MARTIN GARDNER

The Square Root of two = 1.41421 35623 73095 ...

Roses are red,
Violets are blue.
One point 414 ...
Is the square root of two.

I confess that I wrote the above jingle only to have some light verse top this article. The dots at the end of the third line indicate that the decimal fraction is endless and nonrepeating. In other words, $\sqrt{2}$ is irrational. Although its decimal digits, like those of other famous irrationals such as pi and *e*, *look* like a sequence of random digits, they are far from random because if you know what the number is you can always calculate the next digit after any break in the sequence. Such irrationals also should not be called "patternless" because they have a pattern provided by any formula that calculates them. The square root of two, for example, is the limit of the following continued (endless) fraction:

$$\sqrt{2} = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + 1}}}$$

From this continued fraction one can derive rational fractions (fractions with integers above and below the line) that give $\sqrt{2}$ to any desired accuracy. The sequence 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, 577/408, 1303/985 ... is sometimes called "Eudoxus' ladder" after an ancient Greek astronomer and geometrician. The fractions are alternately higher and lower than their limit, which is $\sqrt{2}$. Each fraction is closer to $\sqrt{2}$ than its predecessor. The best approximation with numerator and denominator not exceeding three digits is 577/408. It gives $\sqrt{2}$ to five decimals places. If a fraction in this sequence is represented by a/b, the next fraction will be $(a + 2b)/(a + b)$. Note that on each "rung" of the ladder the numerator is the sum of its denominator and the denominator of the preceding fraction.

David Wells, in his *Penguin Dictionary of Curious and Interesting Numbers* (pages 34–35) gives some strange properties of the multiples of $\sqrt{2}$. For example, write in a line the multiples, omitting the fractional part. For example, 1 times $\sqrt{2}$, ignoring the decimal digits, is 1. Twice $\sqrt{2}$, ignoring the decimals, is 2. In this way you obtain the following sequence: 1, 2, 4, 5, 7, 8, ...

Beneath this sequence put down the numbers *missing* from the first sequence:

1 2 4 5 7 8 9 11 12 ...
3 6 10 13 17 20 23 27 30 ...

The difference between the top and bottom numbers at each n'th position is always twice n.

Normal Numbers

Any nth root of a positive integer (in all that follows "integer" will mean a positive integer) not itself an nth power is irrational. Although all such irrational roots have decimal digits that are neither random nor patternless, they are all, so far as anyone knows, "normal." This means that if you specify any pattern of digits, such as a single digit, pairs of adjacent digits, triplets of adjacent digits, and so on, in the long run the pattern will appear with just the frequency you would expect on the assumption that the probability of finding any given digit at any given place is always 1/10.

The pattern need not involve adjacent digits. They can be spaced any way you like. For example, you might consider the pattern *abc*, where *a* and *b* are separated by, say, seven digits, and *b* and *c* are separated by, say, 100 digits. All tests so far to determine the frequency of such patterns have shown that all irrational roots, in any base notation, are normal.

The most extensive tests for the normalcy of certain irrationals have been made for pi because pi has now been calculated to hundreds of millions of digits, but similar tests of other famous irrationals such as *e* and the golden ratio have shown no deviations from normalcy. I do not know how far $\sqrt{2}$ has been calculated, though I have a reference to it having been carried to more than a million digits in 1971 by Jacques Dutka, then a Columbia University mathematician.

One might imagine that all irrationals are normal, but it is easy to see that this is not the case. A popular example is the binary fraction .10100100010000.... The number clearly is not rational and just as clearly is far from normal.

MARTIN GARDNER (aka "Mr. Mathematical Games") has written more than 60 books on mathematics, magic, Alice in Wonderland, and scores of other topics.

