The Circle-Squaring Problem Decomposed

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Many of us have encountered the famous Tangram puzzle (see Figure 1), which asks the player to arrange a specific set of geometric pieces to form a square, and then rearrange the same pieces to form other shapes, such as a swan or a candle.

What if we restrict the movements of the pieces? If the pieces are allowed to move only by translation, with no rotations or reflections, then it becomes more difficult to generate a specific shape. We will use this idea of moving pieces by translation only to approach a very famous problem in mathematics—the Circle-Squaring problem.

In 1925, Alfred Tarski asked whether a circle can be partitioned into finitely many pieces that can be rearranged to form a square. More technically, is a closed disk in the plane *equidecomposable* with a square (together with its interior)? While you might imagine cutting a circular piece of paper into pieces and then rearranging these pieces in the hopes of obtaining a square, it turns out that the problem is not solvable in this manner. A 1964 publication by Dubins, Hirsch, and Karush informs us that a circular disk is "scissor-congruent" to no other strictly convex body. We can never physically achieve a solution to the circle-squaring problem with scissors and paper.

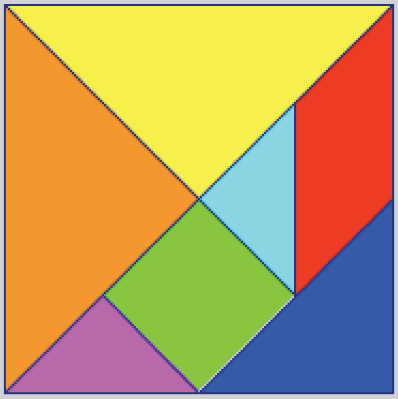

Figure 1. The Tangram Puzzle.

Laczkovich gives an upper bound of 10^{50} for the number of pieces that are required in his Circle-Squaring decomposition, and he shows that the rearrangement of the pieces can be accomplished using translations alone.

Miklós Laczkovich, however, shocked mathematicians around the world with an affirmative response to Tarski's question. In 1990, Laczkovich proved that any circle in the plane is equidecomposable with a square of equal area. He succeeded because he allowed pieces that are difficult to imagine—dustings of points that are selected using the controversial Axiom of Choice. Laczkovich's proof shows that such a decomposition is theoretically possible, but there is no picture to help us understand how this is accomplished. He gives an upper bound of 10^{50} for the number of pieces that are required in this decomposition, and he shows that the rearrangement of the pieces can be accomplished using translations alone. None of the pieces require a rotation or a reflection.

In an effort to shed some light on this remarkable achievement, we ask: How closely can we approximate the circle-squaring process using pieces that are easy to visualize? In the spirit of Laczkovich's proof, we add the further restriction that our pieces must move by translation only.

Dissections vs. Decompositions

To clarify the problem, we must make a distinction between a dissection and a decomposition. A dissection occurs when a planar figure is cut and the pieces are rearranged to form another shape such as in the Tangram puzzle. The study of dissections has fascinated both amateur and professional mathematicians for centuries and has led to many interesting puzzles and pictures. A terrific account of this story is Greg Frederickson's 1997 book, *Dissections: Plane and Fancy*.

For dissections we do not concern ourselves with what happens to the points on the boundaries of the

What's So Controversial About the Axiom of Choice?

Given a collection of non-empty disjoint sets, the Axiom of Choice asserts that it is possible to construct a new "transversal" set that consists of exactly one element from each set in the original collection. What's the harm in that? Well, none at all if there are only a finite number of original sets. The trouble comes when we start with an infinite number of sets and there is no general way to assert how the actual "choosing" is to be done. To take a classic example, if we have ten pairs of socks, no one would object to the existence of a new 10-element set that consists of one sock from each pair. If we are presented with an infinite number of pairs of shoes, very few mathematicians would object to the existence of the transversal set that consists of all the right-footed shoes.

In the first example we have only a finite number of choices to enact, and in the second example, although we have an infinite number of choices to carry out, it is possible to define precisely how each choice is to be made. But what about the situation where we are presented with an infinite number of pairs of socks? Can we create a transversal set that consists of one sock from each pair? The Axiom of Choice says "yes we can!" but we must confess that there is a *non-constructive* aspect to this assertion. Making an infinite number of choices isn't physically viable, and there doesn't seem to be any other way to make it clear exactly which socks are to become the members of our new transversal set.

The Axiom of Choice became an indispensible tool after it was articulated by Zermelo in 1904, as evidenced by its long list of equivalent formulations. The most familiar of these is probably Zorn's Lemma, which is the crucial step in proving many intuitive results such as, "Every vector space has a basis." But beware! Not all of the consequences of the Axiom of Choice are so self-evident—or even reasonable. Perhaps the most counter-intuitive consequence of all appeared in 1924 when Banach and Tarski used the Axiom of Choice to show that it was possible to decompose a solid sphere into a finite number of pieces that could then be rearranged to form a new sphere of *any* size.

pieces. The Circle-Squaring Problem, however, asks whether there exists a *decomposition* of the disk to the square, and this decomposition must take into account each and every point of the disk. In this case, points lying on the boundaries of the pieces become something of an issue. Although it is possible to account for the movements of each boundary piece, for now we focus our attention on dissections only.

