Lewis Carroll's
Condensation Method for Evaluating Determinants

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At first sight, the following poem looks like a fairly standard, slightly overly melodramatic, example of Victorian verse. But there is more to it than meets the eye:

> I often wondered when I cursed,
> Often feared where I would be—
> Wondered where she'd yield her love
> When I yield, so will she,
> I would her will be pitied!
> Cursed be love! She pitied me…

This poem has an unusually clever structure which enables it to be read both horizontally and vertically. Although it has been largely forgotten for many years, its author is still very well known, for he was none other than Lewis Carroll, most famous as the writer of *Alice in Wonderland* and *Through the Looking Glass*.

As many people know, 'Lewis Carroll' was a pseudonym. His real name was the Reverend Charles Lutwidge Dodgson (1832-1898), a mathematics lecturer and Church of England clergyman from Oxford. For Dodgson, writing children's books, puzzles, and verses was just a hobby. His day-job was teaching mathematics at Christ Church, the largest of the 39 colleges in the University of Oxford.

In addition to this, he also wrote and published a considerable number of books on mathematical subjects under his real name, including *A Syllabus of Plane Algebraical Geometry* (1860) and *Euclid and his Modern Rivals* (1879). Although more of a recreational mathematician than a serious researcher, Dodgson did make a number of small contributions to the subject, most of which are largely unknown to today's mathematicians. One of these is of direct relevance to undergraduates, dealing as it does with a concept fundamental to linear algebra: the determinant.

Determinants emerged gradually during the 18th century through the theory of equations in the work of Leibniz, Maclaurin, Cramer, and Laplace. By the 19th century, the subject had become a mathematical area of increasing significance. Gauss (who invented the name 'determinant'), Cauchy, and Cayley all produced important results on the subject, and in 1841, the German mathematician Carl Jacobi published three major papers that finally brought the subject into the mathematical mainstream. An illustration of this was the publication, ten years later, of the very first textbook on the subject, *Elementary Theorems Relating to Determinants*, by Dodgson's Oxford contemporary, the English mathematician William Spottiswoode.

A Brief Introduction to Determinants

As every algebra student knows, given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

its determinant, $\det A$ or $|A|$, is equal to $ad-bc$. Given a matrix A, its determinant provides useful geometric and algebraic information about the matrix. Geometrically, the row entries of an $n \times n$ matrix A define the edges of a parallelepiped in n-dimensional space, of which the volume is simply the absolute value of $\det A$. Algebraically, the same matrix represents the coefficients of a system of n linear equations in n unknowns. The determinant of A determines whether or not this system is solvable. In particular, if $\det A$ is nonzero, we know that the inverse matrix A^{-1} exists, and this in turn promises a unique solution to the system of linear equations represented by matrix A.

Although 2×2 determinants can be calculated very easily, computing determinants is more time-consuming for larger matrices. The standard way of computing a determinant (both today and in Dodgson's time) is to break it down into more determinants of lower degree by taking the product of any row or column entry and the determinant of its complementary minor, then alternately adding and subtracting the results.

For example,

$$\begin{vmatrix} 2 & 1 & -3 \\ 3 & -2 & 1 \\ -1 & 3 & -2 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -14.$$

This method, while perfectly reliable, can get very long and laborious as the dimensions of the determinant increase.

These days, with the help of packages like *Mathematica*, *Maple*, or *Matlab*, such intricate calculations can be done in a matter of seconds. But in the 19th century, of course, they had no such luxury, nor do many of today's math students. It was in order to simplify the work involved in calculating determinants of large matrices that Dodgson introduced his new method—a method which, we believe, can still be of use today.

