HARMONIC ANALYSIS

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1. INTRODUCTORY REMARKS CONCERNING FOURIER SERIES

Classical harmonic analysis deals mainly with the study of Fourier series and integrals. It occupies a central position in that branch of mathematics known as analysis; in fact, it has been described [10, p. xi] as "the meeting ground" of the theory of functions of a real variable and that of analytic functions of a complex variable. Consequently it arises, in a natural way, in several different contexts. Moreover, many basic notions and results in mathematics have been developed by mathematicians working in harmonic analysis. The modern concept of function was first introduced by Dirichlet while studying the convergence of Fourier series; more recently, the theory of distributions (generalized functions) was developed in close connection with the study of Fourier transforms. The Riemann and, later, the Lebesgue integrals were originally introduced in works dealing with harmonic analysis. Infinite cardinal and ordinal numbers, prob-

ably the most original and striking notions of modern mathematics, were developed by Cantor in his attempts to solve a delicate real-variable problem involving trigonometric series.

It is our purpose to present some of the main aspects of classical and, to a lesser extent, modern harmonic analysis. The development of the former uses principally the theories of functions of a real variable and of a complex variable while the latter draws heavily from the ideas of abstract functional analysis. Consequently, we shall assume the reader to be acquainted with the material usually presented in a first course in Lebesgue integration or measure theory and to have an elementary knowledge of analytic function theory; moreover, we will require a minimal knowledge of functional analysis.†

Suppose that we are given a real- or complex-valued function f, defined on the real line, periodic of period 1 (that is, $f(x) = f(x+1)$ for all x) and (Lebesgue) integrable when restricted to the interval $(0, 1)$.‡ Its *Fourier transform* is then the function $\hat{f}$, defined on the integers, whose value at k (the kth *Fourier coefficient*) is

$$\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt}\,dt, \qquad k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

The *Fourier series* of f is the series

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx} \tag{1.1}$$

considered as the sequence of (*symmetric*) *partial sums*

$$s_n(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikx}. \tag{1.2}$$

The reader is undoubtedly familiar with these notions; however, he has probably been introduced to them in a slightly different manner. For example, in classical treatments of Fourier series

† The reader should be acquainted with the elementary properties of Banach spaces and Hilbert spaces, as well as those of linear operators acting on these spaces. He is well advised to glance over the article on functional analysis in Volume 1 of this series.

‡ It follows that f must be integrable over any finite subinterval of $(-\infty, \infty)$. In the sequel we shall use the term "periodic" to mean periodic and of period 1.

the term "Fourier transform" is not used. Generally, the sequence of Fourier coefficients

$$c_k = \int_0^1 f(t)e^{-2\pi ikt}\,dt, \qquad k = 0, \pm 1, \pm 2, \pm 3, \cdots,$$

is introduced without emphasizing that one has really defined a function on the integers (the Fourier transform) and the Fourier series of f is usually denoted, simply, by the series $\sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$. The above notation and emphasis (as will shortly become apparent) are useful in order to give a unified approach to the theory of Fourier series, Fourier integrals, and their many analogs and extensions. Furthermore, when f is real-valued, the Fourier series of f is often introduced as the series

$$(1.1') \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos 2\pi kx + b_k \sin 2\pi kx)$$

considered as the sequence of partial sums

$$(1.2') \qquad s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos 2\pi kx + b_k \sin 2\pi kx),$$

where

$$a_k = 2\int_0^1 f(t)\cos 2\pi kt\,dt \qquad \text{and} \qquad b_k = 2\int_0^1 f(t)\sin 2\pi kt\,dt,$$

$k = 0, 1, 2, 3, \cdots$. It is easy to verify that the two expressions (1.2) and (1.2′) are equal.

In most advanced calculus courses it is shown that the Fourier series (1.1) (or (1.1′)) converges to $f(x)$ provided f is sufficiently well-behaved at the point x. For example, as we shall show in Sec. 3, this occurs whenever f is differentiable at the point x. This is a solution of a very special case of the central problem in the classical study of Fourier series: to determine whether, and in what sense, the series (1.1) represents the function f. Perhaps one of the best ways of penetrating into the subject of harmonic analysis is by studying this problem.

If we pose this problem in the most obvious way by asking if the series (1.1) converges to $f(x)$ for all x or for almost all x (a much more reasonable question, since altering f on a set of measure zero does not alter the Fourier series), we immediately encounter some serious difficulties. In fact, Kolmogoroff [10, p. 310] has shown that there exists a periodic function, integrable on (0, 1), whose Fourier series diverges everywhere. In general, one must impose fairly strong conditions on f in order to obtain the convergence of its Fourier series. Perhaps the most important unsolved problem in the classical theory is the following: does there exist a continuous periodic function whose Fourier series diverges on a set of positive measure?

On the other hand, if we consider only functions in $L^2(0, 1)$, that is, periodic functions f such that

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2} < \infty,$$

we obtain a complete and elegant solution to the central problem announced above restricted to this space and its norm. More precisely, we shall show that in this case the partial sums (1.2) converge to f in the L^2-norm; that is,

$$\lim_{n\to\infty} \|f - s_n\|_2 = \lim_{n\to\infty} \left(\int_0^1 |s_n(x) - f(x)|^2\,dx\right)^{1/2} = 0.$$

This is an immediate consequence of the fact that, with respect to the inner product $(f, g) = \int_0^1 f(x)\overline{g(x)}\,dx$, $L^2(0, 1)$ is "essentially" a Hilbert space† and the exponential functions e_k, $k = 0, \pm 1, \pm 2, \pm 3, \cdots$, where $e_k(x) = e^{2\pi ikx}$, form an *orthonormal basis*; that is,

$$(1.3) \qquad (e_k, e_j) = \int_0^1 e^{2\pi ikx}e^{-2\pi ijx}\,dx = \delta_{kj},$$

where δ_{kj} is the "Kronecker δ" ($\delta_{kj} = 0$ when $k \neq j$ and $\delta_{kk} = 1$), and for each $f \in L^2(0, 1)$

$$(1.4) \qquad \lim_{n\to\infty} \left\| \sum_{k=-n}^{n} c_k e_k - f \right\|_2^2 = \lim_{n\to\infty} \left(\sum_{k=-n}^{n} c_k e_k - f, \sum_{k=-n}^{n} c_k e_k - f \right) = 0,$$

† More precisely, it is the collection of equivalence classes we obtain by identifying functions in $L^2(0, 1)$ that are equal almost everywhere that forms a Hilbert space.

when $c_k = (f, e_k)$ is the kth Fourier coefficient of f, $k = 0, \pm 1, \pm 2, \cdots$. While the orthogonality relations (1.3) are obvious, the convergence (1.4) will require some proof. A very simple argument, however, shows that the partial sums $s_n = \sum_{k=-n}^{n} c_k e_k = \sum_{k=-n}^{n} \hat{f}(k) e_k$ converge in the L^2-norm to some function in L^2. By the Riesz-Fisher theorem, which asserts that all the L^p spaces, $1 \leq p \leq \infty$, are complete with respect to the L^p-norm, this result will hold provided the sequence $\{s_n\}$ is Cauchy in L^2: $\lim_{n,m\to\infty} \|s_n - s_m\|_2 = 0$. But if, say, $m \leq n$, the orthogonality relations (1.3) imply that

$$\|s_n - s_m\|_2^2 = (s_n, s_n) - 2(s_m, s_n) + (s_m, s_m)$$
$$= \sum_{n \geq |k| > m} |c_k|^2 = \sum_{n \geq |k| > m} |\hat{f}(k)|^2.$$

That this last sum tends to zero as m and n tend to ∞ follows from *Bessel's inequality* for functions f in $L^2(0, 1)$,

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \leq \|f\|_2^2 = \int_0^1 |f(x)|^2\, dx, \tag{1.5}$$

which is an easy consequence of the orthogonality relations (1.3) and the definition of the Fourier coefficients $c_k = \hat{f}(k)$: since, for any g in $L^2(0, 1)$, $(g, g) \geq 0$, we have

$$0 \leq \left(\sum_{k=-n}^{n} c_k e_k - f, \sum_{k=-n}^{n} c_k e_k - f\right)$$
$$= \left(\sum_{k=-n}^{n} c_k e_k, \sum_{k=-n}^{n} c_k e_k\right)$$
$$- \left(\sum_{k=-n}^{n} c_k e_k, f\right) - \left(f, \sum_{k=-n}^{n} c_k e_k\right) + (f, f)$$
$$= \sum_{k=-n}^{n} |c_k|^2 - \sum_{k=-n}^{n} |c_k|^2 - \sum_{k=-n}^{n} |c_k|^2 + \|f\|_2^2.$$

That is.

$$\sum_{k=-n}^{n} |c_k|^2 \leq \|f\|_2^2,$$

and (1.5) follows by letting $n \longrightarrow \infty$. These arguments give a flavor of the elegance and simplicity of the L^2-theory.

Another satisfactory solution of the problem of representation of functions by their Fourier series is to consider, instead of convergence, some methods of summability of Fourier series at individual points. The two best known types of summability (and the only ones we shall consider) are *Cesàro* and *Abel summability*. The former (often also referred to as the *method of summability by the first arithmetic means* or, simply, as *(C, 1) summability*) is defined in the following way: suppose we are given a numerical series $u_0 + u_1 + u_2 + \cdots$ with partial sums $s_0, s_1, s_2, \cdots$. We then form the *(C, 1) means* (or *first arithmetic means*)

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n + 1} = \sum_{\nu=0}^{n} \left(1 - \frac{\nu}{n+1}\right) u_\nu$$

and say that the series is $(C, 1)$ summable to l if $\lim_{n\to\infty} \sigma_n = l$. The *Abel means* of the series are defined for each r, $0 \leq r < 1$, by setting

$$A(r) = u_0 + u_1 r + u_2 r^2 + \cdots = \sum_{k=0}^{\infty} u_k r^k$$

and we say that the series is Abel summable to l if $\lim_{r\to 1-} A(r) = l$. It is not hard to show that if $u_0 + u_1 + u_2 + \cdots$ is convergent to the sum l then it must be both $(C, 1)$ and Abel summable to l. On the other hand, there are many series that are summable but not convergent. An illustrative example is the series $1 - 1 + 1 - 1 + \cdots = \sum_{k=0}^{\infty} (-1)^k$, whose $(C, 1)$ and Abel means are easily seen to converge to $\frac{1}{2}$. It can also be shown that $(C, 1)$ summability implies Abel summability. Thus, many results involving Abel summability follow from corresponding theorems that deal with Cesàro summability. Nevertheless, an independent study of the former is of interest, particularly when we consider Fourier series of functions f. This is true, not only because such series may be Abel summable under weaker conditions on f than are necessary to guarantee their $(C, 1)$ summability, but also because Abel summability has special properties, related to the theory of harmonic and analytic functions, that are not enjoyed by Cesàro summability.

We now describe, briefly, how these concepts apply to the study of Fourier series. The two most important results in connection with the problem of representing functions by their Fourier series are the following:

(1.6) *If f is periodic and integrable on* $(0, 1)$ *then the* $(C, 1)$ *means and the Abel means of the Fourier series of f converge to*

$$\tfrac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}$$

at every point x_0 where the limits $f(x_0 \pm 0)$ exist. In particular, they converge at every point of continuity of f.

(1.7) *If f is periodic and integrable on* $(0, 1)$ *then the* $(C, 1)$ *means and the Abel means of the Fourier series of f converge to $f(x)$ for almost every x in* $(0, 1)$.†

We can obtain more insight into these results by examining more closely the first arithmetic means and the Abel means of the Fourier series of a function f. We first obtain an expression for the partial sums (1.2):

$$\begin{aligned} s_n(x) &= \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x} \\ &= \sum_{k=-n}^{n} \left(\int_0^1 f(t) e^{-2\pi i k t}\, dt \right) e^{2\pi i k x} \\ &= \int_0^1 \left(\sum_{k=-n}^{n} e^{2\pi i k (x-t)} \right) f(t)\, dt. \end{aligned}$$

By multiplying $D_n(\theta) = \sum_{k=-n}^{n} e^{2\pi i k\theta}$ by $2 \sin \pi\theta = i(e^{-i\pi\theta} - e^{i\pi\theta})$ all but the first and last term of the resulting sum cancel and we obtain

$$2\, D_n(\theta) \sin \pi\theta = i(e^{-(2n+1)\pi i\theta} - e^{(2n+1)\pi i\theta}) = 2 \sin (2n+1)\pi\theta;$$

† When restricted to Cesàro summability (1.6) is known as the theorem of Fejér and (1.7) as the theorem of Lebesgue. That these results then hold for Abel summability follows from the fact, mentioned above, that $(C, 1)$ summability implies Abel summability.

that is,

$$D_n(\theta) = \frac{\sin (2n+1)\pi\theta}{\sin \pi\theta}. \tag{1.8}$$

Hence,

$$\begin{aligned} s_n(x) &= \int_0^1 f(t) D_n(x - t)\, dt \\ &= \int_0^1 f(t) \frac{\sin (2n+1)\pi(x-t)}{\sin \pi(x-t)}\, dt. \end{aligned} \tag{1.9}$$

