of a subject or proof—usually makes it hard for students to see into the minds of mathematicians at work. Theorems and arguments seem to come from nowhere. Very seldom in textbooks or in research papers is there a hint of the original questions that motivated the researchers, or what special turns their understanding took in the middle of developing their subject. For us, that is one of the most exciting things about doing mathematics. We hope that students will see that the thought processes mathematicians engage in are much the same as the normal human process of asking questions and being alert to hints suggested by the subject itself. This questioning and following leads are at the heart of successful mathematical endeavors.

A secondary purpose of this paper is to provide an introduction to the subtleties of the Golay-Rudin-Shapiro sequence, a sequence that has motivated many interesting developments in the last 25 years. (See [1], [4], [5], [7], and the references contained in those papers.)

We kept an undergraduate audience in mind as we wrote this study. We envision it being used for its examples in elementary real analysis: for the empirical investigation of maxima and minima, arguments involving limit points, lim sups and lim infs, experimenting with inequalities, even for the experience of a frustrated attempt to solve a problem. This paper also serves as an introduction to topics that more advanced students can read in [4]. We think it might be suitable as an introduction to research methods for students involved in summer research programs or independent study.

[1]The sequence was originally named after Shapiro and Rudin, who were the first to study its properties (see [9] and [8]). Golay's contribution was recently pointed out to us by Andrew Odlyzko. See [6], bottom of page 469.

2. GETTING STARTED: THE INITIAL QUESTION. Many investigations begin with a question. In our case, we are looking at the terms of the Golay-Rudin-Shapiro sequence $\{a(n)\}$. This sequence can be defined recursively by the equations

$$\begin{aligned} a(2n) &= a(n), \\ a(2n+1) &= (-1)^n a(n), \quad n \geq 0, \\ a(0) &= 1. \end{aligned} \tag{1}$$

We know from work completed five years earlier [2] that the solution to this recurrence is

$$a(n) = (-1)^{e_0 e_1 + \cdots + e_{k-1} e_k}, \text{ where } n = \sum_{r=0}^{k} e_r 2^r, \quad e_r = 0 \text{ or } 1, \tag{2}$$

so this is clearly a sequence of ± 1's. The exponent on -1 in (2) counts the number of pairs of consecutive 1's in the binary representation of n. Thus if $n = 115_{10} = 1110011_2$, we have $a(115) = (-1)^3 = -1$.

The first eight terms of the sequence, starting with $n = 0$, are $1, 1, 1, -1, 1, 1, -1, 1$, and the obvious question is: does the number of $+1$'s exceed the number of -1's as we go out in the sequence? It's fine to ask this question, but now what? We might try rephrasing the question: "If we add up the terms, do the successive sums remain positive?" Being mathematicians, we make up some notation. Let

$$s(n) = \sum_{k=0}^{n} a(k), \quad n \geq 0. \tag{3}$$

The question now becomes: "Is $s(n) > 0$ for $n \geq 0$?"

To get some idea of what is going on, we do some computing. Working by hand and using the binary for n in (2), we readily find the values listed in Table 1. Since the answer to our question is "yes" for n up to 15, let's use a computer to extend Table 1 to, say, $n = 32{,}000$, and look at $s(n)$ over a larger range. We find when we

TABLE 1

n	$a(n)$	$s(n)$	n	$a(n)$	$s(n)$
0	1	1	8	1	5
1	1	2	9	1	6
2	1	3	10	1	7
3	−1	2	11	−1	6
4	1	3	12	−1	5
5	1	4	13	−1	4
6	−1	3	14	1	5
7	1	4	15	−1	4

do this that $s(n) > 0$ up to $n = 32{,}000$, so we begin to believe that the answer is "yes" for all n. Now what do we do?[2] Let's look at the long table more closely, and see if it suggests an obvious next question. The first thing we notice is a general growth in size of $s(n)$, with minor local variations. We can see this in the next part of the table (see Table 1A). Besides staying positive, the values of $s(n)$ roughly rise up to a peak of 15 (at $n = 42$) and then drop back down again, like a wave. Examining the long table to the end, we find that $s(n)$ cycles through four more such "waves," and that these waves seem to increase in "amplitude" and "wavelength."

[2] This question is the hallmark of having temporarily run out of steam.

TABLE 1A

n	$s(n)$	n	$s(n)$	n	$s(n)$	n	$s(n)$	n	$s(n)$	n	$s(n)$
16	5	24	7	32	9	40	13	48	11	56	9
17	6	25	6	33	10	41	14	49	10	57	10
18	7	26	5	34	11	42	15	50	9	58	11
19	6	27	6	35	10	43	14	51	10	59	10
20	7	28	7	36	11	44	13	52	9	60	9
21	8	29	8	37	12	45	12	53	8	61	8
22	7	30	7	38	11	46	13	54	9	62	9
23	8	31	8	39	12	47	12	55	8	63	8

The obvious next question is: does this wavelike behavior continue? How do we turn this qualitative question into a precise mathematical question that we can actually work with?

Perhaps a first step would be to focus on one aspect of this wave, say its "wavelength." What does "wavelength" mean? In a strictly periodic wave, the wavelength is the distance between consecutive abscissae at which high points occur, for example. How shall we think about wavelength in the present context, where the wave is not periodic? The high points may still give us some indication, so let's look at the high points in the long table, and see what they tell us. (At this point the reader may want to make his or her own table; it may also be helpful to create a plot of the values $s(n)$, say, for n in the interval $[1, 64]$.)

As we look at the table, we notice a "strong" local maximum at various places. For example, $s(10) = 7$ is a clear local maximum. If we look further in the table, we notice an obvious sequence of these strong local maxima: $s(42) = 15$, $s(170) = 31$, and $s(682) = 63$. We see the beginning of a pattern. The s-values at these strong local maxima are 1 less than consecutive powers of 2, and each corresponding n-value is 4 times the previous one plus 2. Now we're getting somewhere! Next question: Does this pattern continue? We think it might, so we state our guess more formally.

Conjecture 1. The n-value for a strong local maximum is 2 more than 4 times the previous one; the s-values at these points are 1 less than successive powers of 2.

The first thing to do after making the conjecture is to test it. According to the conjecture, the next strong local maximum should occur at $n = 4 \cdot 682 + 2 = 2730$, and we should have $s(2730) = 2^7 - 1 = 127$. An examination of the table shows this is true, and this strengthens the conjecture. We now find ourselves being distracted from the original question by our conjecture, which is interesting in its own right. Let's indulge ourselves and pursue this pattern question, ignoring the original question for the time being. Experience shows that such side questions often connect back to the original question and give information that is important to the overall development.