Dodgson's Method of "Condensation"

Dodgson's method originally appeared as his first and only research paper, published by the Royal Society of London in 1866. The following year he included it in a textbook entitled *An Elementary Treatise on Determinants, with their application to simultaneous linear equations and algebraical geometry*. This work (the only book that Dodgson ever published on algebra) prompted an appealing—but entirely unfounded—anecdote. The story goes that Queen Victoria enjoyed *Alice in Wonderland* so much that she asked to be sent the author's next book, and was slightly bemused when she received *An Elementary Treatise on Determinants*! Amusing though this story is, according to Dodgson, "it is utterly false in every particular: nothing even resembling it has ever occurred." Dodgson called his method "Condensation of Determinants, being a new and brief method for computing their arithmetical values," basing it in part on the following definition:

Given an $n \times n$ matrix A, with $n \geq 3$, the *interior* of A, or int A, is the $(n-2) \times (n-2)$ matrix that results when the first row, last row, first column, and last column are deleted.

The method then consisted of four steps:

1. Use elementary row and column operations to remove all zeros from the interior of A.

2. Find the 2×2 determinant for every four adjacent terms to form a new $(n-1) \times (n-1)$ matrix B.

3. Repeat this step to produce an $(n-2) \times (n-2)$ matrix, and then divide each term by the corresponding entry in the interior of the original matrix A, to obtain matrix C.

4. Continue "condensing" the matrix down, until a single number is obtained. This final number will be det A.

As an illustration of Dodgson's method, let's consider the 5×5 matrix

$$A = \begin{pmatrix} 2 & -3 & 1 & 2 & 5 \\ 4 & 1 & -2 & -3 & 2 \\ 5 & -4 & 2 & 2 & -3 \\ 3 & -1 & 5 & 2 & 1 \\ -4 & 1 & 5 & -1 & 2 \end{pmatrix}.$$

Finding all 2×2 determinants of adjacent terms, we 'condense' the matrix into the following 4×4 matrix:

$$B = \begin{pmatrix} 14 & 5 & 1 & 19 \\ -21 & -6 & 2 & 5 \\ 7 & -18 & -6 & 8 \\ -1 & -10 & -15 & 5 \end{pmatrix},$$

where, for example, the first two entries of row 1 come from

$$\begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} = 14,$$

and

$$\begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} = 5.$$

This matrix is, in turn, similarly condensed into the following 3×3 matrix:

$$\begin{pmatrix} 21 & 16 & -33 \\ 420 & 72 & 46 \\ -88 & 210 & 90 \end{pmatrix}.$$

For example, the first entry comes from

$$\begin{vmatrix} 14 & 5 \\ -21 & -6 \end{vmatrix} = 21.$$

We now divide each entry of this matrix by the corresponding term in the interior of matrix A. Since

$$\text{int } A = \begin{pmatrix} 1 & -2 & -3 \\ -4 & 2 & 2 \\ -1 & 5 & 2 \end{pmatrix}$$

we obtain

$$C = \begin{pmatrix} 21 & -8 & 11 \\ -105 & 36 & 23 \\ 88 & 42 & 45 \end{pmatrix}.$$

This matrix is again condensed to form

$$D = \begin{pmatrix} -84 & -580 \\ -7578 & 654 \end{pmatrix},$$

and this time the entries are divided by the corresponding terms in

$$\text{int } B = \begin{pmatrix} -6 & 2 \\ -18 & -6 \end{pmatrix}$$

to obtain

$$E = \begin{pmatrix} 14 & -290 \\ 421 & -109 \end{pmatrix}.$$

The determinant of E is 120,564, which, when divided by the interior term of matrix C, 36, gives the correct answer of 3,349.

But why does this method work, and how did Dodgson come up with it in the first place?

Why Dodgson's Method Works

As Dodgson himself acknowledged, his method was an application of a well-established theorem on determinants, which had been proved by Jacobi several years earlier. This result uses the following definition.

If $A=[a_{ij}]$ is an $n \times n$ matrix, then its **adjugate** matrix is

$$\begin{pmatrix} a'_{11} & a'_{12} & \cdot & \cdot & \cdot & a'_{1n} \\ a'_{21} & a'_{22} & \cdot & \cdot & \cdot & a'_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & \cdot & \cdot & a'_{nn} \end{pmatrix}.$$

where $a'_{ij} = (-1)^{i+j} \det[A_{ij}]$ is the cofactor of a_{ij} in A, and where $[A]_{ij}$ is the minor matrix obtained by deleting the ith row and jth column in A.