The expression (1.8) is called the *Dirichlet kernel.* We can now express the $(C, 1)$ means in terms of it:

$$\begin{aligned} \sigma_n(x) &= \frac{s_0(x) + s_1(x) + \cdots + s_n(x)}{n+1} \\ &= \frac{1}{n+1} \int_0^1 f(t) \left(\sum_{k=0}^{n} D_k(x-t) \right) dt. \end{aligned}$$

By multiplying the numerator and denominator of

$$K_n(\theta) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(\theta) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{\sin (2k+1)\pi\theta}{\sin \pi\theta}$$

by $\sin \pi\theta$ and replacing the products of sines in the numerator by differences of cosines we obtain

$$\begin{aligned} K_n(\theta) &= \frac{1}{n+1} \sum_{k=0}^{n} \frac{\cos 2k\pi\theta - \cos 2(k+1)\pi\theta}{2 \sin^2 \pi\theta} \\ &= \frac{1}{n+1} \frac{1 - \cos 2(n+1)\pi\theta}{2 \sin^2 \pi\theta} \\ &= \frac{1}{n+1} \left[\frac{\sin (n+1)\pi\theta}{\sin \pi\theta} \right]^2. \end{aligned} \tag{1.10}$$

Consequently,

$$\begin{aligned} \sigma_n(x) &= \int_0^1 f(t) K_n(x-t)\, dt \\ &= \frac{1}{n+1} \int_0^1 f(t) \left\{ \frac{\sin (n+1)\pi(x-t)}{\sin \pi(x-t)} \right\}^2 dt. \end{aligned} \tag{1.11}$$

$K_n(\theta)$ is called the *Fejér kernel.*

The result (1.6) follows easily from three basic properties of this kernel:

(A) $\int_0^1 K_n(\theta)\,d\theta = 1$;

(B) $K_n(\theta) \geq 0$;

(C) *for each* $\delta > 0$, $\max_{\delta \leq \theta \leq 1-\delta} K_n(\theta) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Property (B) is obvious. Property (A) is a consequence of the corresponding property for the Dirichlet kernel (which is immediate:

$$\int_0^1 D_n(\theta)\,d\theta = \sum_{k=-n}^{n} \int_0^1 e^{2\pi i k\theta}\,d\theta = 1)$$

and the representation $K_n(\theta) = \sum_{k=0}^{n} D_k(\theta)/(n+1)$. Finally, (C) follows from the inequality (see (1.10))

$$\max_{\delta \leq \theta \leq 1-\delta} K_n(\theta) \leq \frac{1}{n+1} \sin^2 \pi\delta.$$

Now, to obtain (1.6) we argue as follows: suppose x_0 is a point at which the limits $f(x_0 \pm 0)$ exist and let $a = \frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$. Then, using the periodicity of the functions involved, the change of variables $t = x - s$, and property (A),

$$\begin{aligned}\sigma_n(x_0) - a &= \int_{-1/2}^{1/2} f(s)K_n(x_0 - s)\,ds - a \cdot 1 \\ &= \int_{-1/2}^{1/2} f(x_0 - t)K_n(t)\,dt - a\int_{-1/2}^{1/2} K_n(t)\,dt \\ &= 2\int_0^{\delta}\left\{\frac{f(x_0 - t) + f(x_0 + t)}{2} - a\right\}K_n(t)\,dt \\ &\quad + \int_{\delta \leq |t| \leq 1/2} \{f(x_0 - t) - a\}K_n(t)\,dt.\end{aligned}$$

Hence, if $\delta > 0$ is so chosen that $|f(x_0 - t) + f(x_0 + t) - 2a| \leq \epsilon$ if $|t| \leq \delta$, we have, by (B) and (A),

$$\begin{aligned}|\sigma_n(x_0) - a| &\leq \epsilon \int_0^{\delta} K_n(t)\,dt \\ &\quad + \left\{\max_{\delta \leq |t| \leq 1/2} K_n(t)\right\} \int_{\delta \leq |t| \leq 1/2} |f(x_0 - t) - a|\,dt \\ &\leq \epsilon \int_{-1/2}^{1/2} K_n(t)\,dt \\ &\quad + \left\{\max_{\delta \leq |t| \leq 1/2} K_n(t)\right\} \int_{-1/2}^{1/2} |f(x_0 - t) - a|\,dt \\ &= \epsilon\cdot 1 + \left\{\max_{\delta \leq |t| \leq 1/2} K_n(t)\right\} \int_{-1/2}^{1/2} |f(x_0 - t) - a|\,dt;\end{aligned}$$

but, by (C), the last term tends to 0 as $n \longrightarrow \infty$. Since $\epsilon > 0$ is arbitrary we can conclude that $\lim_{n\to\infty} |\sigma_n(x_0) - a| = 0$ and (1.6) is proved.

The theorem of Lebesgue, result (1.7) for the $(C, 1)$ means, is somewhat deeper and we postpone its proof until later. Since $(C, 1)$ summability implies Abel summability, as remarked above, both the results (1.6) and (1.7) follow once we establish them for Cesàro means. We commented before, however, that an independent study of Abel summability is of interest since it has special properties not enjoyed by $(C, 1)$ summability. This is easily made clear by examining the Abel means of Fourier series more closely; we do this by showing how the study of Fourier series is intimately connected with analytic function theory.

That this connection should exist is not surprising once we make the observation, *when f is real-valued*, that the series (1.1) is the real part of the power series

$$\hat{f}(0) + \sum_{k=1}^{\infty} 2\hat{f}(k)z^k \tag{1.12}$$

restricted to the unit circle $z = e^{2\pi i x}$. We note that this series defines an analytic function in the interior of the unit circle since the coefficients $\hat{f}(k)$ are uniformly bounded (in fact,

$$|\hat{f}(k)| \leq \int_0^1 |f(t)|\,dt = ||f||_1).$$

Thus, the real part of (1.12) is a harmonic function when $r =$

$|z| < 1$. But this real part is nothing more than the Abel mean of the Fourier series (1.1):

$$A(r, x) = A_f(r, x) = \hat{f}(0) + \sum_{k=1}^{\infty} r^k \hat{f}(k)(e^{2\pi ikx} + e^{-2\pi ikx})$$

$$= \sum_{k=-\infty}^{\infty} r^{|k|}\hat{f}(k)e^{2\pi ikx}.$$

The imaginary part of (1.10), when $z = e^{2\pi ix}$, is (formally),

$$(1.13) \qquad -i \sum_{k=-\infty}^{\infty} (\operatorname{sgn} k)\hat{f}(k)e^{2\pi ikx},$$

where, for any nonzero complex number z, $\operatorname{sgn} z = z/|z|$ and $\operatorname{sgn} 0 = 0$. This series is called the series *conjugate* to the Fourier series (1.1). Though it is not, in general, a Fourier series, this conjugate series is closely connected (see Sec. 4) to a (not necessarily integrable) function, the *conjugate function*, $\tilde{f}$.

As in the case of the $(C, 1)$ means, the Abel means $A(r, x)$ have an integral representation; that is, a representation similar to (1.11). We have, for $0 \le r < 1$,

$$A(r, x) = \sum_{k=-\infty}^{\infty} r^{|k|}\hat{f}(k)e^{2\pi ikx}$$

$$= \sum_{k=-\infty}^{\infty} r^{|k|}\left(\int_0^1 f(t)e^{-2\pi ikt}\,dt\right)e^{2\pi ikx}$$

$$= \int_0^1 \left(\sum_{k=-\infty}^{\infty} r^{|k|}e^{2\pi ik(x-t)}\right)f(t)\,dt,$$

the change in the order of integration and summation being justifiable by the uniform convergence of the series

$$P(r, \theta) = \sum_{k=-\infty}^{\infty} r^{|k|}e^{2\pi i\theta}$$

for $0 \le r < 1$. But, setting $z = re^{2\pi i\theta}$, $P(r, \theta)$ is simply the real part of

$$1 + \sum_{k=1}^{\infty} 2r^k e^{2\pi ik\theta} = 1 + 2\sum_{k=1}^{\infty} z^k = \frac{1+z}{1-z}.$$

Consequently,

$$(1.14) \qquad P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos 2\pi\theta + r^2}$$

and we obtain the desired integral representation for the Abel means

$$(1.15) \quad A(r, x) = \int_0^1 P(r, x - t)f(t)\,dt$$

$$= \int_0^1 \frac{1 - r^2}{1 - 2r\cos 2\pi(x - t) + r^2} f(t)\,dt.$$

$P(r, \theta)$ is called the *Poisson kernel* and the integral (1.15) is called the *Poisson integral* of f. The reader can easily verify that this kernel satisfies the three properties, completely analogous to those of the Fejér kernel:

(A′) $\int_0^1 P(r, \theta)\,d\theta = 1$;

(B′) $P(r, \theta) \ge 0$;

(C′) *for each* $\delta > 0$, $\max_{\delta \le \theta \le 1-\delta} P(r, \theta) \longrightarrow 0$ *as* $r \longrightarrow 1$.

From this we see that to the proof of (1.6) given above in the case of the Cesàro means there corresponds a practically identical proof of this result for the Abel means.

Let us observe that the imaginary part of $\frac{1+z}{1-z}$ has the form

$$Q(r, \theta) = \frac{2r\sin 2\pi\theta}{1 - 2r\cos 2\pi\theta + r^2}$$

and one readily obtains the Abel mean of the conjugate Fourier series (1.13) by the integral

$$(1.16) \quad \tilde{A}(r, x) = \int_0^1 Q(r, x - t)f(t)\,dt$$

$$= \int_0^1 \frac{2r\sin 2\pi(x - t)}{1 - 2r\cos 2\pi(x - t) + r^2} f(t)\,dt,$$

This integral is called the *conjugate Poisson integral* of f and $Q(r, \theta)$ is known as the *conjugate Poisson kernel*.

In this discussion we assumed that f was real-valued. It is clear.

however, that the Poisson integral formula (1.15) for the Abel means of the Fourier series of f holds in case f is complex-valued as well. To see this one need only apply it to the real and imaginary parts of f.

Before passing to other aspects of harmonic analysis let us examine more closely the integrals (1.9), (1.11), (1.15), and (1.16) that gave us the partial sums, the $(C, 1)$ means, the Abel means of the Fourier series of an integrable periodic function f, and the Abel means of the conjugate Fourier series of f. All these integrals have the form

$$(g * f)(x) = \int_0^1 g(x - t)f(t)\, dt, \tag{1.17}$$

where g is a periodic integrable function. In fact, in all these cases g is much better than merely integrable; for example, it is a bounded function and, consequently, it is obvious that, for each x, the integrand in (1.17) is integrable and $(g * f)(x)$ is well defined. We shall see, however, that the latter is well defined for almost all x when g is integrable. We therefore obtain a function, $g * f$ (defined almost everywhere), by forming the integral (1.17) whenever g and f belong to $L^1(0, 1)$ and are periodic. This operation, that assigns to each such pair (g, f) the function $g * f$, is called *convolution* and plays an important role in the theory of Fourier series. The most important elementary properties of convolution are the following:

(i) *If f and g are periodic and in $L^1(0, 1)$ so is $f * g$ and*

$$\begin{aligned}\|f * g\|_1 &= \int_0^1 |(f * g)(x)|\, dx \\ &\le \Big(\int_0^1 |f(t)|\, dt\Big)\Big(\int_0^1 |g(t)|\, dt\Big) \\ &= \|f\|_1\, \|g\|_1;\end{aligned}$$

(ii) $f * g = g * f$;

(iii) $(f * g) * h = f * (g * h)$ *whenever f, g, and h are periodic and in $L^1(0, 1)$;*

(iv) *For f, g, and h as in* (iii) *and any two complex numbers a and b*

$$f * (ag + bh) = a(f * g) + b(f * h).$$

That $(f * g)(x)$ is well defined for almost all x, as well as property (i), is an easy consequence of Fubini's theorem: since

$$|(f * g)(x)| \le \int_0^1 |f(x - t)|\, |g(t)|\, dt$$

we have, using the periodicity of f,

$$\begin{aligned}\int_0^1 |(f * g)(x)|\, dx &\le \int_0^1 \Big(\int_0^1 |f(x - t)|\, |g(t)|\, dt\Big)\, dx \\ &= \int_0^1 |g(t)| \Big(\int_0^1 |f(x - t)|\, dx\Big)\, dt \\ &= \int_0^1 |g(t)|\, \|f\|_1\, dt = \|f\|_1\, \|g\|_1.\dagger\end{aligned}$$

The remaining three properties follow from simple transformations of integrals and we omit their proofs.

The relation between Fourier transformation and convolution is very simple and elegant:

Suppose f and g are periodic and in $L^1(0, 1)$, then for all integers k

$$(f * g)^\wedge(k) = \hat{f}(k)\hat{g}(k).\ddagger \tag{1.18}$$

In order to see this we use Fubini's theorem and the periodicity of the functions involved:

$$\begin{aligned}(f * g)^\wedge(k) &= \int_0^1 \Big(\int_0^1 f(x - t)g(t)\, dt\Big)e^{-2\pi ikx}\, dx \\ &= \int_0^1 g(t)e^{-2\pi ikt} \Big(\int_0^1 f(x - t)e^{-2\pi ik(x-t)}\, dx\Big)\, dt \\ &= \int_0^1 g(t)e^{-2\pi ikt}\hat{f}(k)\, dt \\ &= \hat{f}(k)\hat{g}(k).\end{aligned}$$

† We are using the following version of Fubini's theorem: If $h \ge 0$ is a measurable function in the square $\{0 \le x \le 1,\ 0 \le t \le 1\}$ and the iterated integral $\int_0^1 (\int_0^1 h(x, t)\, dt)\, dx$ is finite, then h is integrable, $\int_0^1 h(x, t)\, dt$ is finite for almost every x and $\int_0^1 (\int_0^1 h(x, t)\, dt)\, dx = \int_0^1 (\int_0^1 h(x, t)\, dx)\, dt$. We have tacitly assumed, when $h(x, t) = |f(x - t)g(t)|$, that h is measurable. We leave the proof of this fact to the reader.

