> **"By finding so clever a recipe to generate amicable pairs, Euler provided another indication—if another were needed—of why he is so esteemed in the world of mathematics."**

Euler's Amicable Numbers

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Anyone familiar with number theory has heard of amicable pairs. These are two whole numbers, each of which is the sum of the proper whole number divisors of the other. The simplest example is the pair 220 and 284. Here, the sum of the proper divisors of 220 (i.e., those divisors less than 220 itself) is

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284,$$

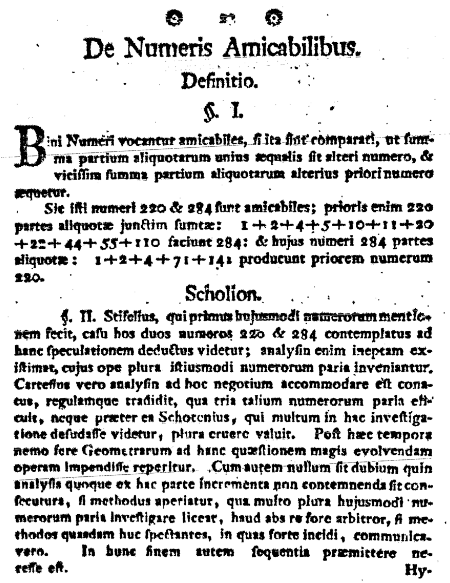
and the corresponding sum for 284 is $1 + 2 + 4 + 71 + 142 = 220$. Sure enough, each is the sum of the other's proper divisors—a phenomenon as fascinating as it is useless.

Prior to 1700, only three such pairs were known. The example above dates back to the ancient Greeks. For them, the Pythagorean tradition of imparting to numbers a near-mystical status would have made numerical superstars out of 220 and 284. Across the following millennia, the work of the 9th century Islamic mathematician Thãbit ibn Qurra and the 17th century French scholars Pierre de Fermat and René Descartes uncovered two additional pairs: the large 17,296 and 18,416 and the stupendous 9,363,584 and 9,437,056. (Their amicability may be checked at your leisure.)

So matters stood when Leonhard Euler (1707–1783) arrived upon the scene. The collective genius of mankind had dredged up three and only three pairs in the previous 2000 years. It thus boggles the mind that Euler found 58 more all by himself! Put another way, he single-handedly increased the world's supply of amicable numbers twenty-fold. As a *percentage* increment, this ranks high among history's great scientific achievements. Consider, by contrast, the astronomer William Herschel, whose discovery of Uranus in 1781 increased the known planets from six to seven—a meager advance by Eulerian standards.

So how did Euler do it?

In a nutshell, he recognized patterns that made the generation of amicable pairs more or less an automatic exercise. We shall follow his reasoning to the point of finding a "new" amicable pair—that is, one unknown to the world until Euler came along. The method presented here is taken from his 1750 paper with the mellifluous title, "De numeris amicabilibus."



De Numeris Amicabilibus.

Definitio.

§. I.

Bini Numeri vocantur amicabiles, si ita sint comparati, ut summa partium aliquotarum unius aequalis sit alteri numero, & vicissim summa partium aliquotarum alterius priori numero aequetur.

Sic isti numeri 220 & 284 sunt amicabiles; prioris enim 220 partes aliquotae junctim sumtae: 1+2+4+5+10+11+20+22+44+55+110 faciunt 284: & hujus numeri 284 partes aliquotae: 1+2+4+71+142 producunt priorem numerum 220.

Scholion.

§. II. Stifelius, qui primus hujusmodi numerorum mentionem fecit, casu hos duos numeros 220 & 284 contemplatus ad hanc speculationem deductus videtur; analysin enim ineptam existimat, cujus ope plura istiusmodi numerorum paria inveniantur. Cartesius vero analysin ad hoc negotium accommodare est conatus, regulamque tradidit, qua tria talium numerorum paria eruit, neque praeter ea Schotenius, qui multum in hac investigatione desudasse videtur, plura eruere valuit. Post haec tempora nemo fere Geometrarum ad hanc quaestionem magis evolvendam operam impendisse reperitur. Cum autem nullum sit dubium quin analysis quoque ex hac parte incrementa non contemnenda sit consecutura, si methodus aperiatur, qua multo plura hujusmodi numerorum paria investigare liceat, haud abs re fore arbitror, si methodos quasdam huc spectantes, in quas forte incidi, communicavero. In hunc finem autem sequentia praemittere necesse est.