To state a more precise conjecture, we need to develop a formula for the n-values at which these strong local maximum values seem to occur. Extending the sequence $n = 10, 42, 170, 682, 2730$ backwards, we find that $s(2) = 3$ is also a local maximum. If we set $M_0 = 2$, $M_1 = 10$, and in general, M_k equal to the n-value of the k-th local maximum, then the sequence $\{M_k\}$ is defined recursively by

$$M_{k+1} = 4M_k + 2, \quad M_0 = 2. \tag{4}$$

Using standard techniques we find that the solution to this recurrence is $M_k = 2(2^{2k+2} - 1)/3$, for $k \geq 0$. We can now make our previous conjecture more precise.

Conjecture 1′. The points $M_k = 2(2^{2k+2} - 1)/3$, $k \geq 0$, are the n-values of local maxima for $s(n)$, and $s(M_k) = 2^{k+2} - 1$.

Now what? Now that we have a conjecture about the high points, what can we say about the wavelength? If we take our definition of wavelength to be the difference between abscissae of successive strong local maxima, then the wavelength of the k-th wave cycle is $M_{k+1} - M_k$. Using (4) (still unproved), we find that $M_{k+2} - M_{k+1} = 4(M_{k+1} - M_k)$. Hence, our wavelengths increase by a factor of 4 from one wave to the next!

While staring at Conjecture 1′, it occurs to us that we should probably examine local minima, as well. It seems natural to think of a wave cycle beginning at a high point and ending at the next high point, so we decide to create a table (Table 2) of the absolute minima between high points, that is, in intervals of the form $[M_k, M_{k+1} - 1]$, $k \geq 0$. In successive intervals there appears to be a doubling of the number of n-values at which the minima occur. At this point we begin to wonder if this choice of interval will really lead to the simplest description of the behavior of the function. What would happen if we considered a different set of intervals? We notice that the last minimum in each interval of Table 2 occurs just before a power of 4. Putting this together with the fact that the wavelength increases by a factor of 4 from one wave to the next, we decide to make a list (Table 3) of the extrema in intervals of the form $[4^k, 4^{k+1} - 1]$, $k \geq 0$.

TABLE 2. n-VALUES FOR MINIMA IN $[M_k, M_{k+1} - 1]$

Interval	Minimum of $s(n)$	n-values at which minimum occurs
[2,9]	2	3
[10,41]	4	13, 15
[42,169]	8	53, 55, 61, 63
[170,681]	16	213, 215, 221, 223, 245, 247, 253, 255
[682,2729]	32	853, 855, 861, 863, 885, 887, 893, 895, 981, 983, 989, 991, 1013, 1015, 1021, 1023

TABLE 3. n-VALUES FOR MINIMA AND MAXIMA IN $[4^k, 4^{k+1} - 1]$

Interval	Minimum of $s(n)$	n-values at which minimum occurs	Maximum of $s(n)$	n-values at which maximum occurs
[1,3]	2	1, 3	3	2
[4,15]	3	4, 6	7	10
[16,63]	5	16, 26	15	42
[64,255]	9	64, 106	31	170
[256,1023]	17	256, 426	63	682
[1024,4095]	33	1024, 1706	127	2730
[4906,16383]	65	4096, 6826	255	10922

There is a surprising simplification here: there are now exactly two minima and one maximum in each power of 4 interval. Moreover, the sequence of n-values $6, 26, 106, 426, \ldots$, at which the second minimum occurs in each interval, satisfies the same recursion formula (4) that the numbers M_k did. This choice of interval has yielded gold!

Note the irrational element in the step we have just taken. There was no reason other than a desire for simplicity (or curiosity, or laziness?) for changing the interval. It turned out to be a good guess, but such a simplification may not always happen. In the present case, it seems quite remarkable that a simple shift of the interval has such a dramatic effect in cutting down the number of minima.

The first n-value of the pair giving the minimum in Table 3 is a power of 4. If we denote the second n-value of this pair by m_k, for $k \geq 1$ (starting with the second interval), we have the recursion $m_{k+1} = 4m_k + 2$, with $m_1 = 6$. Then we find easily that $m_k = (5 \cdot 4^k - 2)/3$, for $k \geq 1$. This leads to a more complete conjecture.

Conjecture 2. a) For $k \geq 1$, the minimum value of $s(n)$ in $[4^k, 4^{k+1} - 1]$ is $2^k + 1$, which occurs at just the points $n = 4^k$ and $n = m_k$. b) For $k \geq 0$, the maximum value of $s(n)$ in $[4^k, 4^{k+1} - 1]$ is $2^{k+2} - 1$, which occurs only at the point $n = M_k$.

Notice that if Conjecture 2 is correct, then it follows immediately that $s(n) > 0$ for $n \geq 0$. Notice also that we haven't yet proved *anything*!

How shall we go about trying to prove Conjecture 2? The fact that M_k and m_k satisfy the same recursion suggests that we try finding a formula for $s(4n + 2)$. Such a formula might be useful in proving the conjecture. Finding this formula turns out to be fairly simple, if we take a hint from (1) and first look for formulas for $s(2n)$ and $s(2n + 1)$. Using (1) to manipulate the sum which defines $s(2n)$ gives

$$s(2n) = \sum_{k=0}^{2n} a(k) = \sum_{k=0}^{n} a(2k) + \sum_{k=0}^{n-1} a(2k+1)$$

$$= \sum_{k=0}^{n} a(k) + \sum_{k=0}^{n-1} (-1)^k a(k) = s(n) + t(n-1),$$

where $t(n)$ is the new function defined by

$$(5) \qquad t(n) = \sum_{k=0}^{n} (-1)^k a(k), \quad n \geq 0.$$

In the same way we can establish the following recursion formulas.

Lemma 1.

$$(6) \qquad s(2n) = s(n) + t(n-1), \quad n \geq 1;$$

$$(7) \qquad s(2n+1) = s(n) + t(n), \quad n \geq 0;$$

$$(8) \qquad t(2n) = s(n) - t(n-1), \quad n \geq 1;$$

$$(9) \qquad t(2n+1) = s(n) - t(n), \quad n \geq 0.$$

Uşing this lemma, we can work out a formula for $s(4n + 2)$. Replacing n by $2n + 1$ in (6) gives

$$s(4n+2) = s(2n+1) + t(2n) = s(n) + t(n) + s(n) - t(n-1)$$

$$= 2s(n) + (-1)^n a(n).$$

The lucky thing is that this recursion involves only the s-function, the t-function having dropped out. While we're at it, we give the formulas for $s(4n + d)$, where $d = 0, 1, 2, 3$ (there are similar formulas for $t(4n + d)$). The proofs are equally simple.