‡ In general, we shall let $(\quad)^\wedge$ denote the Fourier transform of the expression in the parentheses.

It is this result, that the Fourier transform of the convolution of two functions is, simply, the product of their Fourier transforms, that makes convolution play such an important role in the study of Fourier series. This, as we shall see, becomes clear very early in the development of harmonic analysis.

2. HARMONIC ANALYSIS ON THE INTEGERS AND ON THE REAL LINE

Up to this point we have considered only functions that were periodic of period 1. It is often useful to think of such functions as defined on the *additive group of real numbers modulo* 1† or on the perimeter of the unit circle $\{z \text{ complex}; z = e^{2\pi i\theta}\}$ of the complex plane. Consequently, the theory of Fourier series is often referred to as the harmonic analysis associated with this circle, or the reals modulo 1. Toward the end of this monograph we shall describe how harmonic analysis can be associated to a wide variety of domains. In this section we shall consider two of these, the integers and the entire real line. The harmonic analysis corresponding to these domains is intimately connected with the theory of Fourier series.

In the case of the real line we obtain the theory of Fourier integrals, a topic that is as important and as well known as the theory of Fourier series. The harmonic analysis associated with functions defined on the integers, however, is not generally studied per se. The main reason is that its elementary aspects (but by no means its deeper ones) consist of results that are essentially on the surface. But precisely this property, this elementary nature of the subject, makes its study very worthwhile for the nonspecialist as it provides a great deal of motivation for the theories of Fourier series and integrals. Furthermore, some remarks about

† If we say that two real numbers are equivalent when their difference is an integer we obtain a partition of the reals into equivalence classes. Let $[x]$, $[y]$, $[z]$, $\cdots$ denote the equivalence classes containing the real numbers x, y, z, $\cdots$. Then the additive group of real numbers modulo 1 consists of these equivalence classes together with the operation defined by $[x] + [y] = [x + y]$.

this topic are necessary in order to understand better the general picture of our subject. Accordingly, we shall not consider this part of harmonic analysis in any detail, but will treat it briefly and use it to motivate the introduction of the inverse Fourier transform, which will enable us to consider the problem of representation of functions by their Fourier series from a more general point of view. Also, we shall use it to motivate our introductory remarks concerning Fourier integrals. We strongly urge the reader, however, to find the analogs, for functions defined on the integers, of results we shall present in the theories of Fourier series and integrals.

In the last section we started out with a periodic function belonging to $L^1(0, 1)$ and obtained, by means of a certain integral, a function defined on the integers, the Fourier transform. We then asked if it were possible to obtain the original function from the latter by means of a certain series, the Fourier series (1.1). This indicates a duality between the interval (0, 1) and the integers and it is not unreasonable to expect that, by considering originally a function defined on the integers, we can introduce, in analogy to the Fourier transform, a periodic function by means of an appropriate series. Furthermore, we should be able to recapture the original function from this periodic function and a suitable integral. It is only natural, in view of these remarks, to hope that this can be done by interchanging the roles played by the interval (0, 1) and the integers. More explicitly, let us examine the result of systematically replacing, in the definitions made at the beginning of the last section,

$$\sum_{k=-\infty}^{\infty} \text{ by } \int_0^1, \qquad \int_0^1 \text{ by } \sum_{k=-\infty}^{\infty},$$

the continuous variable $x \in (0, 1)$ by the integral variable k and k by x.

Let us consider, then, the integers as a measure space in which each point has measure 1 and an integrable function, f, defined on this measure space; that is, f satisfies

$$\sum_{k=-\infty}^{\infty} |f(k)| < \infty. \tag{2.1}$$

The space of such integrable functions is usually denoted by l^1. For f in l^1 we introduce the periodic function $\hat{f}$ whose value at x is

$$\hat{f}(x) = \sum_{j=-\infty}^{\infty} f(j)e^{-2\pi ijx}. \tag{2.2}$$

We shall call $\hat{f}$ the *Fourier transform* of f in this case as well. Because of the convergence (2.1) the series (2.2) converges uniformly; consequently, $\hat{f}$ is a continuous function. Corresponding to the Fourier series (1.1) we have the integral

$$\int_0^1 \hat{f}(x)e^{2\pi ikx}\,dx. \tag{2.3}$$

But the uniform convergence of (2.2), allowing us to integrate term-by-term, and the orthogonality relations (1.3) immediately imply that

$$\int_0^1 \hat{f}(x)e^{2\pi ikx}\,dx = f(k). \tag{2.4}$$

We therefore see that in the present case we do not encounter any of the difficulties described in the first section when we try to express the original function in terms of its Fourier transform. This illustrates the simplicity of the elementary aspects of the harmonic analysis associated with the integers.

In particular, we see that the mapping that assigns to $f \in l^1$ its Fourier transform is one-to-one and, thus, it has an inverse. This inverse, in view of (2.4), has an obvious extension to all of $L^1(0, 1)$; namely, the operator, called the *inverse Fourier transform*, that takes a function g in $L^1(0, 1)$ into the function $\check{g}$ whose value at $k = 0, \pm 1, \pm 2, \cdots$ is

$$\check{g}(k) = \int_0^1 g(x)e^{2\pi ikx}\,dx. \tag{2.5}$$

We can, therefore, rewrite (2.4) in the following way:

$$(\hat{f})^\vee = f, \tag{2.6}$$

whenever f belongs to l^1.

These considerations lead us to a useful and more general restatement of the problem we studied in the last section concerning the representation of functions by their Fourier series. Suppose

we again interchange the roles played by the interval (0, 1) and the integers; it is then natural to try to define the inverse Fourier transform of a function, g, whose domain is the integers, by the expression

$$\check{g}(x) = \sum_{k=-\infty}^{\infty} g(k)e^{2\pi ikx}. \tag{2.7}$$

When g is in l^1 the series (2.7) is convergent and we obtain a well-defined mapping, $g \longrightarrow \check{g}$, from l^1 into the class of continuous periodic functions. This mapping, however, is insufficient for our purposes. For example, in view of (2.6), we should expect that whenever g is the Fourier transform of an integrable function f it then follows that $\check{g} = (\hat{f})^\vee = f$. But, because of Kolmogoroff's example of an integrable function with an everywhere divergent Fourier series, this equality cannot be valid if we let $\check{g}$ be defined as the function which, at each x (or at almost every x), satisfies (2.7) in the usual sense (that is, the sequence of partial sums $s_n(x) = \sum_{k=-n}^{n} g(k)e^{2\pi ikx}$ converges). On the other hand, using (1.7), we do obtain an almost everywhere defined $\check{g}(x) = (\hat{f})^\vee(x) = f(x)$ if we interpret (2.7) to mean that the $(C, 1)$ or Abel means of the series on the right converge to $\check{g}(x)$. Similarly, if g is the Fourier transform of an f belonging to $L^2(0, 1)$, (2.7) gives us a well-defined function $\check{g} = f$ if we interpret the series on the right to be convergent in the L^2-norm. In each of these cases we obtain a mapping which is an inverse to the Fourier transform mapping when the latter is restricted to some important domain of functions (such as $L^2(0, 1)$ or $L^1(0, 1)$). Thus, a general formulation of the problem we discussed in the last section is the following: given a class C of periodic functions for which the Fourier transform is defined, does there exist a mapping, $g \longrightarrow \check{g}$, defined on a class of functions, whose common domain is the integers, such that $(\hat{f})^\vee$ is defined for all f in C and $(\hat{f})^\vee = f$? We shall refer to the problem stated in this form as the *Fourier inversion problem*.

Let us now turn to the harmonic analysis related to the real line, the theory of Fourier integrals. Suppose that f is integrable over $(-\infty, \infty)$, then its *Fourier transform* is defined for all real x by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ixt}\,dt.$$

The integral on the right is usually also called the Fourier integral. Since the Fourier transform of a function defined on the entire real line is again such a function it should not surprise the reader, in view of our discussion in this section, that a good heuristic approach for obtaining the basic notions and results in the theory of Fourier integrals is to let the real line play the roles that the interval (0, 1) and the integers played in the theory of Fourier series. Let us examine the Fourier inversion problem from this point of view.

Motivated by the expressions (2.5), (2.6), and (2.7) we would expect that the inverse Fourier transform of a function, g, defined on the real line should be given by the formula

$$\check{g}(x) = \int_{-\infty}^{\infty} g(t)e^{2\pi ixt}\,dt, \tag{2.8}$$

and that for each f in $L^1(-\infty, \infty)$ we would then have $(\hat{f})^\vee = f$. Just as in the case of Fourier series, however, we immediately encounter the problem of giving relation (2.8) a suitable interpretation. Though $\hat{f}$ has several nice properties (the reader can easily check that it is uniformly continuous and bounded; in fact,

$$||\hat{f}||_\infty = \sup_{-\infty<x<\infty} |\hat{f}(x)| \le \int_{-\infty}^{\infty} |f(t)|\,dt = ||f||_1 \tag{2.9}$$

whenever f is in $L^1(-\infty, \infty)$) it is not always true that it is integrable. For example, if f is the characteristic function, $X_{(a,b)}$, of the finite interval (a, b) then

$$\hat{X}_{(a,b)}(x) = \int_a^b e^{-2\pi ixt}\,dt = \frac{e^{-2\pi ixa} - e^{-2\pi ixb}}{2\pi ix}, \tag{2.10}$$

when $x \neq 0$ and $\hat{X}_{(a,b)}(0) = b - a$. Here, as in the last section, we obtain a satisfactory solution of the Fourier inversion problem if we consider $(C, 1)$ and Abel summability. We see this easily if we let ourselves be guided by the above mentioned heuristic principle of substituting the real line for the interval (0, 1) and the integers. Let us first examine briefly the case of Cesàro summability.

If u is integrable in the intervals $[-R, R]$, for all $R > 0$, the

Cesàro means, or *(C, 1) means*, of $\int_{-\infty}^{\infty} u(t)\,dt$ are defined by the integrals

$$\sigma_R = \int_{-R}^{R} \left(1 - \frac{|t|}{R}\right) u(t)\,dt.$$

We say that $\int_{-\infty}^{\infty} u(t)\,dt$ is $(C, 1)$ summable to l if $\lim_{R\to\infty} \sigma_R = l$. It is easy to see that if $u \in L^1(-\infty, \infty)$ and its integral is l then $\sigma_R \longrightarrow l$ as $R \longrightarrow \infty$.

Let us now consider the Cesàro means of the integral in (2.8) when g is the Fourier transform of an integrable function f. We have

$$\begin{aligned}\sigma_R(x) &= \int_{-R}^{R} \left(1 - \frac{|t|}{R}\right) e^{2\pi ixt}\hat{f}(t)\,dt \\ &= \int_{-R}^{R} \left(1 - \frac{|t|}{R}\right) e^{2\pi ixt}\left\{\int_{-\infty}^{\infty} f(y)e^{-2\pi ity}\,dy\right\} dt \\ &= \int_{-\infty}^{\infty} f(y)\left\{\int_{-R}^{R} \left(1 - \frac{|t|}{R}\right) e^{2\pi it(x-y)}\,dt\right\} dy.\end{aligned}$$

It is not hard to obtain a simple expression for the inner integral. Using the fact that the sine function is odd and integrating by parts we obtain:

$$\begin{aligned}K_R(\theta) &= \int_{-R}^{R} \left(1 - \frac{|t|}{R}\right) e^{2\pi i\theta t}\,dt = 2\int_0^R \left(1 - \frac{t}{R}\right) \cos(2\pi\theta t)\,dt \\ &= 2\int_0^R \frac{1}{R}\,\frac{\sin(2\pi\theta t)}{2\pi\theta}\,dt = \frac{1}{2\pi^2 R}\cdot\frac{1-\cos(2\pi R\theta)}{\theta^2}\end{aligned} \tag{2.11}$$

Consequently,

$$\begin{aligned}\sigma_R(x) &= \int_{-\infty}^{\infty} f(y)K_R(x-y)\,dy \\ &= \frac{1}{2\pi^2 R}\int_{-\infty}^{\infty} f(y)\,\frac{1-\cos(2\pi R(x-y))}{(x-y)^2}\,dy.\end{aligned} \tag{2.12}$$

$K_R(\theta)$ is called the *Fejér kernel* and it satisfies the following three basic properties:

(a) $\int_{-\infty}^{\infty} K_R(\theta)\, d\theta = 1;$

(b) $K_R(\theta) \geq 0;$

(c) *for each* $\delta > 0$, $\int_{|\theta| \geq \delta} K_R(\theta)\, d\theta \longrightarrow 0$ *as* $R \longrightarrow \infty$.