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If one had to identify a single idea upon which the argument turned, it would be what we call the "Euler sigma function." Because this is critical to all that follows, we provide the formal definition.

Definition: If n is a whole number, then $\sigma(n)$ is the sum of all whole number divisors of n.

For example, $\sigma(15) = 1 + 3 + 5 + 15 = 24$. In his paper, Euler denoted this function by $\int n$, a choice that seems blasphemous in assigning a non-standard meaning to that most standard of symbols, the integral sign. Modern number theorists prefer to use σ.

We get an immediate characterization of primality in

Theorem 1. A number p is prime if and only if $\sigma(p) = p + 1$.

More intriguing is

Theorem 2. If $p \neq q$ are different primes, then $\sigma(pq) = \sigma(p) \cdot \sigma(q)$.

Proof. The only divisors of pq are 1, p, q, and pq, and so, by Theorem 1,

$$\sigma(pq) = 1 + p + q + pq = (1 + p) + q(1 + p) = (1 + p)(1 + q) = \sigma(p) \cdot \sigma(q).$$

Q.E.D.

We revisit $15 = 3 \cdot 5$ and see again that $\sigma(15) = \sigma(3) \cdot \sigma(5) = 4 \cdot 6 = 24$.

Euler recognized Theorem 2 as a specific instance of a more general result. The multiplicative rule holds not just for different primes but for any whole numbers whose greatest common divisor is 1. To state this explicitly, we offer (without proof)

Theorem 3. If $\gcd(a, b) = 1$, then $\sigma(ab) = \sigma(a) \cdot \sigma(b)$.

Thus, to determine the sum of all divisors of 585, we decompose it into relatively prime factors and apply Theorem 3:

$$\sigma(585) = \sigma(5 \cdot 9 \cdot 13) = \sigma(5) \cdot \sigma(9) \cdot \sigma(13) = 6 \cdot 13 \cdot 14 = 1092.$$

(We'll see this example again ... stay tuned.)

Euler could recast the definition of amicable pairs by observing that the sum of the *proper* divisors of a whole number n is just $\sigma(n) - n$. Consequently, M and N are amicable if and only if $\sigma(M) - M = N$ and $\sigma(N) - N = M$, which leads to the elegant characterization that M and N are amicable if and only if

$$\sigma(M) = M + N = \sigma(N). \tag{1}$$

It was this relationship that provided Euler his entrée into the world of amicability.

We are ready to examine his strategy. Logically, he employed what the Greek philosophers called "analysis." That is, he stipulated that M and N are amicable; used (1) to deduce consequences of amicability, thereby restricting the search to a manageable number of cases; and then checked that the process was reversible.

As a first case, Euler assumed that an amicable pair has the form $M = apq$ and $N = ar$, where p, q, and r are different primes and a (not necessarily prime) is the greatest common divisor of M and N. From (1), he knew that

$$\sigma(apq) = \sigma(M) = \sigma(N) = \sigma(ar),$$

and so by the multiplicative property above, $\sigma(a)\sigma(p)\sigma(q) = \sigma(a)\sigma(r)$. It follows that $\sigma(p)\sigma(q) = \sigma(r)$ or, because of primality, that

$$(p + 1)(q + 1) = r + 1. \tag{2}$$

Euler, who never met a substitution he didn't like, let $x = p+1$ and $y = q + 1$, so that by (2), $xy = r + 1$. He then could express the three primes p, q, and r in terms of the two variables x and y as

$$p = x - 1,\ q = y - 1, \text{ and } r = xy - 1.$$

But from (1) Euler also knew that $\sigma(M) = M + N$, or

$$\sigma(apq) = apq + ar = a(pq + r).$$

Consequently, $\sigma(a)\sigma(p)\sigma(q) = a(pq + r)$, and so $\sigma(a)(p + 1)(q + 1) = a(pq + r)$. Using x and y above, we now have $\sigma(a)xy = a[(x-1)(y-1) + (xy-1)] = a(2xy - x - y)$, which Euler solved for

$$y = \frac{ax}{[2a - \sigma(a)]\,x - a}. \tag{3}$$

It was time for another substitution (after all, the alphabet was not yet *completely* exhausted!), so Euler let