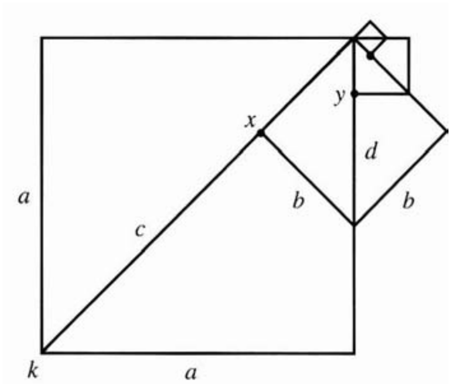

Figure 1. An infinite descent proof that $\sqrt{2}$ is irrational

$\sqrt{2}$ and Drowning at Sea

The discovery of irrational roots was first made by the Pythagoreans, a secret brotherhood that flourished in ancient Greece. Their discovery of the first irrational number, the square root of 2, was a milestone in the history of mathematics. In geometrical form this says that the diagonal of a unit square is incommensurable with the square's side. No ruler, no matter how finely graduated, can accurately measure the two line segments. If the side of a square is rational, the diagonal will be irrational, and vice versa.

There are two legends about the explosive effect of this discovery. One is that a Pythagorean named Hippasus was sworn not to reveal the discovery because it shattered the Pythagorean belief that integers accurately measure all things. Hippasus broke the vow. As a result he was drowned at sea either by suicide, murder, or by the wrath of the gods—the legend has many variations. The other legend has the Pythagoreans celebrating their great discovery by sacrificing many oxen to the gods. The discovery of incommensurable line segments had a profound influence on Plato who writes about it in his *Laws*.

Infinite Descent

The Greeks proved the incommensurability of a square's side and diagonal by a clever "infinite descent" proof using the diagram shown in Figure 1. Assume that the side of the largest square is commensurable with its diagonal. If so, each of the two line segments will be multiples of a unit which we call k. Draw a smaller square of side b, choosing point x so that $a = c$. Side b of this square will be commensurable with its diagonal because each is a multiple of k. Next we select point y so that $d = b$. Again, the side and diagonal of this smaller square will be commensurable with respect to k.

This process can be continued to infinity as suggested by the fourth tiny square. The sides of all these squares cannot be zero, but at some point in the endless construction we reach a square with a side less than k. A length less than k cannot be a multiple of k, so we have encountered a contradiction proving that our assumption, that the side and diagonal of a square are commensurable, is false. If the square's side is 1, the diagonal is $\sqrt{2}$. We have shown that $\sqrt{2}$ is irrational.

We can express the proof another way. We seem to get an infinite series of integers (multiples of k) each smaller than the previous one, but such a series obviously must be finite.

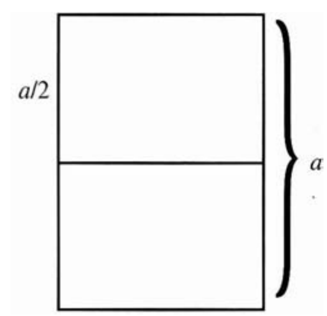

Figure 2. Another infinite descent proof.

Hugo Steinhaus, in the first chapter of *Mathematical Snapshots*, gives a different geometrical proof by infinite descent. It is based on the rectangle shown in Figure 2. Its sides are in a ratio such that if the rectangle is sliced in half as shown, each half will be a rectangle similar to the original one. If the sides are labeled as indicated, a and b will be in the same ratio as $a/2$ and b. The equation reduces to $a^2 = 2b^2$, so if $b = 1$, a will be $\sqrt{2}$.

Assume that a and b are commensurable, each side a multiple of unit k. Of course, k can be any unit, inches, centimeters, or whatever.

In Figure 3 we have attached to the long side of rectangle ab a congruent rectangle that has been given a quarter turn clockwise. This produces a larger rectangle of sides b and $(a + b)$. by cutting two squares of side b from this large rectangle we produce the smaller shaded rectangle. Its sides are b and $(a - b)$. Because a and b are integers, $(a-b)$ must also be an integer. Therefore the shaded rectangle must have sides that are multiples of k.