The second property is obvious. The first property follows easily from the well-known result: $\lim_{N\to\infty} \int_0^N (\sin t/t)\, dt = \pi/2$. To see this we change variables and integrate by parts:

$$\int_{-\infty}^{\infty} K_R(\theta)\, d\theta = 2\int_{-\infty}^{\infty} \frac{1 - \cos 2\pi R\theta}{R(2\pi\theta)^2}\, d\theta = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1 - \cos s}{s^2}\, ds$$

$$= \frac{2}{\pi} \lim_{N\to\infty} \int_0^N \frac{1 - \cos s}{s^2}\, ds = \frac{2}{\pi} \lim_{N\to\infty} \int_0^N \frac{\sin t}{t}\, dt = 1.$$

In order to prove (c) let us first observe that $K_R(\theta) \leq 1/R\theta^2$ (thus,

(2.13) $$\max_{|\theta| \geq \delta} K_R(\theta) \leq \frac{1}{R} \max_{|\theta| \geq \delta} \frac{1}{\theta^2} \longrightarrow 0$$

as $R \longrightarrow \infty$). Consequently,

$$\int_{|\theta| \geq \delta} K_R(\theta)\, d\theta \leq \frac{1}{R} \int_{|\theta| \geq \delta} \frac{d\theta}{\theta^2} = \frac{2}{R\delta} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$

If we replace (c) by (2.13) we have three properties that are completely analogous to the properties (A), (B), and (C) of the Fejér kernel obtained in the periodic case. Precisely the same argument that is used in establishing the theorem of Fejér (see (1.6)) will then give us the corresponding result for Fourier integrals. We introduce property (c), however, to show how $(C, 1)$ summability can be used in yet another way in order to obtain a solution of the Fourier inversion problem. More precisely, we shall prove the following result:

(2.14) *If f is integrable then the $(C, 1)$ means*

$$\sigma_R(x) = \int_{-R}^{R} \left(1 - \frac{|t|}{R}\right) e^{2\pi i x t} \hat{f}(t)\, dt = \int_{-\infty}^{\infty} f(t) K_R(x - t)\, dt$$

of the integral defining the inverse Fourier transform of $\hat{f}$ converge to f in the L^1 norm. That is,

$$\lim_{R\to\infty} \|f - \sigma_R\|_1 = \lim_{R\to\infty} \int_{-\infty}^{\infty} |\sigma_R(x) - f(x)|\, dx = 0.$$

It is convenient, at this point, to introduce the L^p *modulus of continuity* of a function f in $L^p(-\infty, \infty)$:

$$\omega_p(\delta) = \max_{0 \leq t \leq \delta} \left\{ \int_{-\infty}^{\infty} |f(x + t) - f(x)|^p\, dx \right\}^{1/p}.$$

It is an elementary fact that $\omega_p(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$ when $1 \leq p < \infty$. For, given $\delta > 0$, we can write $f = f_1 + f_2$, where f_1 is continuous, vanishes outside a finite interval, and $\|f_2\|_p < \epsilon/3$. Thus, by Minkowski's inequality,

$$\left\{ \int_{-\infty}^{\infty} |f(x + t) - f(x)|^p\, dx \right\}^{1/p}$$
$$\leq \left\{ \int_{-\infty}^{\infty} |f_1(x + t) - f_1(x)|^p\, dx \right\}^{1/p}$$
$$+ \left\{ \int_{-\infty}^{\infty} |f_2(x + t)|^p\, dx \right\}^{1/p} + \left\{ \int_{-\infty}^{\infty} |f_2(x)|^p\, dx \right\}^{1/p}.$$

Each of the last two terms is less than $\epsilon/3$; since f_1 is uniformly continuous and vanishes outside a finite interval, the last term is also smaller than $\epsilon/3$ provided t is close enough to 0. Thus, $\omega_p(\delta) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ if δ is close enough to 0.

We now prove (2.14). Using the change of variables $t = x - s$ and property (a),

$$\sigma_R(x) - f(x) = \int_{-\infty}^{\infty} f(x) K_R(x - s)\, ds - f(x) \cdot 1$$
$$= \int_{-\infty}^{\infty} [f(x - t) - f(x)] K_R(t)\, dt.$$

Thus,

$$\int_{-\infty}^{\infty} |\sigma_R(x) - f(x)|\, dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} [f(x - t) - f(x)] K_R(t)\, dt \right| dx$$
$$\leq \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x - t) - f(x)|\, K_R(t)\, dt \right\} dx$$
$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x - t) - f(x)|\, dx \right\} K_R(t)\, dt$$
$$= \int_{|t| \leq} \left\{ \int_{-\infty}^{\infty} |f(x - t) - f(x)|\, dx \right\} K_R(t)\, dt$$
$$+ \int_{|t| > \delta} \left\{ \int_{-\infty}^{\infty} |f(x - t) - f(x)|\, dx \right\} K_R(t)\, dt.$$

The first of these last two terms is clearly dominated by

$$\omega_1(\delta) \int_{-\infty}^{\infty} K_R(t)\, dt = \omega_1(\delta),$$

which tends to 0 as δ tends to 0. The second term is majorized by

$$\int_{|t|>\delta} \left\{ \int_{-\infty}^{\infty} |f(x-t)|\, dx + \int_{-\infty}^{\infty} |f(x)|\, dx \right\} K_R(t)\, dt$$
$$= 2\|f\|_1 \int_{|t|>\delta} K_R(t)\, dt.$$

Thus, given $\epsilon > 0$, let us first choose $\delta > 0$ so that $\omega_1(\delta) < \epsilon/2$; then, with this δ fixed, property (c) can be used to find $R_0 > 0$ so that

$$\int_{|t|>\delta} K_R(t)\, dt < \frac{\epsilon}{4\|f\|_1}$$

when $R \geq R_0$. This shows that

$$\int_{-\infty}^{\infty} |\sigma_R(x) - f(x)|\, dx < \frac{\epsilon}{2} + \frac{2\|f\|_1}{4\|f\|_1}\epsilon = \epsilon,$$

provided $R \geq R_0$, and (2.14) is proved.

We recall that in the case of Fourier series the Abel means behaved very much like the $(C, 1)$ means, yet an independent study of them was of interest, particularly when we examined the relation between the theory of Fourier series and the theory of harmonic and analytic functions of a complex variable. This is equally true for Fourier integrals; consequently it is worthwhile, at this point, to devote a few words to Abel summability and its relation to the Fourier inversion problem.

Guided by our heuristic principle of replacing the integers by the real line and sums by integrals, we would expect the Abel means of the integral $\int_{-\infty}^{\infty} u(t)\, dt$ to be defined by the expression $\int_{-\infty}^{\infty} r^{|t|} u(t)\, dt$ with $0 \leq r < 1$. For technical reasons, which will become apparent shortly, it is convenient to put $r = e^{-2\pi y}$, $0 < y < \infty$; thus the Abel means of $\int_{-\infty}^{\infty} u(t)\, dt$ have the form

$$A(y) = \int_{-\infty}^{\infty} e^{-2\pi y|t|} u(t)\, dt, \qquad y > 0,$$

and we say that our integral is Abel summable to l if $\lim_{y\to 0+} A(y) = l$.

Let us now examine the Abel means of the integral (2.8) when g is the Fourier transform of an integrable function f. We then have

$$f(x, y) = \int_{-\infty}^{\infty} e^{-2\pi y|t|} e^{2\pi i x t} \hat{f}(t)\, dt$$
$$= \int_{-\infty}^{\infty} e^{-2\pi y|t|} e^{2\pi i x t} \left\{ \int_{-\infty}^{\infty} f(s) e^{-2\pi i t s}\, ds \right\} dt$$
$$= \int_{-\infty}^{\infty} f(s) \left\{ \int_{-\infty}^{\infty} e^{-2\pi y|t|} e^{2\pi i t(x-s)}\, dt \right\} ds.$$

As in the case of the $(C, 1)$ means we can easily obtain a simple expression for the inner integral:

$$\int_{-\infty}^{\infty} e^{-2\pi y|t|} e^{2\pi i t x}\, dt = \int_0^{\infty} e^{2\pi t(ix-y)}\, dt + \int_{-\infty}^{0} e^{2\pi t(ix+y)}\, dt$$
$$= \frac{1}{2\pi(y-ix)} + \frac{1}{2\pi(y+ix)} = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

Hence,

(2.15) $$f(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2 + y^2}\, dt$$
$$= \int_{-\infty}^{\infty} f(t) P(x-t, y)\, dt$$

where

(2.16) $$P(x, y) = \frac{1}{\pi}\frac{y}{x^2+y^2},$$

for $y > 0$ and $-\infty < x < \infty$. $P(x, y)$ is called the *Poisson kernel* and the integral (2.15) is called the *Poisson integral of f*.

It is clear that result (2.14) still holds if we replace the Cesàro means of the integral $\int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i x t}\, dt$ by the Abel means, provided we can show that the Poisson kernel satisfies

(a′) $\int_{-\infty}^{\infty} P(x, y)\, dx = 1$;

(b′) $P(x, y) \geq 0$;

(c′) *for each* $\delta > 0$ $\int_{|x|\geq\delta} P(x, y)\, dy \longrightarrow 0$ *as* $y \longrightarrow 0$.

But the last two of these properties are obvious. Property (a′) is also easy to establish: if we let $s = x/y$ then

$$\int_{-\infty}^{\infty} P(x, y)\, dx = \frac{1}{x}\int_{-\infty}^{\infty} \frac{ds}{1 + s^2}$$
$$= \lim_{N\to\infty} \frac{1}{\pi}\,[\tan^{-1} N - \tan^{-1}(-N)] = 1.$$

We note that the Poisson kernel is a harmonic function in the upper half-plane $\{z = x + iy; y > 0\}$. This can be seen either by computing its Laplacian directly or by observing that it is the real part of the analytic function

$$\frac{i}{\pi}\frac{1}{z} = \frac{1}{\pi}\frac{y}{x^2 + y^2} + i\,\frac{1}{\pi}\frac{x}{x^2 + y^2}.$$

The imaginary part,

$$Q(x, y) = \frac{1}{\pi}\frac{x}{x^2 + y^2},$$

is called the *conjugate Poisson kernel.*

Now suppose f belongs to $L^1(-\infty, \infty)$ and is real-valued. Let us form the integral

$$(2.17)\qquad F(z) = F(x + iy) = \frac{i}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1}{(x - t) + iy}\,dt,$$

where $y > 0$ and $-\infty < x < \infty$. It is easy to see that F is an analytic function in the upper half-plane† and that its real part is given by the Poisson integral of f. Thus the latter defines a harmonic function in the upper half-plane. The imaginary part,

$$(2.18)\qquad \tilde{f}(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{x - t}{(x - t)^2 + y^2}\,dt$$
$$= \int_{-\infty}^{\infty} f(t)Q(x - t, y)\,dt,$$

† Perhaps the easiest way of proving this fact is to use Morera's theorem: given a simple closed contour C in the upper half-plane we can easily check that

$$\int_C F(z)\,dz = \frac{i}{\pi}\int_{-\infty}^{\infty} f(t)\left\{\int_C \frac{dz}{z - t}\right\} dt = \frac{i}{\pi}\int_{-\infty}^{\infty} f(t)\cdot 0\,dt = 0.$$

This then implies the analyticity of F. We leave the details of this argument to the reader.

is called the *conjugate Poisson integral* of f. We see from these observations that there must be a strong connection between Poisson integrals and the theory of harmonic and analytic functions of a complex variable.

By now we have given a good deal of evidence that the theory of Fourier integrals is not only intimately connected with the theory of Fourier series but is very similar to it. This is indeed the case. We shall see in more detail in the next section, for example, how the L^2-theory of Fourier integrals is as elegant as its analog, the L^2-theory of Fourier series, which we described briefly in the first section. We shall not, however, always discuss a result concerning, say, Fourier series and then also describe the corresponding result in the theory of Fourier integrals. On the contrary, we shall often discuss an aspect of harmonic analysis in one of the two theories, but not in both. The reader should be aware that there exist parallel results in the other theory as well. For example, the operation of convolution of two functions in $L^1(-\infty, \infty)$ plays an equally important role on the real line. Its definition is the obvious one: if f and g are integrable on $(-\infty, \infty)$, then $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t)\,dt.$$

We leave it to the reader to check that $(f * g)(x)$ is defined for almost all real x and that the properties (i), (ii), (iii), and (iv) announced in the first section hold in this case as well. Moreover, the argument that was used to establish the important relation (1.18) between convolution and Fourier transformation can be used, after some obvious changes, to show that the same relation holds in the case of Fourier integrals.

3. THE L^1 AND L^2 THEORIES

In the last two sections we introduced several concepts but have not studied any of them very deeply. In this section we shall examine in much greater detail the convergence and summability of Fourier series, the Fourier inversion problem, and the L^2-theory. Moreover, we shall describe some of the better-known theorems in the harmonic analysis associated with the real line and the circle.