Lemma 2.

$$(10) \qquad s(4n) = 2s(n) - a(n), \qquad n \geq 1;$$

$$(11) \qquad s(4n+1) = s(4n+3) = 2s(n), \quad n \geq 0;$$

$$(12) \qquad s(4n+2) = 2s(n) + (-1)^n a(n), \; n \geq 0.$$

It seems now that we might have enough relations to attempt an inductive proof of Conjecture 2. Since the assertions we have to prove concern the interval $[4^k, 4^{k+1} - 1]$, it makes sense to carry out the induction on the variable k. Let's look at the simpler statement in part b) of Conjecture 2: "For $k \geq 0$, the maximum value of $s(n)$ in $[4^k, 4^{k+1} - 1]$ is $2^{k+2} - 1$, which occurs only at the point $n = M_k$."

For $k = 0$ this assertion is true, since the maximum of $s(n)$ in $[1, 3]$ is 3, which occurs only at $M_0 = 2$. Assume the assertion is true for the interval $I_k = [4^k, 4^{k+1} - 1]$ and try to prove it for I_{k+1}.

We first show that $2^{k+3} - 1$ is an upper bound for $s(n)$ in I_{k+1}. If n lies in $I_{k+1} = [4^{k+1}, 4^{k+2} - 1]$, then we can write $n = 4m + d$, for some m in $[4^k, 4^{k+1} - 1]$ and for some d in the set $\{0, 1, 2, 3\}$. Formulas (10)–(12) of Lemma 2 give:

$$(13) \quad s(n) = s(4m + d) \leq 2s(m) + 1 \leq 2(2^{k+2} - 1) + 1 = 2^{k+3} - 1,$$

which establishes the upper bound. Note that equality holds in (13) at most when $m = M_k$, by the induction assumption.

To finish the proof of b), we have to prove that M_{k+1} is the only place in I_{k+1} where the value $2^{k+3} - 1$ is actually achieved. This is really an "if and only if" statement. First we note from (4) and (12) that

$$(14) \qquad s(M_{k+1}) = s(4M_k + 2) = 2s(M_k) + (-1)^{M_k} a(M_k).$$

Using (4) it is not hard to see that the binary expansion of M_k is $1010\ldots10$, with $k + 1$ occurrences of the pattern "10," an expansion that contains no consecutive 1's. Thus, $a(M_k) = +1$, and M_k is even, so the last term in (14) is $+1$, and the induction hypothesis leads to $s(M_{k+1}) = 2(2^{k+2} - 1) + 1 = 2^{k+3} - 1$.

Conversely, suppose that $s(n) = 2^{k+3} - 1$, for some n in I_{k+1}. We have to show that $n = M_{k+1}$. From the comment following (13) we get that $m = M_k$, so $n = 4M_k + d$, for some $d = 0$, 1, 2, or 3. Since $s(n)$ is odd, equation (11) shows that $d \neq 1$ or 3. If $d = 0$, formula (10) gives the contradiction

$$2^{k+3} - 1 = s(n) = s(4M_k) = 2s(M_k) - a(M_k) = 2(2^{k+2} - 1) - 1 = 2^{k+3} - 3.$$

Hence we must have $d = 2$, so $n = 4m + 2 = M_{k+1}$. This proves everything!

We leave it to the reader to prove part a)—the same technique works. So we have a theorem:

Theorem 1. *a) For $k \geq 1$, the minimum value for $s(n)$ in $[4^k, 4^{k+1} - 1]$ is $2^k + 1$, and this value occurs only at the points $n = 4^k$ and $n = m_k = (5 \cdot 4^k - 2)/3$. b) For $k \geq 0$, the maximum value for $s(n)$ in $[4^k, 4^{k+1} - 1]$ is $2^{k+2} - 1$, and this value occurs only at the point $n = M_k = \frac{2}{3}(2^{2k+2} - 1)$.*

Corollary. *The sum $s(n)$ is positive, for all $n \geq 1$.*

3. GOING FURTHER: A NEW QUESTION. Having proved more than we needed to settle the original question, we're hooked! Finding the answer to one question often suggests new questions. In our case the first new question was: what is the asymptotic behavior of $s(n)$; that is, can we find an elementary function $f(n)$ so that the ratio $s(n)/f(n)$ stays bounded away from 0 and ∞, or even approaches a non-zero limit, as $n \to \infty$?

Finding such a function is not difficult, since Theorem 1 shows that $s(n)$ is roughly 2^k when n lies in the interval $I_k = [4^k, 4^{k+1} - 1]$. Hence, a reasonable choice is $f(n) = \sqrt{n}$. Looking at Theorem 1 more closely gives a good bit more. If n lies in the interval I_k, part a) shows that

$$\frac{s(n)}{\sqrt{n}} > \frac{2^k + 1}{\sqrt{4^{k+1}}} = \frac{1}{2} + \frac{1}{2^{k+1}} > \frac{1}{2},$$

while part b) gives that

$$\frac{s(n)}{\sqrt{n}} \le \frac{2^{k+2} - 1}{\sqrt{4^k}} = 4 - \frac{1}{2^k} < 4.$$

Thus, $\frac{1}{2} < s(n)/\sqrt{n} < 4$, for $n \ge 1$, and so $s(n)$ is roughly a constant times $\sqrt{n}$. This shows that $\sqrt{n}$ is the "right" order of magnitude of $s(n)$. However, this leaves open the question whether $s(n)/\sqrt{n}$ actually approaches a limit as $n \to \infty$. We can investigate this question using the sub-sequences $\{m_k\}$ and $\{M_k\}$. Using Theorem 1 we easily calculate that

$$\lim_{k\to\infty} \frac{s(m_k)}{\sqrt{m_k}} = \lim_{k\to\infty} \frac{(2^k + 1)\sqrt{3}}{\sqrt{5 \cdot 4^k - 2}} = \sqrt{\frac{3}{5}} = .7745\ldots$$

and

$$\lim_{k\to\infty} \frac{s(M_k)}{\sqrt{M_k}} = \lim_{k\to\infty} \frac{(2^{k+2} - 1)\sqrt{3}}{\sqrt{2(2^{2k+2} - 1)}} = \sqrt{6} = 2.4494\ldots.$$

Therefore $s(n)/\sqrt{n}$ certainly does not approach a limit as $n \to \infty$.