A natural way to approximate a circle is to use an inscribed regular polygon, or n-gon. By increasing the number of sides of the polygon we can approximate the circle with as much precision as we desire. (See Figure 2.)

Can we dissect a regular n-gon and rearrange the pieces to form a square? This is certainly possible because the Bolyai-Gerwein Theorem of 1832 tells us that any polygon in the plane can be dissected and rearranged to form any other polygon of equal area. It turns out, however, that because we wish to use translations only, we must restrict ourselves to dissecting polygons with an even number of sides. Polygons with an odd number of sides always require at least one rotation in order to generate a square. (See Boltyanski's book listed in the "Further Reading" section.)

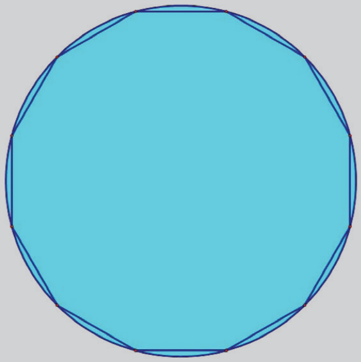

Figure 2. The 12-gon's area is only 4.5% less than the area of a circle.

The Algorithm

Here is our procedure for completing the even n-gon-to-square dissection using only translations.

1. Orient the n-gon so that one pair of sides is horizontal. This is always possible because n is even.
2. Decompose the n-gon into strips by drawing horizontal line segments that connect each vertex on the left with its corresponding vertex on the right. (See Figure 4.)
3. Translate each piece above the center strip so that it sits next to its corresponding piece below the center strip in order to form a parallelogram. (See Figure 5.)

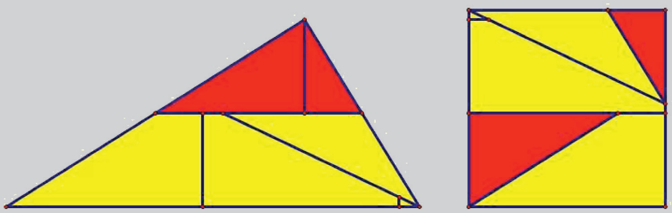

Figure 3. Rearranging the pieces of a triangle to form a square requires at least one rotation (rotated pieces shown in red).

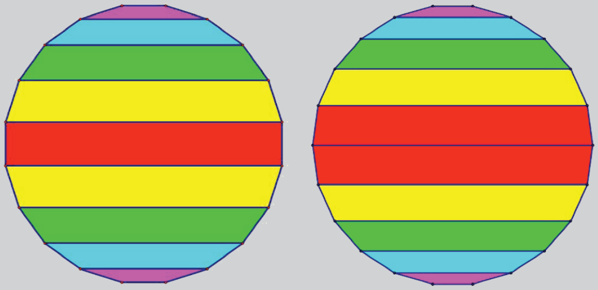

Figure 4 (on left). Creating horizontal strips when $n/2$ is even and odd respectively.

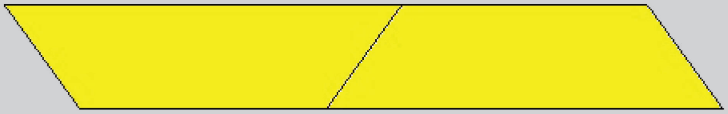

Figure 5. Forming parallelograms by matching corresponding sides.

4. Turn each parallelogram into a rectangle by slicing off a triangle from the right-hand side of the parallelogram and translating that piece to the left side. (See Figure 6.) When $n/2$ is even, the center strip is already in the desired format, so steps 3 and 4 are unnecessary.

5. Now we form a new rectangle from the current rectangle. The width of the new rectangle should be the width of the square that is our final product. We use a known technique here called a *parallelogram slide* or *P-slide* (the details can be found in Fredrickson's book) and we overlay the new cuts onto the cuts that already exist. (See Figure 7.)

6. Finally, we stack the new rectangles to form a square whose area is equal to that of the original n-gon. (See Figure 8.)

What's the Count?

As we look at these particular dissections, it is natural to ask how many pieces are required to form a square from a regular n-gon. Since the regular 4-gon is already a square, its dissection requires only one piece. Using the method just outlined we find that the regular 6-gon requires six pieces, the 8-gon needs nine pieces, and the 10-gon uses 12 pieces. Let $P(n)$ denote the number of pieces generated when dissecting an n-gon using

Figure 6. Turning parallelograms into rectangles.