Let us begin with a closer look at the Fourier coefficients of an integrable and periodic function f. We have seen that when f belongs to $L^2(0, 1)$ the Fourier coefficients $c_k = \hat{f}(k)$ satisfy Bessel's inequality (see (1.5)). Thus, in particular,

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty.$$

It follows, therefore, that $c_k \longrightarrow 0$ as $|k| \longrightarrow \infty$. But this result is true even when f is in $L^1(0, 1)$. For, suppose $\epsilon > 0$, we can then write $f = g + h$ where g is in $L^2(0, 1)$ and $||h||_1 < \epsilon/2$. Since $|\hat{h}(k)| \leq ||h||_1$ for $k = 0, \pm 1, \pm 2, \cdots$, we have $|\hat{f}(k)| \leq |\hat{g}(k)| + \epsilon/2$. Now, using the result just established for $L^2(0, 1)$, we can find an $N > 0$ such that $|\hat{g}(k)| < \epsilon/2$ if $|k| \geq N$. Thus, $|\hat{f}(k)| < \epsilon/2 + \epsilon/2 = \epsilon$ if $|k| \geq N$. We have proved the following result:

(3.1) THE RIEMANN-LEBESGUE THEOREM: *If f is an integrable and periodic function then* $\lim_{|k| \to \infty} \hat{f}(k) = 0$.

An immediate application of the Riemann-Lebesgue theorem is the following convergence test for Fourier series.

(3.2) DINI TEST: *If a periodic and integrable function, f, satisfies the condition*

$$\int_{-1/2}^{1/2} \left| \frac{f(x-t) - f(x)}{\tan \pi t} \right| dt < \infty \tag{3.3}$$

at a point x, then the partial sums $s_n(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x}$ *converge to $f(x)$ as $n \longrightarrow \infty$.*

To see this we let g be the function whose value at $t \in (-\frac{1}{2}, \frac{1}{2})$ is $[f(x-t) - f(x)]/\tan \pi t$, then the integrability of f and condition (3.3) imply that g is integrable. Using (1.9) and the fact that

$$\int_{-1/2}^{1/2} D_n(t)\, dt = 1$$

we have, for $n \geq 1$,

$$\begin{aligned} s_n(x) - f(x) &= \int_{-1/2}^{1/2} f(x-t) \frac{\sin(2n+1)\pi t}{\sin \pi t}\, dt - f(x) \int_{-1/2}^{1/2} D_n(t)\, dt \\ &= \int_{-1/2}^{1/2} \{f(x-t) - f(x)\} \frac{\sin(2n+1)\pi t}{\sin \pi t}\, dt \\ &= \int_{-1/2}^{1/2} \{f(x-t) - f(x)\} \left\{ \frac{e^{2\pi i n t} - e^{-2\pi i n t}}{2i \tan \pi t} + \frac{e^{2\pi i n t} + e^{-2\pi i n t}}{2} \right\} dt \\ &= \frac{\hat{g}(-n) - \hat{g}(n)}{2i} + \frac{e^{-2\pi i n x} \hat{f}(-n) + e^{2\pi i n x} \hat{f}(n)}{2}. \end{aligned}$$

But it follows from the Riemann-Lebesgue theorem that this last expression tends to 0 as $n \longrightarrow \infty$. This proves (3.2).

The Dini test is probably the most useful of the various convergence criteria in the literature. One of its consequences is the fact that the *Fourier series of an integrable and periodic function converges to the value of the function at each point of differentiability.* We see this by first noting that, since $\lim_{t \to 0} \frac{\tan \pi t}{\pi t} = 1$, condition (3.3) is equivalent to the condition

$$\int_{-1/2}^{1/2} \left| \frac{f(x-t) - f(x)}{t} \right| dt < \infty. \tag{3.3$'$}$$

But it is obvious that if f is differentiable at x then (3.3′) must hold.

Before passing to the topic of summability of Fourier series we state, without proof, what is probably the best-known convergence test in the theory of Fourier series:

(3.4) THE DIRICHLET-JORDAN TEST: *Suppose a periodic function f is of bounded variation over* $(0, 1)$. *Then*

(a) *the partial sums $s_n(x)$ converge to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ at each real number x. In particular, they converge to $f(x)$ at each point of continuity of f;*

(b) *if f is continuous on a closed interval then $s_n(x)$ converges uniformly on this interval.*

We now pass to a more detailed study of summability of Fourier series. Let us observe that in the proof of the theorem of Fejér (result (1.6) restricted to Cesàro summability) we really have shown that the convergence of $\sigma_n(x)$ to $f(x)$ is uniform in any interval where f is uniformly continuous. From this we easily obtain the following classical result:

(3.5) WEIERSTRASS APPROXIMATION THEOREM: *Suppose f is a continuous periodic function and $\epsilon > 0$. Then there exists a trigonometric polynomial T, that is, a finite linear combination of the exponentials $e^{2\pi inx}$, $n = 0, \pm 1, \pm 2, \cdots$, such that*

$$|f(x) - T(x)| < \epsilon \qquad \textit{for all } x.$$

For we may take $T(x) = \sigma_n(x)$ for n large enough since the Cesàro means converge to f uniformly in this case.

One important consequence of these considerations is that *the system $\{e^{2\pi inx}\}$ is complete;* that is, *if all the Fourier coefficients of an integrable periodic function f vanish then f must be 0 almost everywhere.* We first note that if f is continuous and $\hat{f}(k) = 0$ for all integers k then $\sigma_n(x) \equiv 0$ for all n. But, since $\sigma_n(x) \longrightarrow f(x)$ at all x, we must have $f(x) \equiv 0$. If f is merely integrable and periodic we form the indefinite integral $F(x) = \int_0^x f(t)\,dt$. The condition $\hat{f}(0) = 0$ implies that

$$F(x+1) - F(x) = \int_x^{x+1} f(t)\,dt = \int_0^1 f(t)\,dt = 0.$$

Thus, F is a continuous periodic function. We claim that the hypothesis $\hat{f}(k) = 0$ for $k = \pm 1, \pm 2, \pm 3, \cdots$ implies that $\hat{F}(k) = 0$ for $k = \pm 1, \pm 2, \pm 3, \cdots$. For, integrating by parts,

$$\hat{F}(k) = \int_0^1 F(t)e^{-2\pi ikt}\,dt = F(t)\,\frac{e^{-2\pi ikt}}{-2\pi ik}\Big]_0^1 + \frac{1}{2\pi ik}\int_0^1 f(t)e^{-2\pi ikt}\,dt$$

$$= 0 + \frac{\hat{f}(k)}{2\pi ik} = 0 + 0 = 0.$$

From this and the orthogonality relations (1.3) we then conclude that the continuous and periodic function G whose value at x is $F(x) - \hat{f}(0)$ must satisfy $\hat{G}(k) = 0$ for all integers k. But we have shown that this implies that $G(x) \equiv 0$. Since, by Lebesgue's theorem on the differentiation of the integral $F'(x) = f(x)$ for almost every x, it follows that $0 = G'(x) = F'(x) = f(x)$ almost everywhere. This proves the completeness of the system $\{e^{2\pi inx}\}$.

We now show how to obtain the theorem of Lebesgue (see result (1.7)) that asserts that the $(C, 1)$ means of the Fourier series of an integrable and periodic function converge almost everywhere to the values of the function. In order to do this we will have to introduce the *Lebesgue set* of such a function f. We have just used the well-known fact that $F'(x) = f(x)$ for almost every x when $F(x) = \int_0^x f(t)\,dt$. We can rewrite this fact in the following way:

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \{f(x+t) - f(x)\}\,dt = 0$$

for almost every x. It turns out that a stronger result is true:

$$(3.6) \qquad \lim_{h\to 0} \frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt = 0$$

for almost every x. It is not hard to show this: For a fixed rational number r let E_r be the set of all x such that

$$\lim_{h\to 0} \frac{1}{h}\int_0^h |f(x+t) - r|\,dt = |f(x) - r|$$

fails to hold. Applying Lebesgue's theorem on the differentiation of the integral to $g(t) = |f(x+t) - r|$ we conclude that E_r has measure 0. Let $E = \bigcup E_r$, the union being taken over all rational numbers r. Then E also has measure 0. We claim that if x does not belong to E then (3.6) holds. For let $\epsilon > 0$. Choose a rational number r_0 such that $|f(x) - r_0| < \epsilon/2$. Then

$$\frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt \le \frac{1}{h}\int_0^h |f(x+t) - r_0|\,dt + \frac{1}{h}\int_0^h |f(x) - r_0|\,dt.$$

But the first term of this sum is less than $\epsilon/2$ if h is close to 0 while

$$\frac{1}{h}\int_0^h |f(x) - r_0|\,dt < \frac{1}{h}\int_0^h \frac{\epsilon}{2}\,dt = \frac{\epsilon}{2}.$$

Thus

$$\frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt < \epsilon$$

if h is small.

The set of all x such that (3.6) holds is called the *Lebesgue set* of f. We shall show that the $(C, 1)$ means of the Fourier series of f converge to $f(x)$ whenever x is a member of the Lebesgue set.

We shall need the following two estimates on the Fejér kernel:

(a) $K_n(t) \leq n + 1$;

(b) $K_n(t) \leq \dfrac{A}{(n+1)t^2}$, $|t| \leq \frac{1}{2}$, where A is an absolute constant.

The first one follows from the obvious estimate on the Dirichlet kernel $|D_k(t)| \leq 2k + 1$: for

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) \leq \frac{1}{n+1}\sum_{k=0}^{n}(2k+1)$$
$$= \frac{(n+1)^2}{n+1} = n + 1.$$

The second one is a consequence of formula (1.10) and the well-known fact $\lim\limits_{t\to 0}\dfrac{\sin t}{t} = 1$.

Now suppose x belongs to the Lebesgue set of f. As in the proof of (1.6), we have, using property (A) of the Fejér kernel,

$$\sigma_n(x) - f(x) = \int_{-1/2}^{1/2}\{f(x-t) - f(x)\}K_n(t)\,dt.$$

Thus, using the estimates (a) and (b),

$$|\sigma_n(x) - f(x)| \leq \int_{-1/2}^{1/2}|f(x-t) - f(x)|K_n(t)\,dt$$
$$\leq (n+1)\int_{|t|\leq 1/(n+1)}|f(x-t) - f(x)|\,dt$$
$$+ \frac{A}{n+1}\int_{1/(n+1)\leq|t|\leq 1/2}\frac{|f(x-t) - f(x)|}{t^2}\,dt.$$

Given $\epsilon > 0$ let $\delta > 0$ be such that $\dfrac{1}{h}\displaystyle\int_{|t|\leq h}|f(x-t) - f(x)|\,dt < \epsilon$ if $h \leq \delta$. Then the first term in the above sum is less than ϵ whenever $(n+1)^{-1} \leq \delta$. In order to estimate the second term we write the integral as the sum of the two integrals $\displaystyle\int_{(n+1)^{-1}}^{1/2}$ and $\displaystyle\int_{-1/2}^{-(n+1)^{-1}}$. We shall show that the first integral tends to 0 as $n \longrightarrow \infty$; a similar estimate will then show that the same is true for the second.

Let $G(t) = \int_0^t |f(x-s) - f(x)|\,ds$. Then, integrating by parts, we have

$$\frac{A}{(n+1)}\int_{1/(n+1)}^{1/2}\frac{|f(x-t) - f(x)|}{t^2}\,dt$$
$$\leq \frac{A}{4(n+1)}G(1/2) + \frac{2A}{n+1}\int_{1/(n+1)}^{\delta}\frac{G(t)}{t^3}\,dt + \frac{2A}{n+1}\int_{\delta}^{1/2}\frac{G(t)}{t^3}\,dt.$$

The first and third terms tend to 0 as $n \longrightarrow \infty$. Since $(1/t)G(t) < \epsilon$ for $|t| \leq \delta$ the second term is dominated by

$$\frac{2A\epsilon}{n+1}\int_{1/(n+1)}^{\delta}\frac{dt}{t^2} < 2A\epsilon.$$

Thus, $|\sigma_n(x) - f(x)|$ can be made as small as we wish by choosing n large enough. This proves the theorem of Lebesgue.

An application of this theorem is that we can reverse the inequality in Bessel's inequality (see (1.5)). For, if $f \in L^2(0, 1) \subset L^1(0, 1)$ and $c_k = \hat{f}(k)$, $k = 0, \pm 1, \pm 2, \cdots$, then the $(C, 1)$ means of f have the form

$$\sigma_n(x) = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)c_k e^{2\pi ikx}.$$

Using the orthogonality relations (1.3) we have

$$\int_0^1 |\sigma_n(x)|^2\,dx = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)^2|c_k|^2 \leq \sum_{k=-\infty}^{\infty}|c_k|^2.$$

Since $\sigma_n(x) \longrightarrow f(x)$ almost everywhere, Fatou's lemma implies that

$$\int_0^1 |f(x)|^2\,dx \leq \varliminf_{n\to\infty}\int_0^1|\sigma_n(x)|^2\,dx.$$

Consequently,

$$\int_0^1 |f(x)|^2\,dx \leq \sum_{k=-\infty}^{\infty}|c_k|^2.$$

Together with Bessel's inequality this gives us the following relation, known as *Parseval's formula:*

$$\int_0^1 |f(x)|^2\,dx = \sum_{k=-\infty}^{\infty}|\hat{f}(k)|^2. \tag{3.7}$$

We can also show easily that the partial sums of the Fourier series of a function f in $L^2(0, 1)$ converge to f in the L^2-norm. We have already seen that they do converge to a function g in $L^2(0, 1)$ (see the argument preceding (1.5)). This implies that g and f have the same Fourier coefficients $\{c_k\} = \{\hat{f}(k)\}$; for

$$\int_0^1 g(t)e^{-2\pi ikt}\,dt = \int_0^1 [g(t) - s_n(t)]e^{-2\pi ikt}\,dt + \int_0^1 s_n(t)e^{-2\pi ikt}\,dt.$$

The first term of this sum is dominated, in absolute value, by $\|g - s_n\|_2$ (use Schwarz's inequality) and, thus, tends to 0 as $n \longrightarrow \infty$. The second term equals c_k as long as $n \geq k$. From this we conclude that the Fourier coefficients of the function $f - g$ are all 0. But, since the system $\{e^{2\pi inx}\}$ is complete, this implies $f(x) - g(x) = 0$ almost everywhere.