Having determined that the sequence has at least two limit points, two more questions occur to us: 1) how many limit points does the sequence have; and 2) what are the extremal limit points, i.e., the lim inf and lim sup of $\{s(n)/\sqrt{n}\}_{n=1}^{\infty}$? Such questions are often difficult to answer explicitly for the usual garden variety, pathological objects in real analysis. However, we have an intuition about the first question: since the denominator of $s(n)/\sqrt{n}$ is steadily increasing with n, while the numerator varies up and down by steps of 1, the difference between consecutive terms of the sequence is roughly

$$\frac{s(n+1)}{\sqrt{n+1}} - \frac{s(n)}{\sqrt{n}} \approx \frac{s(n+1) - s(n)}{\sqrt{n}} = \frac{\pm 1}{\sqrt{n}},$$

so the difference tends to 0 as n increases. Thus, an increasing number of smaller and smaller steps will be required to pass from the neighborhood of one limit point to the other. It is reasonable to guess, then, that *every* point of the interval $[\sqrt{3/5}, \sqrt{6}]$ is a limit point of $\{s(n)/\sqrt{n}\}_{n=1}^{\infty}$. This is true, in fact, and it isn't hard to turn our intuition into a formal proof.

Theorem 2. *Every point of the interval* $[\sqrt{3/5}, \sqrt{6}]$ *is a limit point of the sequence* $\{s(n)/\sqrt{n}\}_{n=1}^{\infty}$.

Proof: Let $\sqrt{3/5} < \xi < \sqrt{6}$, and suppose $\delta > 0$ and $N > 0$ are given. The assertion is: there is some $n > N$ for which

$$\left| \frac{s(n)}{\sqrt{n}} - \xi \right| < \delta. \tag{15}$$

Choose $m_2 > m_1 > N$ so that both $s(m_1)/\sqrt{m_1} < \xi < s(m_2)/\sqrt{m_2}$ and $\sqrt{n} > 1/\delta$ for $n \geq m_1$. Then, as we now show, (15) will be satisfied for some n between m_1 and m_2. Suppose to the contrary that there is no n with $m_1 \leq n \leq m_2$ for which (15) is satisfied, i.e., for which $\xi - \delta < s(n)/\sqrt{n} < \xi + \delta$. Then there must be a largest n_1 in $[m_1, m_2)$ for which $s(n_1)/\sqrt{n_1} \leq \xi - \delta$. This implies $s(n_1+1)/\sqrt{n_1+1} \geq \xi + \delta$, and therefore, $s(n_1+1)/\sqrt{n_1+1} - s(n_1)/\sqrt{n_1} \geq 2\delta$. On the other hand,

$$\frac{s(n_1+1)}{\sqrt{n_1+1}} - \frac{s(n_1)}{\sqrt{n_1}} \leq \frac{s(n_1+1) - s(n_1)}{\sqrt{n_1}} \leq \frac{1}{\sqrt{n_1}} < \delta.$$

The contradiction $2\delta < \delta$ shows our supposition to be false, i.e., there does exist an integer $n > N$ for which (15) is true. ∎

This answers question 1). To get some idea about question 2), we go back to the computer and compute the terms $s(n)/\sqrt{n}$ to $n = 32{,}000$. (Again, the reader may want to create a table or a plot of $s(n)/\sqrt{n}$ to follow along in this discussion.) We see the high points at $n = 10, 42, 170$ and low points at $n = 15, 26, 106$. The maximum of $s(n)/\sqrt{n}$ in I_k occurs at exactly the same point $n = M_k$ that it does for $s(n)$. Except for $n = 15$—an unruly initial exception—the minimum in I_k occurs at $n = m_k$ and is unique.

If extremal points for $s(n)$ remain extremal for the quotient $s(n)/\sqrt{n}$, it is plausible to think that the limit points $\sqrt{3/5}$ and $\sqrt{6}$ might be the lim inf and lim sup of the quotient sequence. Moreover, as is easily shown, the terms $s(m_k)/\sqrt{m_k}$ decrease monotonically to $\sqrt{3/5}$ and the terms $s(M_k)/\sqrt{M_k}$ increase monotonically to $\sqrt{6}$. Hence it is also plausible that $\sqrt{3/5}$ and $\sqrt{6}$ might be upper and lower bounds as well. Polya encourages us to be bold,[3] so we take the leap:

Conjecture 3. For $n \geq 1$, $\sqrt{3/5} < s(n)/\sqrt{n} < \sqrt{6}$.

If this is true, then it certainly follows that $\sqrt{3/5}$ and $\sqrt{6}$ are the lim inf and lim sup, respectively, since we already know they are limit points. Hence we can focus all of our attention on Conjecture 3.

4. THE UPPER BOUND.

We decide to focus on the upper bound first, since the maxima of $s(n)/\sqrt{n}$ seem to occur for the same n-values that they do for $s(n)$. In taking this route, we're letting ourselves be guided by the sense of internal or hidden logic that the subject seems to have.

What should we try? Suppose we focus on the interval $I_k = [4^k, 4^{k+1} - 1]$, as before. We would like to show that $s(n)/\sqrt{n} < \sqrt{6}$ for n in I_k, and we know that

[3] See George Polya's lecture in "Let Us Teach Guessing," an MAA video.

$s(M_k)/\sqrt{M_k} < \sqrt{6}$. It is natural to try to show that

$$(16) \qquad s(n)/\sqrt{n} \le s(M_k)/\sqrt{M_k}, \text{ for } n \text{ in } I_k.$$

We know that $s(n) \le s(M_k)$ on the whole interval, but we can only say that $\sqrt{n} \ge \sqrt{M_k}$ when $n \ge M_k$; hence the inequality in (16) is true in the sub-interval $[M_k, 4^{k+1} - 1]$. This is a start.

This is a common situation in mathematics. Working on a problem can be very much like putting together a jigsaw puzzle. You first try to find pieces you can fit together, so as to form islands of connections. Having found some of these islands, you then want to see how they fit together to solve the larger puzzle. The trouble with a mathematical puzzle is that you don't always know what all the pieces *are*; sometimes you have to create them. Sometimes, they don't all fit together to make the picture you want.

To create the next part of the argument we have to try imagining a new piece of the puzzle, something that will give us another way to look at the inequality (16). We have proved the inequality in the interval $[M_k, 4^{k+1} - 1]$, and we want to prove it in the interval $[4^k, M_k]$. We need some formulas to work with.