We can repeat the procedure by cutting two squares from the shaded rectangle to create a still smaller rectangle, similar to the shaded one, with sides that also must be multiples of k. As in the previous proof, if this process is continued we soon produce a rectangle with sides smaller than k. We have reached a contradiction. The procedure can be carried to infinity, but one cannot have an infinite sequence of integers that keep getting smaller and smaller. Therefore a and b are incommensurable, and $\sqrt{2}$ is irrational. Infinite descent proofs can be given algebraic forms, many of

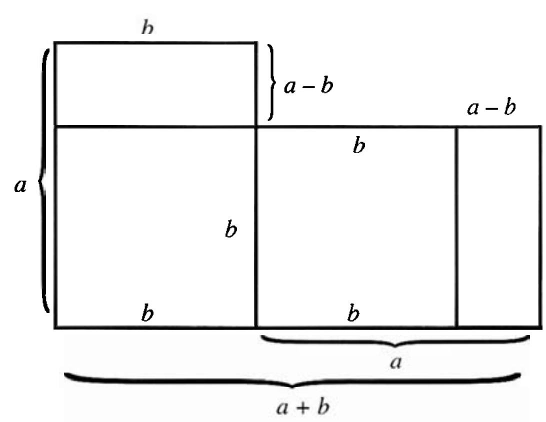

Figure 3

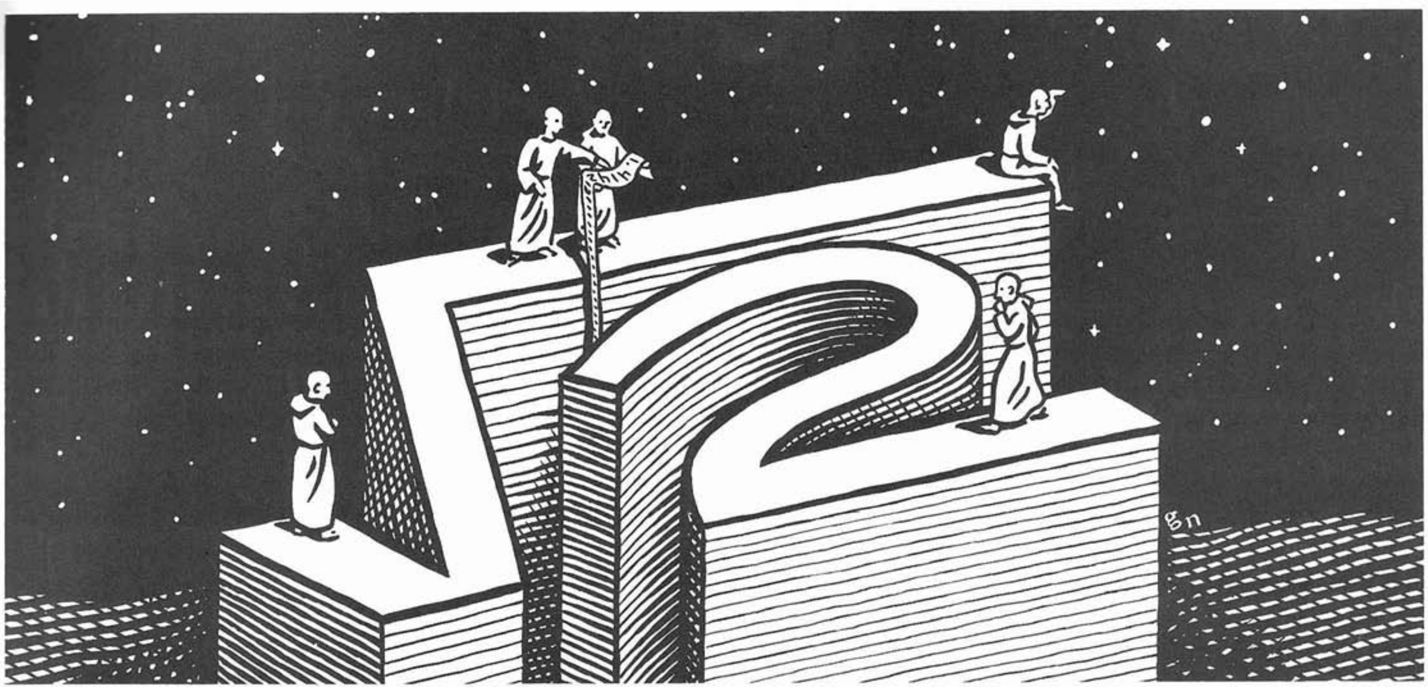

Illustration by Greg Nemec

which generalize to proving that any nth root not an nth power is irrational.