Let us observe that if we had started with a square summable sequence $\{c_k\}$ (that is, $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$), then, by the orthogonality relations (1.3), the partial sums $s_n(x)$ of $\sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$ converge in the L^2-norm to a function g. The argument just used shows that $\hat{g}(k) = c_k$ for $k = 0, \pm 1, \pm 2, \cdots$.

We collect these facts together in the following statement:

(3.8) *Suppose f belongs to $L^2(0, 1)$ then its Fourier series converges to f in the L^2-norm; that is,*

$$\|f - s_n\|_2 = \left(\int_0^1 |f(x) - s_n(x)|^2\,dx\right)^{1/2}$$
$$= \left(\int_0^1 \left| f(x) - \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikx}\right|^2 dx\right)^{1/2}$$

tends to 0 as n tends to ∞. Furthermore,

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2} = \left(\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2\right)^{1/2} = \|\hat{f}\|_2.$$

If a sequence $\{c_k\}$ satisfies $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$ then there exists a function f in $L^2(0, 1)$ such that $c_k = \hat{f}(k)$ for all integers k.

Except for not having proved (1.7) in the case of Abel summability we have now established all the results announced in the

first section in connection with the Fourier inversion problem. We leave it to the reader to show that essentially the argument used above for Cesàro summability, using the estimates

(a′) $P(r, t) \leq \dfrac{1}{1 - r}$ and

(b′) $P(r, t) \leq \dfrac{A(1 - r)}{t^2}$, $|t| \leq \dfrac{1}{2}$, $0 \leq r < 1$, where A is an absolute constant,

gives us (1.7) in full.

As we stated toward the end of the last section, we shall not essentially repeat all this material by giving the corresponding results in the theory of Fourier integrals. The reader should have no trouble, for example, in stating and proving the analogs of results (1.6) and (1.7). Nevertheless, some discussion of what happens when we carry over the above material to the case of the real line is in order.

First of all, we cannot adapt the argument we gave to establish the Riemann-Lebesgue theorem to the case of functions in $L^1(-\infty, \infty)$. For one thing, we have not even defined the Fourier transform for functions in $L^2(-\infty, \infty)$; moreover, as we shall see shortly after we define it, it is *not* true in general that $\hat{f}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ when $f \in L^2(-\infty, \infty)$. We shall show, however, that the Riemann-Lebesgue theorem does extend to the case of the real line, and the simple argument we shall give can be adapted to prove (3.1) as well. We shall prove

(3.9) *If $f \in L^1(-\infty, \infty)$ then $\hat{f}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.*

Since

$$-\hat{f}(x) = \int_{-\infty}^{\infty} (-1)e^{-2\pi ixt} f(t)\,dt$$
$$= \int_{-\infty}^{\infty} e^{-2\pi ix[t-(1/2x)]} f(t)\,dt$$
$$= \int_{-\infty}^{\infty} e^{-2\pi ixt} f\left(t + \frac{1}{2x}\right) dt,$$

we have

$$|\hat{f}(x)| = \left|\frac{1}{2}\int_{-\infty}^{\infty}\left\{f(t) - f\left(t + \frac{1}{2x}\right)\right\} e^{-2\pi i x t}\, dt\right|$$
$$\leq \frac{1}{2}\omega_1\left(\frac{1}{2x}\right) \longrightarrow 0 \qquad \text{as } x \longrightarrow \infty.$$

Let us now state the result that corresponds to (3.8):

(3.10) THE PLANCHEREL THEOREM: *If f belongs to $L^2(-\infty, \infty)$ then there exists a function $\hat{f}$, also in $L^2(-\infty, \infty)$, such that*

$$\int_{-\infty}^{\infty}\left|\hat{f}(x) - \int_{-N}^{N} e^{-2\pi i x t} f(t)\, dt\right|^2 dx \longrightarrow 0$$

as $N \longrightarrow \infty$. The function $\hat{f}$ is called the Fourier transform of f and it agrees a.e. with the previously defined Fourier transform whenever $f \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty)$. Furthermore, Parseval's formula holds

$$\|\hat{f}\|_2 = \|f\|_2.$$

Fourier inversion is possible in the L^2-norm:

$$\int_{-\infty}^{\infty}\left|f(t) - \int_{-N}^{N} e^{2\pi i x t}\hat{f}(x)\, dx\right|^2 dt \longrightarrow 0$$

as $N \longrightarrow \infty$. Finally, each f in $L^2(-\infty, \infty)$ has the form $f = \hat{g}$ for an (almost everywhere) unique g in $L^2(-\infty, \infty)$.

To prove (3.10) let us choose an f in $L^1(-\infty, \infty) \cap L^2(-\infty, \infty)$ and form the convolution $h = f * g$ where $g(t) = \overline{f(-t)}$. It follows from the remarks made at the very end of Sec. 2 that h, being the convolution of two functions in $L^1(-\infty, \infty)$, is integrable and, also, $\hat{h}(x) = \hat{f}(x) \cdot \overline{\hat{f}}(x) = |\hat{f}(x)|^2 \geq 0$ (by the real-line analog of (1.18)). Moreover, we claim that 0 is a point of the Lebesgue set of h; in fact, it is a point of continuity of h. For

$$|h(\delta) - h(0)| = \left|\int_{-\infty}^{\infty}\{g(\delta - t) - g(-t)\}f(t)\, dt\right|$$
$$\leq \left[\int_{-\infty}^{\infty}|g(\delta - t) - g(-t)|^2\, dt\right]^{1/2}\|f\|_2.$$

But $\left[\int_{-\infty}^{\infty}|g(\delta - t) - g(-t)|^2\, dt\right]^{1/2}$ is dominated by the L^2 modulus of continuity evaluated at δ, $\omega_2(\delta)$, of the function whose value at t is $g(-t)$. Since the latter belongs to $L^2(-\infty, \infty)$, we can conclude that $\lim_{\delta\to 0}|h(\delta) - h(0)| = 0$. Consequently, the $(C, 1)$ means of the integral $\int_{-\infty}^{\infty}\hat{h}(x)e^{2\pi i x t}\, dx$, defining $(\hat{h})^{\vee}(x)$, converge to $h(t)$ when $t = 0$. That is,

$$\int_{-R}^{R}\left(1 - \frac{|x|}{R}\right)\hat{h}(x)\, dx \longrightarrow h(0).$$

But $\hat{h}(x) \geq 0$ and the integrand in this integral increases monotonically to $\hat{h}(x)$. Thus, by the Lebesgue monotone convergence theorem $\hat{h}$ is integrable and

$$\int_{-\infty}^{\infty}\hat{h}(x)\, dx = h(0).$$

Since $h(0) = \int_{-\infty}^{\infty} f(t)\overline{f(t)}\, dt = \|f\|_2^2$ and $\hat{h} = |\hat{f}|^2$ this shows

$$\int_{-\infty}^{\infty}|\hat{f}(x)|^2\, dx = \int_{-\infty}^{\infty}|f(x)|^2\, dx. \tag{3.11}$$

Thus, Parseval's formula holds when $f \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty)$.

In particular, (3.11) tells us that the mapping $f \longrightarrow \hat{f}$ is bounded, in the L^2-norm, as a linear operator on the dense subset $L^1 \cap L^2$ of the Hilbert space L^2 into L^2. It is well known that in such a case there exists a unique, bounded extension of the operator on all of the Hilbert space. Using the same notation for this extension we then can conclude that (3.11) holds for all $f \in L^2(-\infty, \infty)$.

If we let χ_N be the characteristic function of the interval $[-N, N]$ we set $f_N = \chi_N f$, for $f \in L^2(-\infty, \infty)$. Then $f_N \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty)$ and $\|f - f_N\|_2 \longrightarrow 0$ as $N \longrightarrow \infty$. Because of the boundedness of the operator we have just defined we then must have $\|\hat{f} - \hat{f}_N\|_2 \longrightarrow 0$. This proves the first part of (3.10).

The Fourier inversion part of Plancherel's theorem follows easily from the relation

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\, dx = \int_{-\infty}^{\infty}\hat{f}(x)g(x)\, dx \tag{3.12}$$

whenever $f, g \in L^2(-\infty, \infty)$. The proof of (3.12) for functions in $L^1 \cap L^2$ is straightforward; thus, we first establish (3.12) for f_N and g_N and obtain the general result by letting $N \longrightarrow \infty$. It is an immediate consequence of (3.12) that

$$\bar{f} = (\bar{\hat{f}})^\wedge \tag{3.13}$$

for all $f \in L^2(-\infty, \infty)$. For

$$\|\bar{f} - (\bar{\hat{f}})^\wedge\|_2^2 = \left\{\int f\bar{f} - \int f(\bar{\hat{f}})^\wedge\right\} - \overline{\left\{\int f(\bar{\hat{f}})^\wedge - \int (\bar{\hat{f}})^\wedge\overline{(\bar{\hat{f}})^\wedge}\right\}}.$$

But both expressions in the brackets are 0; applying (3.12) to the first one we obtain $\int f\bar{f} - \int \hat{f}\bar{\hat{f}}$, which is 0 by Parseval's formula. A similar argument shows the second expression is 0 also. Now, because of (3.13) we have that $\bar{f}$ is the limit in the L^2-norm, as $N \longrightarrow \infty$, of the functions given by the integrals

$$\int_{-N}^{N} e^{-2\pi ixt}\bar{\hat{f}}(t)\,dt = \overline{\int_{-N}^{N} e^{2\pi ixt}\hat{f}(t)\,dt}.$$

By taking complex conjugates we have the Fourier inversion result announced in (3.10).

The last statement follows from this inversion applied to $g = f$, where $\check{f}$ is the limit in L^2, as $N \longrightarrow \infty$, of

$$\int_{-N}^{N} e^{2\pi ixt}f(t)\,dt = \overline{(\bar{f}_N)^\wedge(x)}.$$

The material presented up to this point belongs to the foundation of the theories of Fourier series and integrals. It is desirable also to describe some of the directions in which harmonic analysis has been developed. This subject, however, is so rich with results, covering such a wide field of mathematics, that it is impossible to present something that approximates a survey of the highlights in the space that we have available. For this reason we shall, from time to time, select certain topics and present them, mostly without proofs. The bases for our selections are the possibility of extending these topics to other parts of harmonic analysis, the simplicity of the concepts involved, and their applicability to other branches of mathematics. We conclude this section with what is perhaps the one result that best fits this description, a

celebrated theorem of Wiener. As we shall see in Sec. 5, this theorem extends to the harmonic analysis associated with any locally compact abelian group, no new concepts are needed to understand it, and from it one can prove rather easily the prime number theorem [4, p. 303]!

Suppose f belongs to $L^1(-\infty, \infty)$ then the collection of all finite linear combinations of translates of f will be denoted by T_f. That is, g belongs to T_f if and only if it has the form

$$g(x) = \Sigma\, a_k f(x + t_k) \tag{3.14}$$

for some finite set of real numbers t_k and complex numbers a_k. The theorem of Wiener asserts the following:

(3.15) *Suppose f belongs to $L^1(-\infty, \infty)$ and that $\hat{f}(x)$ is never* 0, *then the closure, in the L^1 topology, of T_f is all of $L^1(-\infty, \infty)$. In other words, any function in $L^1(-\infty, \infty)$ can be approximated arbitrarily closely in the L^1-norm by functions of the form* (3.14).

It is easy to find functions f whose Fourier transforms never vanish. For example, the proof of (2.16), with $y = 1$, consisted, simply, of showing that when $f(t) = e^{-2\pi|t|}$ then $\hat{f}(x) = \frac{1}{\pi}\frac{1}{1 + x^2}$.

That the condition $\hat{f}(x) \neq 0$ for all real x is necessary is clear. For, if $\hat{f}(x_0) = 0$ for some x_0 and $g \in T_f$ then g has the form (3.14) and, thus,

$$\hat{g}(x) = \Sigma\, a_k e^{2\pi it_k}\hat{f}(x).$$

Therefore $\hat{g}(x_0) = 0$. Now suppose h is in the L^1 closure of T_f; then there exists a sequence $\{g_n\}$ in T_f such that $\|g_n - h\|_1 \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, by (2.9),

$$|\hat{g}_n(x_0) - \hat{h}(x_0)| \le \|\hat{g}_n - \hat{h}\|_\infty \le \|g_n - h\|_1 \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Since $g_n(x_0) = 0$ for all n we must have $\hat{h}(x_0) = 0$. We have shown that the Fourier transforms of all h in the closure of T_f vanish at x_0. Since there are integrable functions whose Fourier transforms never vanish, this closure cannot comprise all of $L^1(-\infty, \infty)$.

We shall not give a proof of (3.15). We would like to point out, however, that this proof uses strongly the fact that whenever $f \in L^1(-\infty, \infty)$ then the closure of T_f is a (closed) *ideal* in

$L^1(-\infty, \infty)$. By the term "ideal" we mean a linear subspace, I, of $L^1(-\infty, \infty)$ such that $g * h \in I$ whenever $g \in I$ and $h \in L^1(-\infty, \infty)$.