What do we know? We have the formulas (10)–(12), which relate values of s in I_k to values in I_{k-1}. It might be worth trying to use these as part of an induction proof. We try formula (10) first:

$$\frac{s(4n)}{\sqrt{4n}} = \frac{2s(n) - a(n)}{\sqrt{4n}} \le \frac{2s(n) + 1}{2\sqrt{n}} = \frac{s(n)}{\sqrt{n}} + \frac{1}{2\sqrt{n}}.$$

This almost works, except for that annoying term $1/2\sqrt{n}$. On the other hand, we find that formula (11) really does work, giving us $s(4n+1)/\sqrt{4n+1} < \sqrt{6}$ and $s(4n+3)/\sqrt{4n+3} < \sqrt{6}$, if $s(n)/\sqrt{n} < \sqrt{6}$. We get the desired inequality for *odd* values of n in I_k if we know it for *all* values in I_{k-1}. Close, but not good enough for an induction proof.

It seems as if these formulas don't quite give us what we want. We decide to go back to the tables and look for other patterns that might give a clue to some useful relationships. First we notice the similarity between the first 8 values of $s(n)$ in Tables 1 and 1A. We list them side by side:

Table 1 ($0 \le n \le 7$)	1	2	3	2	3	4	3	4
Table 1A ($16 \le n \le 23$)	5	6	7	6	7	8	7	8

The difference between corresponding numbers in the two rows is always 4. The next eight values from each table are:

Table 1 ($8 \le n \le 15$)	5	6	7	6	5	4	5	4
Table 1A ($24 \le n \le 31$)	7	6	5	6	7	8	7	8

Now the *sum* of corresponding numbers is always 12. We summarize these patterns in equation form:

$$s(n+16) = s(n) + 4, \qquad 0 \le n \le 7;$$

$$s(n+16) = -s(n) + 12, \quad 8 \le n \le 15.$$

When we consider the last four columns in Table 1A we find similar patterns:

$$s(n+32) = s(n) + 8, \qquad 0 \le n \le 15;$$

$$s(n+32) = -s(n) + 16, \quad 16 \le n \le 31.$$

These four equations show that all the values of $s(n)$ for n in $[16, 63]$ are obtainable from the values in $[0, 15]$. Looking further in the table we see that these elaborate patterns continue, and we are able to guess the general form of the last term in each of the four equations.

Lemma 3.

(17) $\quad s(n+2^{2k}) = s(n) + 2^k, \qquad 0 \le n \le 2^{2k-1} - 1, k \ge 1;$

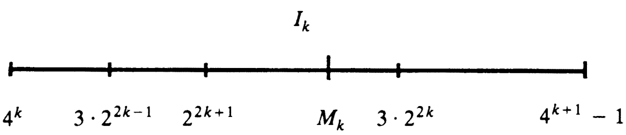
(18) $\quad s(n+2^{2k}) = -s(n) + 3\cdot 2^k, \qquad 2^{2k-1} \le n \le 2^{2k} - 1, k \ge 1;$

(19) $\quad s(n+2^{2k+1}) = s(n) + 2^{k+1}, \qquad 0 \le n \le 2^{2k} - 1, k \ge 0;$

(20) $\quad s(n+2^{2k+1}) = -s(n) + 2^{k+2}, \qquad 2^{2k} \le n \le 2^{2k+1} - 1, k \ge 0.$

The proof of the formulas, once we have found them, is not hard. The idea is to prove them by induction on n, for a fixed k (see [**3**, Satz 5]).

These formulas show that the values of the sequence $s(n)$, for n in $I_k = [4^k, 4^{k+1} - 1]$, can be generated from the values in $[0, 4^k - 1]$. In this process the interval I_k is divided into four pieces, the last two of which are twice as long as the first two. The integer $M_k = 2(2^{2k+2} - 1)/3$ is contained in the third of these subintervals.



This method of generating the sequence gives us some hope that we can prove the inequalities we want to. Formula (19) catches our eye first because $M_k = 2 + 2^3 + \cdots + 2^{2k+1}$ is a sum of odd powers of 2, so that $M_{k-1} + 2^{2k+1} = M_k$ (see the discussion following (14)). Hence we see that if $n + 2^{2k+1} \le M_k$, then $n \le M_{k-1}$. This looks promising, but there is a restriction on the values of n for which formula (19) holds: n must lie between 0 and $2^{2k} - 1$. Thus we can make use of (19) only for values of $n + 2^{2k+1}$ lying in the interval

$$2^{2k+1} \le n + 2^{2k+1} \le 2^{2k} - 1 + 2^{2k+1} = 3\cdot 2^{2k} - 1.$$

Since M_k is definitely in this interval, we can use formula (19) in the subinterval $[2^{2k+1}, M_k]$. We find that

$$\text{(21)} \quad \frac{s(n+2^{2k+1})}{\sqrt{n+2^{2k+1}}} = \frac{s(n)}{\sqrt{n}}\frac{\sqrt{n}}{\sqrt{n+2^{2k+1}}} + \frac{2^{k+1}}{\sqrt{n+2^{2k+1}}} < \frac{\sqrt{6n}+2^{k+1}}{\sqrt{n+2^{2k+1}}}$$

for $1 \le n \le M_{k-1}$, assuming that the inequality $s(n)/\sqrt{n} < \sqrt{6}$ is known for $n \le M_{k-1}$.[4] We get the inequality we want as long as

$$\text{(22)} \qquad \frac{\sqrt{6n}+2^{k+1}}{\sqrt{n+2^{2k+1}}} < \sqrt{6},$$

i.e., as long as $\sqrt{6n} + 2^{k+1} < \sqrt{6n + 3\cdot 2^{2k+2}}$. The latter inequality is equivalent to $n < 2^{2k+1}/3$, which is true in the interval we're considering $(n \le M_{k-1})$.

[4] Note that we have to assume $n \ge 1$ since we divide by $\sqrt{n}$ in (25).

We now have part of an induction step. This would prove $s(n)/\sqrt{n} < \sqrt{6}$ in $[2^{2k+1}+1, M_k]$.

Note that this analysis is already more complicated than the analysis in Section 2. This is to be expected, since the new question involves a ratio of functions.

The only part of the interval I_k we have yet to consider is $[2^{2k}, 2^{2k+1}]$. We might be able to use (17) in a similar way to handle this subinterval, but this formula would only be applicable for $n + 2^{2k} \leq 3 \cdot 2^{2k-1} - 1$, leaving out the subinterval $[3 \cdot 2^{2k-1}, 2^{2k+1}]$. However, we also notice from our tables that $s(n)$ appears to be bounded by 2^{k+1} in $[2^{2k}, 2^{2k+1}-1]$, which would give us the estimate

$$\frac{s(n)}{\sqrt{n}} \leq \frac{2^{k+1}}{2^k} = 2 < \sqrt{6}, \tag{23}$$

for $k \geq 0$. The one omitted value in (23), $n = 2^{2k+1}$, can be checked using the fact that $s(2^{2k+1}) = 2^{k+1} + 1$:

$$\frac{s(2^{2k+1})}{\sqrt{2^{2k+1}}} = \frac{2^{k+1}+1}{\sqrt{2}\cdot 2^k} < \sqrt{2} + 1 < \sqrt{6}, \tag{24}$$

for $k \geq 0$. So all we have to do to make this work is prove that $s(n) \leq 2^{k+1}$, for n in $[2^{2k}, 2^{2k+1}-1]$. The proof of this last assertion is very similar to the proof of Theorem 1b), as long as we also specify the places where $s(n)$ takes on the value 2^{k+1}. This gives the following statement.