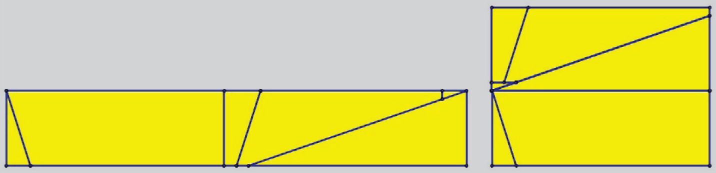

Figure 7. The re-rectangling process.

the "slicing" algorithm presented here. Because we have been dissecting only polygons with an even number of sides, the only acceptable values of n for this function are even natural numbers greater than or equal to 4. The sequence $P_k = P(2k)$ for $k = 1, 2, 3, \ldots$ looks like

1, 6, 9, 12, 15, 19, 22, 25, 28, 31, 35, 38, 42, 47, 50, 53, 56, 60, 63, …

Does this sequence represent the minimum number of pieces required in a $2k$-gon-to-square dissection where the movements are restricted to translations only? The answer is no. Busschop has given a 5-piece hexagon-to-square dissection that uses only translations (which can also be found in Fredrickson's book). We conjecture, however, that this is the only exception. That is, the revised sequence, beginning 1, 5, 9, 12, 15, 19, 22, … represents the minimum number of pieces required in a $2k$-gon-to-square dissection using only translations.

The general formula for generating the terms in this sequence is rather complex, but we can predict the values of $P(n)$ very accurately with a remarkably simple function. For $n > 6$, the curve $f(x) = .212x \cdot \ln(58.906x)$ appears to be an excellent fit for the data. (See Figure 9.)

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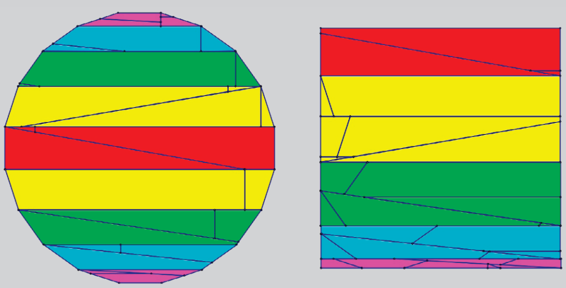

Figure 8: The squared 20-gon requires 28 pieces.

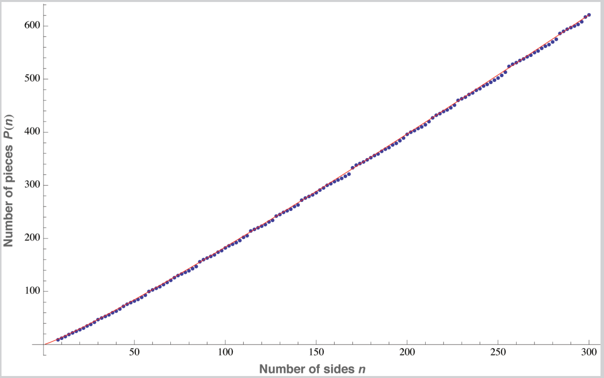

Figure 9. The sequence $P(n)$ grows like $f(x) = .212x \cdot \ln(58.906x)$, which is shown in red.

first player's choices: 17, 13, 9, 5, and 1. Can you now see who has a winning strategy for any n?

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. All problems and/or solutions may be submitted to Derek Smith, Mathematics Department, Lafayette College, Easton, PA 18042. Electronic submissions (preferred) may also be sent to smithder@lafayette.edu. Please include your name, email address, school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is January 10, 2010.

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Dissection to Decomposition

As we mentioned, a dissection is not the same thing as a set-theoretic decomposition. A true decomposition of the n-gon would describe the behavior of each of the vertices and each of the boundary segments that result when we "cut" the n-gon. In his book, *The Banach-Tarski Paradox*, Stan Wagon presents a theorem stating that if two polygons are equidissectable then they are equidecomposable. In other words, if we can find a dissection between the two figures, then we can also find a decomposition between them. The technique involved in the proof of this theorem provides a method for absorbing boundary segments when they are not needed and creating new segments when necessary. Fortunately, each of these procedures can be accomplished using finitely many translations, and so we can use these methods to complete the "translations only" decomposition of an even-sided n-gon to a square of equal area that we were seeking.

Our algorithm provides a starting point for visualizing Tarski's Circle-Squaring Problem and gives a concrete formula for the number of pieces required based on the size of the approximating polygon. If we were to carry out a dissection using the full allotment of the 10^{50} or so pieces that Laczkovich offers as an upper bound, then our n-gon would be essentially indistinguishable from a circle—but of course, it would still not be a circle. The remarkable thing about Laczkovich's proof is not the size of the upper bound on the number of pieces but the fact that there is one. By moving from dissections to decompositions, and trading in his scissors for the Axiom of Choice, Laczkovich was able to move his argument beyond the world of polygons to find a proof for perfect circles. ■

Further Reading

V.G. Boltyanskii, *Equivalent and Equidecomposable Figures*, D.C. Heath and Company, Boston, 1963.

Greg N. Frederickson, *Dissections: Plane and Fancy*, Cambridge, Cambridge UK, 1997.

M. Laczkovich, "Equidecomposability and Discrepancy; a Solution of Tarski's Circle-Squaring Problem," *Journal Für Die Reine Und Angewandte Mathematik*, 404 (1990), 77-117.

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