An important class of ideals in $L^1(-\infty, \infty)$ is the collection of closed *maximal ideals* (an ideal M is said to be maximal if it is not contained in any proper ideal in $L^1(-\infty, \infty)$ other than M itself). This class has a very elegant characterization:

(3.16) *M is a closed maximal ideal if and only if there exists a real number x such that M consists of all $f \in L^1(-\infty, \infty)$ such that $\hat{f}(x) = 0$.*

Thus, we have a one-to-one correspondence between the real numbers and the closed maximal ideals in $L^1(-\infty, \infty)$. We shall denote the closed maximal ideal corresponding to x by $M(x)$. It is not hard to see that the following is a generalization of Wiener's theorem:

(3.15′) *Every proper closed ideal in $L^1(-\infty, \infty)$ is contained in a closed maximal ideal.*

For if $\hat{f}(x)$ is never 0 then f cannot belong to $M(x)$ for any x. Thus, the closed ideal obtained by taking the closure of T_f cannot be included in any closed maximal ideal. Consequently, this ideal is not proper; that is, it must coincide with $L^1(-\infty, \infty)$.

These considerations lead us to the formulation of a well-known problem in harmonic analysis, the problem of *spectral synthesis*. If the closure of T_f is a proper ideal, I_f, then, by (3.15′), it is contained in a certain class of ideals $M(x)$. It is easy to check that the intersection of all closed maximal ideals containing I_f is a closed ideal. The problem of spectral synthesis is to determine for which $f \in L^1(-\infty, \infty)$ it is true that I_f equals this intersection. It has been discovered only recently (in 1959) that there are $f \in L^1(-\infty, \infty)$ for which I_f is not equal to the intersection of all maximal ideals containing it.

It is often useful to rephrase this problem in the following way: *For which f in $L^1(-\infty, \infty)$ is it true that if $\hat{g}(x) = 0$ whenever $\hat{f}(x) = 0$ then g is in the closure of T_f?*

4. SOME OPERATORS THAT ARISE IN HARMONIC ANALYSIS

Suppose $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$ is an analytic function in the interior of the unit circle. Suppose, further, that F is bounded in this domain; say, $|F(z)| \le B < \infty$ for $|z| < 1$. Let us write $z = re^{2\pi i\theta}$, $0 \le r < 1$, $0 \le \theta < 1$. Then, using the orthogonality relations (1.3),

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} = \int_0^1 \left(\sum_{k=0}^{\infty} a_k r^k e^{2\pi ik\theta}\right)\left(\sum_{k=0}^{\infty} \bar{a}_k r^k e^{-2\pi ik\theta}\right) d\theta$$
$$= \int_0^1 |F(re^{2\pi i\theta})|^2\, d\theta \le B^2 \qquad \text{for } 0 \le r < 1.$$

Letting $r \longrightarrow 1$ we therefore obtain $\sum_{k=0}^{\infty} |a_k|^2 < \infty$. By (3.8) we thus can conclude that there exists an f belonging to $L^2(0, 1)$ such that $\hat{f}(k) = a_k$, $k = 0, 1, 2, \cdots$ and $\hat{f}(k) = 0$ for all negative integers k. This shows that

$$F(re^{2\pi i\theta}) = \sum_{k=0}^{\infty} \hat{f}(k) r^k e^{2\pi ik\theta} = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^k e^{2\pi ik\theta}, \qquad 0 \le r < 1,$$

are the Abel means of the Fourier series of f. By (1.7), therefore, $\lim_{r\to 1} F(re^{2\pi i\theta}) = f(\theta)$ for almost every θ. In particular, we have proved

(4.1) Fatou's Theorem: *If F is a bounded analytic function in the interior of the unit circle then the radial limits $\lim_{r\to 1} F(re^{2\pi i\theta})$ exist for almost every θ in $[0, 1)$.*

We shall use this theorem to define an important operator, the *conjugate function mapping*, acting on integrable and periodic functions. Suppose f is such a function. It follows from our discussion concerning the Poisson kernel and the conjugate Poisson kernel that the function G defined by

$$(4.2) \qquad G(z) = \int_0^1 \frac{1 + re^{2\pi i(\theta - t)}}{1 - re^{2\pi i(\theta - t)}} f(t)\, dt$$
$$= \int_0^1 P(r, \theta - t) f(t)\, dt + i \int_0^1 Q(r, \theta - t) f(t)\, dt,$$

$z = re^{2\pi i\theta}$, is analytic in the interior of the unit circle. We already know that the first expression in the last sum has radial limits, as $r \longrightarrow 1$, for almost all θ. The following theorem asserts that this is also true for the second term.

(4.3) *Suppose $f \in L^1(0, 1)$; then the limits, $\tilde{f}(\theta)$, as $r \longrightarrow 1$, of*

$$\tilde{A}(r, \theta) = \int_0^1 Q(r, \theta - t) f(t)\, dt$$
$$= \int_0^1 \frac{2r \sin 2\pi(\theta - t)}{1 - 2r \cos 2\pi(\theta - t) + r^2} f(t)\, dt$$

exist for almost all θ. The function $\tilde{f}$ is called the conjugate function of f.†

By decomposing f into its real and imaginary parts and considering separately the positive and negative parts of each of these, we see that it suffices to prove (4.3) for $f \geq 0$. Thus, letting $A(r, \theta)$ be the Poisson integral and $\tilde{A}(r, \theta)$ the conjugate Poisson integral of f, we obtain an analytic function for $|z| < 1$, $z = re^{2\pi i\theta}$, and its values lie in the right half-plane (by property (B′) of the Poisson kernel). Thus,

$$F(z) = e^{-A(r,\theta) - i\tilde{A}(r,\theta)}$$

† If we let $r \longrightarrow 1$ we obtain, formally,

$$\tilde{f}(\theta) = \int_0^1 \frac{2 \sin 2\pi(\theta - t)}{2(1 - \cos 2\pi(\theta - t))} f(t)\, dt = \int_0^1 \frac{f(t)}{4 \tan \pi(\theta - t)}\, dt.$$

This last integral, however, is not defined even when f is an extremely well-behaved function (for example, if f is constant and nonzero in a neighborhood of θ the integral fails to exist). One can show, however, that if we take the principal value integral

$$(*) \qquad \lim_{\epsilon \to 0+} \int_{\substack{\epsilon \leq |\theta - t| \\ 0 \leq t \leq 1}} \frac{f(t)}{4 \tan \pi(\theta - t)}\, dt$$

we do obtain a value for almost all θ. In fact, an argument not unlike that used to prove (1.7) shows that the existence of these limits, as $\epsilon = 1 - r \longrightarrow 0$, is equivalent almost everywhere to the existence of the limits of (4.3). One may take (*) as the definition of the conjugate function, therefore, and avoid the use of analytic function theory. However, the real-variable proof of the existence of $\tilde{f}$ is by no means easy.

is a bounded ($|F(z)| \leq 1$) analytic function in the interior of the unit circle. By Fatou's theorem (4.1) the radial limits of F exist almost everywhere. Since the radial limits of $A(r, \theta)$ also exist almost everywhere and are finite (they equal $f(\theta)$), the limits of F must be nonzero almost everywhere. But this implies the existence of $\lim_{r \to 1} \tilde{A}(r, \theta)$ for almost all θ, and (4.3) is proved.

The conjugate function mapping is obviously linear. If $f \in L^2(0, 1)$, then, using the fact that $\tilde{A}(r, \theta)$, $0 \leq r < 1$, $0 \leq \theta < 1$, are the Abel means of the conjugate Fourier series of f and, also, the result (3.8), we can show very easily that

$$(4.4) \qquad ||\tilde{f}||_2^2 = \sum_{|k| \geq 1} |\hat{f}(k)|^2 \leq \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 = ||f||_2^2.$$

Thus the mapping $f \longrightarrow \tilde{f}$ is a bounded linear transformation when restricted to the space $L^2(0, 1)$. One can show, however, that there are functions in $L^1(0, 1)$ for which the conjugate function is not integrable. In particular, it follows that this mapping is not bounded as an operator from $L^1(0, 1)$ into $L^1(0, 1)$. We do have the following theorem, however.

(4.5) Theorem of M. Riesz: *If $f \in L^p(0, 1)$, $1 < p < \infty$, then $\tilde{f} \in L^p(0, 1)$ and*

$$||\tilde{f}||_p \leq A_p ||f||_p,$$

where A_p depends only on p.

For exactly the same reasons that we gave in the proof of (4.3) it suffices to consider the case $f \geq 0$. Furthermore, we claim that it is sufficient to show that

$$(4.6) \qquad \int_0^1 |\tilde{A}(r, \theta)|^p\, d\theta \leq c_p \int_0^1 |A(r, \theta)|^p\, d\theta = c_p \int_0^1 [A(r, \theta)]^p\, d\theta$$

for $0 \leq r < 1$, where c_p depends only on p. For, an argument very similar to that used to prove (2.14), shows that the Poisson integrals $A(r, \theta)$ converge to $f(\theta)$ in the $L^p(0, 1)$ norms. In particular,

$$\lim_{r \to 1} \int_0^1 [A(r, \theta)]^p\, d\theta = ||f||_p^p.$$

Since $\tilde{A}(r, \theta) \longrightarrow \tilde{f}(\theta)$ almost everywhere as $r \longrightarrow 1$, an application of Fatou's lemma then gives us the inequality $\|\tilde{f}\|_p^p \leq c_p\|f\|_p^p$, from which the theorem follows.

To show (4.6) we argue in the following manner. Let

$$F(z) = A(r, \theta) + i\tilde{A}(r, \theta) \qquad \text{for } x + iy = z = re^{2\pi i\theta}, 0 \leq r < 1,$$

and let $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denote the Laplacian operator. Treating A, $\tilde{A}$, and F as functions of x and y we have, by a simple calculation which uses the Cauchy-Riemann equations (recall that F is analytic),

$$\Delta A^p = p(p-1)A^{p-2}|F'|^2 \qquad \text{and} \qquad \Delta|F|^p = p^2|F|^{p-2}|F'|^2.$$

Let us assume, first, that $1 < p \leq 2$. Then, since $|F| \geq A$,

$$\Delta|F|^p \leq q\,\Delta A^p,$$

where $(1/q) = 1 - (1/p)$. We claim that this inequality and Green's formula imply

$$\int_0^1 |F(re^{2\pi i\theta})|^p\,d\theta \leq q \int_0^1 [A(r, \theta)]^p\,d\theta$$

for $0 \leq r < 1$, which certainly implies (4.6). The form of Green's formula we need is the following. Suppose u is a continuous function defined in the unit circle which has continuous first and second derivatives, S is the circle $\{(x, y);\ x^2 + y^2 \leq r^2 < 1\}$ and C its circumference. Then

$$\int_C \frac{\partial u}{\partial r}\,ds = \iint_S \Delta u\,dx\,dy,$$

where $\partial/\partial r$ denotes differentiation in the direction of the radius vector and $ds = r\,d\theta$. Applying this formula to $u = A^p$ and $u = |F|^p$ we obtain, because of the inequality $\Delta|F|^p \leq q\,\Delta A^p$,

$$\int_0^1 \left(\frac{\partial}{\partial r}|F(re^{2\pi i\theta})|^p\right) r\,d\theta \leq q \int_0^1 \left(\frac{\partial}{\partial r}[A(r, \theta)]^p\right) r\,d\theta.$$

Thus, because of the smoothness of the functions involved,

$$\frac{d}{dr}\int_0^1 |F(re^{2\pi i\theta})|^p\,d\theta \leq \frac{d}{dr}\,q \int_0^1 A[(r, \theta)]^p\,d\theta.$$

Since $F(0) = A(0, \theta)$ we obtain the desired inequality by integrating with respect to r.

It remains for us to show that the theorem holds for $p \geq 2$. But it is an easy exercise, using the fact that L^p and L^q are dual when $1/p + 1/q = 1$, to show that whenever a bounded operator acting on L^p is given by a convolution, then it is defined on L^q and is a bounded operator on this space as well. The mapping $f \longrightarrow \tilde{A}(r, \theta)$ is such an operator and it satisfies

$$\int_0^1 |\tilde{A}(r, \theta)|^p\,d\theta \leq B_p \int_0^1 |A(r, \theta)|^p\,d\theta \leq B_p\|f\|_p^p \dagger$$

for $1 < p \leq 2$. Thus, it satisfies this inequality for the indices conjugate to p; that is, for p replaced by $q = p/(p-1)$:

$$\int_0^1 |\tilde{A}(r, \theta)|^q\,d\theta \leq C_q\|f\|_q^q$$

whenever $f \in L^q(0, 1)$, $q \geq 2$. But, by Fatou's lemma, this implies

$$\int_0^1 |\tilde{f}(\theta)|^q\,d\theta = \int_0^1 \lim_{r\to 1} |\tilde{A}(r, \theta)|^q\,d\theta \leq C_q\|f\|_q^q$$

and (4.5) is proved.

This development gives us a glimpse of the role that "complex methods" (that is, the use of the theory of analytic functions of a complex variable) play in the theory of Fourier series.