Lemma 4. *For $2^{2k} \leq n \leq 2^{2k+1} - 1$ we have $s(n) \leq 2^{k+1}$, with equality if and only if $n = 2^{2k+1} - 1 - \sum_{r=0}^{k-1} \varepsilon_r 2^{2k+1}$, where $\varepsilon_r = 0$ or $1, k \geq 0$.*

Again we leave the straightforward induction proof to the reader (see [**3**, Satz 9]). The condition for equality in this lemma comes from analyzing the binary representations of the n-values at which equality holds.

We have all the pieces now!

Theorem 3. *For $n \geq 1$, $s(n)/\sqrt{n} < \sqrt{6}$.*

Proof: We just sketch the outline, since we have already given most of the details above. We focus on values of n in the interval $[4^k, 4^{k+1}-1]$. In the subinterval $[M_k, 4^{k+1}-1]$, $k \geq 0$, we use the argument in the paragraph containing (16). In the interval $[2^{2k}, 2^{2k+1}]$, $k \geq 0$, we use Lemma 4, (23) and (24). To prove the inequality in the remaining interval $[2^{2k+1}+1, M_k]$, for $k \geq 1$, we use induction on k. The assertion is true for $k = 1$, since the interval in question is $[9, 10]$, and the inequality can be checked directly. If the assertion is true for $k-1$, then (19), (21) and (22) show that it also holds for k. It is easy to check that this covers all the integers, and the proof is complete.

There is no way of telling whether this proof for the upper bound is the simplest proof. Perhaps the reader can find a better one.

5. THE LOWER BOUND. What about the lower bound? Will a similar approach establish that $s(n)/\sqrt{n} > \sqrt{3/5}$ for $n \geq 1$? Looking over what we've done, we observe that we used three different approaches in three different subintervals of I_k to establish the upper bound. In tackling the lower bound, we wonder if a more unified approach will work, one that just uses the formulas of Lemma 3 on each of the four subintervals of I_k determined by that lemma. We denote these four subintervals by I, II, III, IV.

On interval I, a simple argument using the lower bound $s(n) \geq 1$ in (17) gives

$$\frac{s(n+2^{2k})}{\sqrt{n+2^{2k}}} = \frac{s(n)+2^k}{\sqrt{n+2^{2k}}} \geq \frac{1+2^k}{\sqrt{3 \cdot 2^{2k-1}-1}} > \frac{1+2^k}{\sqrt{3 \cdot 2^{2k-1}}} = \sqrt{\frac{2}{3}}\,\frac{1+2^k}{2^k},$$

and the right hand side is obviously $> \sqrt{3/5}$, for $k \geq 1$. The same argument works in interval III, using (19), for $k \geq 0$. A similar argument using (20) gives the lower bound we want in interval IV (hint: use the upper bound from Lemma 4).

We're almost home. The last subinterval II is the trickiest one, since it contains the minimum point m_k. As it happens, we now run out of luck! It doesn't miss by much, but formula (18) is apparently not strong enough to allow us to prove what we believe to be true in the whole of interval II. (The reader may enjoy performing the calculations.) Unfortunately, we have found no way to finish the proof of the lower bound by this method, i.e., by piecing together inequalities on subintervals of I_k, even though it shows that the inequality we want is true for most integers. The interval idea leads to a dead end. R.I.P.[5]

This is a good example of a proof that fails by the slimmest of margins. However, this was only our first attempt to prove the lower bound. We tried to imitate what worked nicely for the upper bound, and got stuck. It might be that the interval method didn't work to establish the lower bound because the method was simply too crude. We need to go back and look at the problem again, if possible, from a different point of view.

How else can we look at the problem of getting a lower bound for $s(n)/\sqrt{n}$? As we look at the graph of $s(n)$, we see that there are only a finite number of places n where $s(n)$ has a fixed value k, since the values of $s(n)$ go to infinity with n (by Theorem 1a). When $s(n) = k$, for a fixed k, the ratio $s(n)/\sqrt{n}$ will be smallest when n is largest. This leads us to the following idea: let's focus on the last (largest) value of n for which a given integer k appears as $s(n)$ in the sequence $\{s(n)\}_{n=0}^{\infty}$; call it $\omega(k)$. If we could prove the inequality

$$k/\sqrt{\omega(k)} > \sqrt{3/5}\,, \text{ for } k \geq 1, \tag{25}$$

then the lower bound would follow for $s(n)/\sqrt{n}$, for any n, because taking $k = s(n)$ would give $s(n)/\sqrt{n} \geq s(n)/\sqrt{\omega(s(n))} > \sqrt{3/5}$. Thus the subsequence $\{k/\sqrt{\omega(k)},\ k \geq 1\}$ of $\{s(n)/\sqrt{n},\ n \geq 1\}$ is the key subsequence to consider in looking for the best lower bound for $s(n)/\sqrt{n}$. Not knowing what else to do, we set off to see if we can prove (25). First, we state the definition of ω formally.

Definition. For a given $k \geq 1$, let $\omega(k)$ be the largest value of n for which $s(n) = k$.

The next thing to do is to go back to our tables and determine $\omega(k)$ for the values of k up to 255. The beginning of the table is given in Table 4. In the full table we are surprised to find that $\omega(k)$ satisfies recursion formulas much like those satisfied by $s(n)$, but with some interesting wrinkles.

[5]Ending symbol for a proof that didn't work.