$$\frac{a}{2a - \sigma(a)} = \frac{b}{c}$$

in lowest terms. By (3) we see that

$$y = \frac{ax}{\left[\dfrac{ac}{b}\right]x - a} = \frac{bx}{cx - b},$$

and thus

$$cy - b = c\left[\frac{bx}{cx - b}\right] - b = \frac{b^2}{cx - b}.$$

Euler cross-multiplied to get his critical relationship:

$$(cx - b)(cy - b) = b^2. \tag{4}$$

At this point, the reader is forgiven for demanding a payoff. Is Euler lost? Is his argument completely adrift? Where is this cascade of letters taking us?

In fact, Euler now had all he needed to crank out amicable pairs at will. Just follow the four easy steps:

- Start with a value of a.
- From the relation
  $$\frac{a}{2a - \sigma(a)} = \frac{b}{c},$$
  find the values of b and c.
- Use (4) to find values for $cx - b$ and $cy - b$ and hence for x and y.
- Finally, if $p = x - 1$, $q = y - 1$, and $r = xy - 1$ are all prime, consider $M = apq$ and $N = ar$ as the candidates for amicability.

Obviously, some examples are in order. For starters, we choose $a = 4$, so that

$$\frac{a}{2a - \sigma(a)} = \frac{4}{8 - \sigma(4)} = \frac{4}{8 - 7} = \frac{4}{1}.$$

Then $b = 4$ and $c = 1$, and (4) becomes $(x - 4)(y - 4) = 4^2 = 16$. We need only examine the options for writing 16 as the product of whole numbers $x - 4$ and $y - 4$, the results of which are summarized in the following table.

$x-4$	$y-4$	x	y	$p=x-1$	$q=y-1$	$r=xy-1$
16	1	20	5	19	4	99
8	**2**	**12**	**6**	**11**	**5**	**71**
4	4	8	8	7	7	63

Only the middle row (bold) yields prime values for all three of p, q, and r as required by the analysis above. The associated pair is

$$M = apq = 4 \cdot 11 \cdot 5 = 220 \text{ and } N = ar = 4 \cdot 71 = 284.$$

Here we have good news and bad news—good because these are indeed amicable and bad because the Greeks knew them 2000 years before. But Euler was free to start with a different value of a, which might produce hitherto unknown amicable pairs.

This is exactly what happened. We follow his lead by letting $a = 585$. Using the calculation from above, we see that

$$\frac{a}{2a-\sigma(a)} = \frac{585}{1170-\sigma(585)} = \frac{585}{1170-1092} = \frac{585}{78} = \frac{15}{2}.$$

So, $b = 15$ and $c = 2$, and (4) this time yields $(2x-15)(2y-15) = 15^2 = 225$. As before, we write 225 as binary products of whole numbers and fill in a table.

$2x-15$	$2y-15$	x	y	$p=x-1$	$q=y-1$	$r=xy-1$
225	1	120	8	119	7	959
75	3	45	9	44	8	404
45	5	30	10	29	9	299
25	**9**	**20**	**12**	**19**	**11**	**239**
15	15	15	15	14	14	224

Again, only one row—the fourth—gives primes for p, q, and r. So, we have

$M = apq = 585 \cdot 19 \cdot 11 = 122{,}265$ and $N = ar = 585 \cdot 239 = 139{,}815$.

Eureka! Here is a "new"—i.e., never before seen—pair, one of the five dozen (or so) that appeared in Euler's paper.

Besides applying this procedure to other values of a (e.g., to $a = 819$ or $a = 5733$), Euler considered different initial structures for M and N. For instance, beginning with $M = apq$ and $N = ars$, or with $M = agpq$ and $N = ahr$, he derived analogous formulas that allowed him to add more pairs to the pot.

And so it was that Euler expanded the known amicable numbers by a factor of twenty. In following him through this simple argument, which involves little more than elementary algebra, the reader might have two immediate reactions: first, "It wasn't that hard" and, second, "I could have thought of it myself."

If so, the reader is correct about *one* of these.

As a matter of fact, this line of reasoning is genius incarnate. By finding so clever a recipe to generate amicable pairs, Euler provided another indication—if another were needed—of why he is so esteemed in the world of mathematics.