For an application of the 1 by $\sqrt{2}$ rectangle to a magic trick involving the repeated folding of a playing card, see the chapter on rep-tiles in my *Unexpected Hanging and Other Mathematical Diversions*. The rectangle is called an order-2 reptile because it can be cut into two parts each similar to itself. British and European papers usually have sides in 1 to $\sqrt{2}$ ratio so that when halved, quartered, and so on, the sheets remain similar.

Odds and Evens

The ancient Greeks also had an elegant way of using the laws of odd and even numbers to prove $\sqrt{2}$ is irrational. It can be expressed in numerous ways, but the following seems the simplest.

Let a stand for the hypotenuse of a right isosceles triangle and b for its side. We know from the Pythagorean theorem that $a^2 = 2b^2$, or $a^2/b^2 = 2$. The fraction a/b obviously is between 1 and 2. Assume it is reduced to lowest terms—that is, its top and bottom numbers have no common divisor other than 1. We know b is greater than 1, otherwise a/b would be an integer.

The right side of $a^2 = 2b^2$ is even, therefore the left side a^2 is also even, and a is even because the square root of any even number is even. For a we can substitute $2x$ where x is any integer. Squaring $2x$ gives $4x^2$, so we can write $4x^2 = 2b^2$. This reduces to $2x^2 = b^2$. The left side is even, therefore b^2 is even and b is even. Because both a and b are even, each can be divided by 2. This contradicts the assumption that a/b has been reduced to lowest terms. We have proved that a/b cannot be a rational fraction between 1 and 2, therefore $\sqrt{2}$ is irrational.

Euclid gave this proof in Book 10, and Aristotle alludes to it in many places. According to Plato in his dialogue *Theaetetus* (section 147), Theodorus of Cyrene, a brilliant philosopher and geometrician, also proved the irrationality of the square roots of all nonsquares of 3 through 17. Alas, none of his writings survive, so we don't know how he did it, or why he stopped at 17. Incidently, Theodorus was banished from Cyrene because he doubted the existence of the Greek gods. With suitable modifications, parity (odd-even) proofs of $\sqrt{2}$ can be generalized to all nth roots of integers that are not nth powers.

Each of the foregoing proofs is a *reductio ad absurdum* or "indirect" proof in which an assumption is made then later proved false by a contradiction. A whimsical indirect proof of the irrationality of $\sqrt{2}$ is based on the final digit of square numbers. It is easy to see that this digit must be 0, 1, 4, 5, 6, or 9. Consider again the equation $a^2 = 2b^2$, where a/b is reduced to lowest terms, b greater than 1.

The terminal digit of both a^2 and b^2 must be one of the six listed above. On the right side of $a^2 = 2b^2$, b^2 is multiplied by 2, therefore the final digit of $2b^2$ must be 0, 2, or 8. It cannot be 2 or 8 because there is no 2 or 8 as the last digit of a^2. The only match is 0. So a^2 and $2b^2$ must each end in zero. It follows that a must end in 0, and b must end in 0 or 5. In either case both a and b are divisible by 5, contradicting the assumption that a/b is reduced to lowest terms. Hence a/b is irrational and $\sqrt{2}$ is irrational.

Similar terminal digit proofs of the irrationality of $\sqrt{2}$ can be formulated in other base notations. In binary notation, for example, the proof is unusually simple. The left side of $a^2 = 2b^2$ terminates in an even number of zeros and the right side terminates in an odd number of zeros.

Many elegant proofs of the irrationality of $\sqrt{2}$ are based on the fundamental theorem of arithmetic which states that every integer is the product of a unique set of primes. Here is one of the easiest to follow.