TABLE 4

k	1	2	3	4	5	6	7	8	9	10
$\omega(k)$	0	3	6	15	26	27	30	63	106	107

Lemma 5.

$$\omega(2n) = 4\omega(n) + 3, \qquad n \geq 1. \tag{26}$$

$$\omega(2n+1) = 4\omega(n+1) + 2, \quad n \geq 2, n+1 \neq 2^r, r \geq 2. \tag{27}$$

Proof: The proof of (26) is not hard. Note first that $s(n)$ and n have opposite parity, so that $\omega(2n)$ must be odd. Hence we have either that $\omega(2n) = 4m + 1$ or $\omega(2n) = 4m + 3$, for some $m \geq 0$. The first case is impossible, because by (11), $s(4m+1) = s(4m+3) = 2n$, so $4m + 1$ cannot be the largest argument of s to give $2n$. Thus $\omega(2n) = 4m + 3$. Then (11) implies that $2n = s(4m+3) = 2s(m)$, so that $s(m) = n$. If there were an $m_1 > m$ with $s(m_1) = n$, then $s(4m_1 + 3) = 2n$ and $4m_1 + 3 > 4m + 3 = \omega(2n)$ would contradict the definition of $\omega(2n)$. Thus $\omega(n) = m$, and $\omega(2n) = 4\omega(n) + 3$.

The proof of (27) is much trickier. To see how to approach the proof, let's first note one consequence of the formula. If (27) is true, then certainly

$$s(4\omega(n+1)+2) = s(\omega(2n+1)) = 2n+1.$$

How might we prove just this much? Formula (12) gives

$$\begin{aligned} s(4\omega(n+1)+2) &= 2s(\omega(n+1)) + (-1)^{\omega(n+1)}a(\omega(n+1)) \\ &= 2(n+1) + (-1)^n a(\omega(n+1)), \end{aligned}$$

and so $s(4\omega(n+1)+2) = 2n+1$ if and only if $a(\omega(n+1)) = (-1)^{n+1}$ (when $n+1$ is not a power of 2). This shows that to prove (27) we must consider the formula

$$a(\omega(n)) = (-1)^n, \qquad n \geq 3, n \neq 2^r, r \geq 2. \tag{28}$$

Since induction has worked so often before, it is worth trying to prove (28) by induction as well. This is what we do now. Formula (28) holds for $n = 3$, since $a(\omega(3)) = a(6) = -1 = (-1)^3$. Assume that (28) has been proved for all the integers m for which $3 \leq m < 2n+1$, for some $n \geq 2$. We proceed to prove it for $2n+1$ and $2n+2$. There are two cases to consider, because of the excluded values in (28).

Case 1: Suppose that $2n + 2 \neq 2^r$, for any $r \geq 3$. Since we have already proved (26), we can use that formula and the defining formulas (1) for $a(n)$ to compute $a(\omega(2n+2))$:

$$\begin{aligned} a(\omega(2n+2)) &= a(4\omega(n+1)+3) = -a(2\omega(n+1)+1) \\ &= (-1)^{1+\omega(n+1)}a(\omega(n+1)) \\ &= (-1)^{1+n}(-1)^{n+1} = (-1)^{2n+2}; \end{aligned}$$

this computation uses the fact that the parities of $\omega(n+1)$ and $n+1$ are opposite, and the induction assumption ($n+1 < 2n+1$ and $n+1$ is not a power of 2). This proves (28) for $2n+2$. Before considering (28) for $2n+1$, we need a formula for $\omega(2n+1)$. From what we have just shown it is easy to find a good candidate for $\omega(2n+1)$, since

$$s(\omega(2n+2)-1) = s(\omega(2n+2)) - a(\omega(2n+2)) = (2n+2)-1 = 2n+1.$$

Thus we might guess that $\omega(2n+1) = \omega(2n+2)-1$. If there were an $m > \omega(2n+2)-1$ for which $s(m) = 2n+1$, then because the sequence $\{s(m), m \geq 0\}$ goes to infinity by steps of ± 1, there would have to be an integer $m' > m$ for which $s(m') = 2n+2$. But then $m' \geq m+1 > \omega(2n+2)$ would give a contradiction. Hence our guess was correct, and $\omega(2n+1) = \omega(2n+2)-1 = 4\omega(n+1)+2$. This proves (27), since in this case $n+1$ is not equal to a power of 2.

Now (28) follows for the value $2n+1$, since

$$\begin{aligned} a(\omega(2n+1)) &= a(4\omega(n+1)+2) = a(2\omega(n+1)+1) \\ &= (-1)^{\omega(n+1)} a(\omega(n+1)) \\ &= (-1)^n(-1)^{n+1} = (-1)^{2n+1}. \end{aligned}$$

Case 2: If $2n+2 = 2^r$, for some $r \geq 3$, then to complete the induction we have to prove (28) only for the value $2n+1 = 2^r - 1$. Here we need the fact that $\omega(2^r-1) = 2^{2r-1}-2$. To see this, first note that

$$s(2^{2r-1}-2) = s(2^{2r-1}-1) - a(2^{2r-1}-1) = 2^r - 1$$

by Lemma 4 and the fact that there are $2r-2$ pairs of consecutive 1's in the binary expansion of $2^{2r-1}-1$. Furthermore, it can be proved by induction that $s(m) \geq 2^r$ for $2^{2r-1} \leq m \leq 2^{2r}-1$ (with equality if and only if $m = 2^{2r}-1-\sum_{i=0}^{r-2}\varepsilon_i 2^{2i+1}$, where $\varepsilon_i = 0$ or 1). This, together with Theorem 1a (take $k \geq r$), shows that $s(m) \geq 2^r$ when $m > 2^{2r-1}-2$, and hence that $\omega(2^r-1) = 2^{2r-1}-2$, as claimed.

It follows that $a(\omega(2n+1)) = a(\omega(2^r-1)) = a(2^{2r-1}-2) = (-1)^{2r-3} = (-1)^{2n+1}$, and this completes the proof of (28). With (28) we have also completely proved (27) as well. ∎

Looking back over this proof, we see that we were led to (28) by considering possible consequences of (27), but then proving (28) gave us a complete proof of (27) as a by-product: formula (27) is implied by (28) at the value $2n+2$. Actually, if we think of the induction proof as an argument that proceeds step-by-step through the positive integers, then the two formulas (27) and (28) are really *intertwined*, since (28) at $2n+2$ is used to establish (27), which is used in turn to prove (28) at $2n+1$. It is surprising that such intricate arguments are required to establish fairly simple recursion formulas.

After we had found the recursion formulas in Lemma 5, it seemed we were no closer to a proof of (25). However, we started to look for more patterns in the table by taking differences between consecutive values of $\omega(n)$, one of the standard ways of spotting possible formulas. Taking differences of the first 25 terms of the ω sequence gives:

n	1	2	3	4	5	6	7	8	9	10	11	12
$\omega(n+1)-\omega(n)$	3	3	9	11	1	3	33	43	1	3	1	11

n	13	14	15	16	17	18	19	20	21	22	23	24
$\omega(n+1)-\omega(n)$	1	3	129	171	1	3	1	11	1	3	1	43

What strange numbers! For long stretches the difference is 1 at odd integers, and then it skyrockets. At even integers n, the difference takes on the values $3, 11, 3, 43$, except at powers of 2, where it also suddenly increases. Powers of 2! Suddenly we see that the difference depends only on the power of 2 dividing n, except when n is 1 less than a power of 2, a wrinkle that fits with the recursion formulas in Lemma 5. We also see that the values $1, 3, 11, 43, 171$ satisfy a recursion: each value is 4 times the preceding value minus 1. When we solve the recursion for these values and investigate the wrinkle more closely, we find the following remarkable formulas.