For example, if

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & 1 \\ -1 & 3 & -2 \end{pmatrix}$$

then its adjugate would be

$$\begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 7 \\ -7 & -7 & -7 \\ -5 & -11 & -7 \end{pmatrix}.$$

In his *Elementary Treatise on Determinants*, Dodgson stated Jacobi's Theorem as follows:

If there be a square Block of the nth degree, and if in it any Minor of the mth degree be selected: the Determinant of the corresponding Minor in the adjugate Block is equal...to the product of the $(m–1)^{th}$ power of the Determinant of the first Block, multiplied by the Determinant of the Minor complemental to the one selected.

When $m = 2$, we get the following special case of the theorem, which will help to clarify the mechanics of Dodgson's method of condensation:

Theorem (Jacobi): *Let A be an $n \times n$ matrix. Consider the 2×2 minor consisting of*

$$\begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix}.$$

Then

$$\det \begin{pmatrix} a'_{11} & a'_{1n} \\ a'_{n1} & a'_{nn} \end{pmatrix} = \det(\text{int } A) \cdot \det A.$$

Of course, what Dodgson noticed was that, if det A was unknown, Jacobi's Theorem provided a useful algorithm for finding it. For example, let

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & 1 \\ -1 & 3 & -2 \end{pmatrix}.$$

We already know that det $A = -14$, but suppose we didn't. Using the minor matrix

$$\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & -2 \end{pmatrix},$$

the corresponding adjugate would be

$$\begin{pmatrix} a'_{11} & a'_{13} \\ a'_{31} & a'_{33} \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -5 & -7 \end{pmatrix}$$

And since

$$\det \begin{pmatrix} 1 & 7 \\ -5 & -7 \end{pmatrix} = 28 ,$$

by Jacobi's Theorem,

$$\det \begin{pmatrix} 1 & 7 \\ -5 & -7 \end{pmatrix} = \det(\text{int } A) \cdot \det A = a_{22} \cdot \det A = -2 \det A$$

$$\text{So } \det A = \frac{28}{-2} = -14.$$

This gives a simple illustration of why Dodgson's method works in the 3×3 case. For determinants of higher dimension, the idea is much the same. We omit the details, but they can be found in the MAA book *Proofs and Confirmations* by David Bressoud.

Of course the observant reader will have noticed that, although this method is extremely appealing, it does have an Achilles' heel. If at any point in the method the interior contains a zero, one cannot proceed. If one is fortunate, simple row operations on the original matrix will allow one to rearrange the entries so that the offending zeros will move to the exterior rows and columns. However, if one is unfortunate, the method breaks down. Nevertheless, when teaching linear algebra, we have consistently found Dodgson's method to be the most popular method among our students for evaluating large determinants.

Now that we have seen a basic overview of Dodgson's condensation method and the mathematics behind it, one final question remains. Apart from the fact that they were produced by the same person, what does the work of Charles Dodgson on matrices and determinants got to do with the poetry of Lewis Carroll?

The answer is provided by the verse given at the beginning of this article. A closer look reveals that the verse consists of 36 words arranged in six lines, of six words each, so the poem forms the entries of a 6×6 matrix:

$$\begin{bmatrix} I & often & wondered & when & I & cursed \\ Often & feared & where & I & would & be \\ Wondered & where & she'd & yield & her & love \\ When & I & yield & so & will & she \\ I & would & her & will & be & pitied \\ Cursed & be & love & she & pitied & me \end{bmatrix}$$

Since the poem reads the same horizontally and vertically, then the matrix is symmetric. Of course, one would not necessarily need a familiarity with the properties of symmetric matrices to compose such a verse, but such knowledge might help. Could this secret lie behind its composition? If so, it would mark an intriguing intersection between the mathematics of Charles Dodgson and the poetry of Lewis Carroll. ■

Suggested Reading

David M. Bressoud, *Proofs and Confirmations*: *The Story of the Alternating Sign Matrix Conjecture*, MAA Spectrum Series, Cambridge University Press (1999).

R. Wilson, Alice in Numberland: An Informal Dramatic Presentation, *College Mathematics Journal* **33**, No. 5, 2002, 354–377.