As before, we use the equation $a^2 = 2b^2$ where a/b is a rational fraction reduced to lowest terms, b greater than 1. The term a^2 must have an even number of prime factors. Why? Because if a is the product of either an odd or an even number of primes, its square will have twice as many prime factors.

Consider now the right side of $a^2 = 2b^2$. It will have an odd number of prime factors because to the even number of prime factors of b^2 we add the prime factor 2. We have produced a contradiction because the number of prime factors for the two sides of the equation cannot be even on one side and odd on the other. It is not difficult to see that the proof applies to the square root of any prime, or to any integer with an odd number of prime factors.

Prime divisors provide a simple proof that any square root not an integer is irrational. We apply it first to $\sqrt{2}$. From $a^2 = 2b^2$ we can derive the equation $b^2 = a^2/2$ which is the same as a times $a/2$. If a prime divides the product of two integers x and y, it obviously must divide either x or y. Let a^2 and a be the two integers whose product is $a^2/2$. There must be a prime which divides b^2 because b is greater than 1. This same prime must divide the right side of the equation, therefore it must divide $a/2$ or a. In either case it divides a because if it divides half of a, it will also divide a. Contradiction! We have shown that a prime divides both a and b, therefore a/b cannot be a rational fraction reduced to lowest terms.

Substitute for 2 any integer whose square root is not an integer and the foregoing proof holds. With further generalizations the proof will apply to all nth roots of integers that are not nth powers.

Another simple proof of the irrationality of $\sqrt{2}$ is based on inequalities. If a/b is $\sqrt{2}$ reduced to lowest terms, then b is less than a, and a is less than $2b$, therefore $(a-b)$ is less than b. Start with $a^2 = 2b^2$, and make the following changes:

$$a^2 - ab = 2b^2 - ab$$
$$a(a-b) = b(2b-a)$$
$$a/b = (2b-a)/(a-b)$$

As we have seen, $(a-b)$ is smaller than b. We have contradicted the assumption that a/b is reduced to lowest terms. This proof also generalizes to any nth root of any number not an nth power.

There are dozens of other ways to prove the irrationality of the square roots of integers that are not squares, many of which extend easily to nth roots. They all come down to the following theorem: If a/b is a rational fraction in lowest terms, b greater than 1, then any power of a/b will also be a rational fraction that cannot be reduced to lower terms.

This can be proved by the following argument involving prime factors. Assume that a/b, with b greater than 1, is reduced to lowest terms. The prime factors of a will have no factors in common with b, otherwise the common factors cancel out and a/b is reduced. Consider now the square of a/b. The factors above the line will be the same as before, each repeated twice, and the same for the prime factors below the line. There are still no common factors to cancel. This means that the square of a rational fraction reduced to lowest terms is another fraction reduced to lowest terms, so it cannot be an integer. In brief, no integer not a square can have a square root that is rational.

The argument obviously applies to cubes and all higher roots. For example, a^3/b^3 is $(a \times a \times a)/(b \times b \times b)$. This too is a nonreducible fraction because there are no common prime factors above and below the line to be canceled. Is there any simpler, easier to comprehend, way to show that nth roots of integers not nth powers are irrational?

When I was in high school and first learned that $\sqrt{2}$ could not be expressed as a rational fraction, I couldn't believe it. I squandered many hours in study periods trying to find such a fraction. Eventually I convinced myself it couldn't be done, but today I have no memory of how I proved it, if indeed I did. I like to think it was one of the proofs given in this article. It would be interesting to know how many mathematicians, far greater than I, had a similar experience when they were very young.

Note that all the proofs in this article are *reductio ad absurdum* proofs. They illustrate how powerful this type of proof is. As G. H. Hardy put it in his famous *Mathematician's Apology:*

> It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game.* ■

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Hundreds of books contain proofs of the irrationality of $\sqrt{2}$ and more general proofs of the irrationality of any nth root of an integer not an nth power. What follows are references in easily accessible periodicals.

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