Lemma 6. *a) If* $n = 2^{\alpha}(2m+1)$, *for* $m \geq 0$, $\alpha \geq 0$, *then* $\omega(n+1) - \omega(n) = (2^{2\alpha+1} + 1)/3$, *unless* $\alpha = 0$ *and* $n = 2^s - 1$, $s \geq 1$. *b) If* $\alpha = 0$ *and* $n = 2^s - 1$, $s \geq 1$, *then* $\omega(n+1) - \omega(n) = 2^{2s-1} + 1$.

We omit the details of the proof, and note only that part a) can be proved by induction on α, using (26), (27), and the special values $\omega(2^s - 1) = 2^{2s-1} - 2$ and $\omega(2^{s+2} - 2) = 2^{2s+3} - 5$.

Once we have a formula for the difference $\omega(n+1) - \omega(n)$, we are close to finding a formula for $\omega(k)$, since $\sum_{n=1}^{k-1}\{\omega(n+1) - \omega(n)\} = \omega(k) - \omega(1) = \omega(k)$. Summing up the expressions in Lemma 6 leads to the following explicit formula.

Theorem 4. *If* $2^r \leq k \leq 2^{r+1} - 1$, $r \geq 0$, *then*

$$\omega(k) = k - 1 + \frac{1}{3}(2^{2r+1} - 2) + 2\sum_{i=0}^{r-1}\left[\frac{k-1}{2^{i+1}}\right]2^{2i}.$$

We leave the somewhat technical details of the proof to the reader. This formula follows directly from Lemma 6, but may also be proved by a straightforward induction proof (on k) using only Lemma 5 and the fact that $\omega(2^r - 1) = 2^{2r-1} - 2$, $r \geq 0$. (See [3, Satz 1].)

Will this formula give us the lower bound we want?

Theorem 5. *For* $k \geq 1$, $k/\sqrt{\omega(k)} > \sqrt{3/5}$.

Proof: Assume that $2^r \leq k \leq 2^{r+1} - 1$, $r \geq 0$. By the formula for $\omega(k)$ we have

$$\begin{aligned} 3\omega(k) &= 3k - 3 + 2^{2r+1} - 2 + 6\sum_{i=0}^{r-1}\left[\frac{k-1}{2^{i+1}}\right]2^{2i} \\ &\leq 3k - 3 + 2^{2r+1} - 2 + 3(k-1)\sum_{i=0}^{r-1}2^i \\ &= 3k - 3 + 2^{2r+1} - 2 + 3(k-1)(2^r - 1) \\ &< 3k - 3 + 2k^2 - 2 + 3(k-1)k = 5k^2 - 5 < 5k^2 \end{aligned}$$

and the inequality of the theorem follows immediately. It worked!

Corollary. *For* $n \geq 1$, $s(n)/\sqrt{n} > \sqrt{3/5}$.

To summarize, we may combine Theorems 2, 3, and 5 in the following explicit result.

Theorem 6. *For* $n \geq 1$ *we have* $\sqrt{3/5} < s(n)/\sqrt{n} < \sqrt{6}$, *and the sequence* $\{s(n)/\sqrt{n}, n \geq 1\}$ *is dense in the interval* $[\sqrt{3/5}, \sqrt{6}]$. *In particular,*

$$\limsup_{n\to\infty} \frac{s(n)}{\sqrt{n}} = \sqrt{6} \quad \text{and} \quad \liminf_{n\to\infty} \frac{s(n)}{\sqrt{n}} = \sqrt{3/5}.$$

Looking back over our development, we find that our original purposes, which were to introduce the reader to the Golay-Rudin-Shapiro sequence, and to illustrate how mathematicians are led by their questions, have been realized. Sometimes, proofs of conjectures are constructed without difficulty and work more or less on the first attempt, as in Section 2.[6] On the other hand, this was not the case in the investigation of the lower bound, where a leap was required into a whole new investigation to get past a barrier in our first attempt at a proof. In the process we broke through into an area that is interesting in its own right, as is evidenced by the mysterious and elegant properties of the ω-function.

Since our main purpose has been to retrace the development of questions and ideas, we have given priority to these questions over full details of sometimes technical proofs. We hope the reader will find it an interesting challenge to fill in these details, or to read them in [3]. The reader can also obtain an expanded version of this paper by writing to the authors.

REFERENCES

1. J.-P. Allouche and J. Shallit, The ring of k-regular sequences, *J. Theoretical Computer Science* 98 (1992), 163–197.
2. J. Brillhart and L. Carlitz, Note on the Shapiro polynomials, *Proc. Amer. Math. Soc.* 25 (1970), 114–118.
3. J. Brillhart and P. Morton, Über Summen von Rudin-Shapiroschen Koeffizienten, *Illinois J. Math.* 22 (1978), 126–148.
4. J. Brillhart, P. Erdös and P. Morton, On Sums of Rudin-Shapiro Coefficients, II, *Pacific J. Math.* 107 (1983), 39–69.
5. M. Dekking, M. Mendes France, A. J. van der Poorten, Folds! I, II, III, *Math. Intelligencer* 4 (1982), 130–138, 173–181, 190–195.
6. M. J. E. Golay, Static multislit spectrometry and its application to the panoramic display of infrared spectra, *J. Optical Soc. America* 41 (1951), 468–472.
7. P. Morton and W. Mourant, Paper folding, digit patterns, and groups of arithmetic fractals, *Proc. London Math. Soc.* (3) 59 (1989), 253–293.
8. W. Rudin, Some theorems on Fourier coefficients, *Proc. Amer. Math. Soc.* 10 (1959), 855–859.
9. H. S. Shapiro, Extremal problems for polynomials and power series, Master's thesis, M.I.T., Cambridge, Mass., 1951.

Department of Mathematics
University of Arizona
Tucson, AZ 85721
jdb@math.arizona.edu

Department of Mathematics
Wellesley College
Wellesley, MA 02181
pmorton@wellesley.edu

[6] But recall the detailed analysis that led to precise conjectures before any proof had been attempted.