Let us examine some more operators that arise naturally in harmonic analysis. For example, let us study the Fourier transform mapping acting on functions defined on the entire real line. Inequality (2.9) tells us that it is a bounded transformation defined on $L^1(-\infty, \infty)$ with values in $L^\infty(-\infty, \infty)$. The Plancherel theorem (3.10) tells us that it is a bounded transformation from $L^2(-\infty, \infty)$ into itself. A natural question, then, is whether it can be defined on other classes L^p and, if so, whether we obtain a

† One can give several direct proofs of the inequality

$$\int_0^1 |A(r, \theta)|^p\,d\theta \leq \int_0^1 |f(\theta)|^p\,d\theta,\ p \geq 1.$$

Since it is an immediate consequence of Young's inequality (4.8) (for $g(\theta) = P(r, \theta)$ defines a function in L^1 and $A(r, \theta) = (g * f)(\theta)$) we will not prove it here.

bounded transformation with values in some classes L^q. But any function in L^p, $1 < p < 2$, can be written as a sum of a function in L^1 and one in L^2: put $f = f_1 + f_2$, where $f_2(x) = f(x)$ when $|f(x)| \leq 1$ and $f_2(x) = 0$ otherwise; then $f_1 \in L^1$ and $f_2 \in L^2$. Thus, we can write $\hat{f} = \hat{f}_1 + \hat{f}_2$, where $\hat{f}_1$ is defined as the Fourier transform of a function in L^1 while $\hat{f}_2$ is defined by (3.10). The fact that these two definitions agree when a function belongs to $L^1 \cap L^2$ implies that $\hat{f}$ is well defined. The following theorem tells us that Fourier transformation defined on $L^p(-\infty, \infty)$, $1 < p < 2$, is bounded as a mapping into $L^q(-\infty, \infty)$, where q is the conjugate index to p.

(4.7) THE HAUSDORFF-YOUNG THEOREM: *If* $f \in L^p(-\infty, \infty)$, $1 \leq p \leq 2$, *then* $\hat{f} \in L^q(-\infty, \infty)$, *where* $1/p + 1/q = 1$, *and*

$$\|\hat{f}\|_q \leq \|f\|_p.$$

We shall not prove (4.7) immediately. Instead, we shall give examples of some other inequalities that occur in harmonic analysis and then state some general results from which all these inequalities, including (4.5) and (4.7), follow as relatively easy consequences.

(4.8) YOUNG'S THEOREM: *Suppose* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, *where* $\frac{1}{p} + \frac{1}{q} \geq 1$. *If* $f \in L^p(-\infty, \infty)$ *and* $g \in L^q(-\infty, \infty)$ *then* $f * g$ *belongs to* $L^r(-\infty, \infty)$ *and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

The same result holds for periodic functions if we replace the interval $(-\infty, \infty)$ *by the interval* $(0, 1)$.

The operator on functions defined on $(-\infty, \infty)$ that corresponds to the conjugate function operator satisfies the same inequality (4.5). Using (2.18) and arguments that are completely analogous to those we gave at the beginning of this section, we see that this operator, called the *Hilbert transform,* can be defined by letting

$$\tilde{f}(x) = \lim_{y \to 0+} \tilde{f}(x, y) = \lim_{y \to 0+} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{x - t}{(x - t)^2 + y^2}\, dt$$

correspond to $f \in L^p(-\infty, \infty)$, $1 \leq p$, and that the following result holds:

(4.9) *If* $f \in L^p(-\infty, \infty)$, $1 < p < \infty$, *then its Hilbert transform* $\tilde{f}$ *also belongs to* $L^p(-\infty, \infty)$ *and*

$$\|\tilde{f}\|_p \leq A_p \|f\|_p,$$

where A_p *depends only on* p.

All the operators we have encountered up to this point are linear. There are several important transformations in harmonic analysis, however, that are not linear. Perhaps the best-known example of such a transformation is the *Hardy-Littlewood maximal function.* This operator is defined in the following way: if $f \in L^p(-\infty, \infty)$, $1 \leq p \leq \infty$, then its maximal function is the function whose value at $x \in (-\infty, \infty)$ is

$$f^*(x) = \sup_{h \neq 0} \frac{1}{h} \int_x^{x+h} |f(t)|\, dt.$$

Lebesgue's theorem on the differentiation of the integral guarantees that $f^*(x) < \infty$ for almost every x. It can be shown that

(4.10) *If* $f \in L^p(-\infty, \infty)$, $1 < p \leq \infty$, *then* $f^* \in L^p(-\infty, \infty)$ *and*

$$\|f^*\|_p \leq A_p \|f\|_p,$$

where A_p *depends only on* p.

The usefulness of the maximal function lies in the fact that it majorizes several important operators. Thus, it is clear why a theorem like (4.10) is desirable, as it immediately implies the boundedness of these operators.

Although the mapping $f \longrightarrow f^*$ is not linear, it does satisfy the inequality $(f + g)^* \leq f^* + g^*$. This property is generally referred to as *sublinearity.* More generally, we say that an operator T mapping functions into functions is *sublinear* if, whenever Tf and Tg are defined, so is $T(f + g)$ and

$$|T(f + g)| \leq |Tf| + |Tg|.$$

In all these instances special cases of the inequalities involved are fairly easy to establish. For the conjugate function mapping the case $p = 2$ was seen to be an easy consequence of Theorem

(3.8) (see (4.4)). A similar argument, using the Plancherel theorem, shows that the same is true for the Hilbert transform. We have pointed out that the cases $p = 1$, $q = \infty$ and $p = 2 = q$ of the Hausdorff-Young theorem had already been obtained by us in the previous sections. The inequality $\|f * g\|_1 \leq \|f\|_1\|g\|_1$, which was the first result (property (i)) we established after introducing the operation of convolution, is the special case $r = p = q = 1$ of Young's theorem. Another special case of this theorem that is immediate is obtained when p and q are conjugate indices, $1/p + 1/q = 1$, and, thus, $r = \infty$; for this is simply a consequence of Hölder's inequality. Finally, it is clear that (4.10) holds when $p = \infty$.

It was M. Riesz who first discovered (in 1927) a general principle that asserted, in part, that in a wide variety of inequalities of the type we are discussing, special cases, such as those described in the previous paragraph, imply the general case. In order to state his theorem, known as the *M. Riesz convexity theorem*, we need to establish some notation. Suppose (M, μ) and (N, ν) are two measure spaces, where M and N are the point sets and μ and ν the measures. An operator T mapping measurable functions on M into measurable functions on N is said to be of *type* (p, q) if it is defined on $L^p(M)$ and there exists a constant A, independent of $f \in L^p(M)$, such that

$$\text{(4.11)} \qquad \|Tf\|_q = \left(\int_N |Tf|^q \, d\nu\right)^{1/q} \leq A \left(\int_M |f|^p \, d\mu\right)^{1/p} = A\|f\|_p.$$

The least A for which (4.11) holds is called the *bound*, or *norm*, of T. The general principle can then be stated in the following way.

(4.12) THE M. RIESZ CONVEXITY THEOREM: *Suppose a linear operator T is of types (p_0, q_0) and (p_1, q_1), with bounds A_0 and A_1, respectively. Then it is of type (p_t, q_t), with bound $A_t \leq A_0^{1-t}A_1^t$, for $0 \leq t \leq 1$, where*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

The Hausdorff-Young theorem is an immediate consequence of this result. Since $T: f \longrightarrow \hat{f}$ is of types $(1, \infty)$ and $(2, 2)$, it must be of type $\left(\frac{2}{2-t}, \frac{2}{t}\right)$ for $0 \leq t \leq 1$. But if $p = \frac{2}{2-t}$, then the conjugate index is $q = \frac{p}{p-1} = \frac{2}{t}$. Since the "end-point" (i.e. $t = 0$ and $t = 1$) bounds are 1, we have

$$\|\hat{f}\|_q \leq 1^{(1-t)}1^t\|f\|_p = \|f\|_p,$$

which is the inequality in (4.7).

Similarly, Young's theorem follows from (4.12). First, let us fix $g \in L^1$ and define $Tf = f * g$. We have seen that T is of type $(1, 1)$, with bound $\|g\|_1$, and of type (∞, ∞), also with bound $\|g\|_1$. Thus, T is of type $\left(\frac{1}{1-t}, \frac{1}{1-t}\right)$, $0 \leq t \leq 1$, with a bound less than or equal to $\|g\|_1^{1-t}\|g\|_1^t = \|g\|_1$. Putting $p = \frac{1}{1-t}$ this gives us (4.8) with $r = p$ and $q = 1$. To obtain the general case we fix $f \in L^p$ and define $Tg = f * g$. We have just shown that T is of type $(1, p)$ with bound $\|f\|_p$. Letting $g \in L^q$, where q is conjugate to p, we also have T of type (q, ∞) with bound $\|f\|_p$. Thus, T is of type (p_t, q_t), with bound no greater than $\|f\|_p$, where $p_t = \frac{p}{p-t}$ and $q_t = \frac{p}{1-t}$, $0 \leq t \leq 1$. That is,

$$\|f * g\|_{q_t} \leq \|f\|_p\|g\|_{p_t}.$$

Since it follows immediately that $\frac{1}{q_t} = \frac{1}{p} + \frac{1}{p_t} - 1$, and, as t ranges between 0 and 1, $\frac{1}{p} + \frac{1}{p_t}$ ranges from $\frac{1}{p} + 1$ to 1, this is precisely the inequality of (4.8).

Unfortunately none of the other inequalities we stated can be derived from the special cases discussed above and the M. Riesz convexity theorem. For example, the conjugate function mapping, as we have seen, is easily seen to be of type $(2, 2)$. Were we able to show that it is of type $(1, 1)$ it would then follow that it is of type (p, p), $1 < p < 2$, and this, in turn, would imply the result for $p > 2$ (as we saw at the end of the proof of (4.5)). But we have already stated that this operator is not a bounded transformation on $L^1(0, 1)$. Nevertheless, there is a substitute result, due to Kolmogoroff, and an extension of the M. Riesz convexity theorem, due to Marcinkiewicz, that does allow us to obtain Theorem (4.5) much in the same way we obtained (4.7) and (4.8). Furthermore, this method is applicable to Theorems (4.9) and (4.10) as well.

The substitute result of Kolmogoroff is a condition that is weaker than type (1, 1). We shall consider this condition in a more general setting. First, however, we need to introduce the concept of the distribution function of a measurable function. Let g be a measurable function defined on the measure space (N, ν) and, for $y > 0$, $E_y = \{x \in N;\, |g(x)| > y\}$. Then the *distribution function of* g is the nonincreasing function $\lambda = \lambda_g$ defined for all $y > 0$ by

$$\lambda(y) = \nu(E_y).$$

It is an easy exercise in measure theory to show that if $g \in L^q(N)$ then

$$\text{(4.13)} \qquad \|g\|_q = \left(\int_N |g(x)|^q \, d\nu\right)^{1/q} = \left(q \int_0^\infty y^{q-1}\lambda(y)\, dy\right)^{1/q}.$$

Suppose, now, that T is an operator of type (p, q), with bound A, $1 \le q < \infty$, mapping functions defined on M into functions defined on N. Let $f \in L^p(M)$, $g = Tf$, and λ the distribution function of g. Then

$$y^q\lambda(y) = \int_{E_y} y^q \, d\nu \le \int_{E_y} |g(x)|^q \, d\nu \le \int_N |g(x)|^q \, d\nu$$
$$\le \left(A\left[\int_M |f(t)|^p \, d\mu\right]^{1/p}\right)^q.$$

That is,

$$\text{(4.14)} \qquad \lambda_g(y) = \lambda(y) \le \left(\frac{A}{y}\|f\|_p\right)^q.$$

This condition is easily seen to be weaker than boundedness. An operator that satisfies (4.14) for all $f \in L^p(M)$ is said to be of *weak-type* (p, q). If $q = \infty$ it is convenient to identify weak-type with type.

Kolmogoroff showed that the conjugate function mapping is of weak-type (1, 1). It is then immediate that the following theorem can be used to prove (4.5):

(4.15) THE MARCINKIEWICZ INTERPOLATION THEOREM: *Suppose T is a sublinear operator of weak types (p_0, q_0) and (p_1, q_1), where $1 \le p_i \le q_i \le \infty$ for $i = 0, 1$, and $q_0 \ne q_1$, $p_0 \ne p_1$. Then T is of type (p, q) whenever*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \textit{and} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1},$$

$0 < t < 1$.

Similarly, the *Hilbert transform* can be shown to be of weak type (1, 1); thus (4.9) is also a consequence of (4.15). The same is true of (4.10). We shall not prove any of these facts. The reader, however, should have no difficulty in checking that the maximal function mapping cannot be of type (1, 1) (take for f the characteristic function of a finite interval; then f^* is not integrable). The proof that it is of weak type (1, 1) is not hard. The corresponding results for the conjugate function and for the Hilbert transform, however, are somewhat more difficult.

The M. Riesz convexity theorem and the Marcinkiewicz interpolation theorem have many more applications. The examples discussed in this section, however, are sufficient to illustrate the role they play in harmonic analysis.

5. HARMONIC ANALYSIS ON LOCALLY COMPACT ABELIAN GROUPS

We have discussed harmonic analysis associated with three different domains, the circle group (or the group of reals modulo one), the group of integers, and the (additive) group of real numbers. All of these are examples of *locally compact abelian groups*. These are abelian (commutative) groups G, with elements $x, y, z, \cdots$, endowed with a locally compact Hausdorff topology in such a way that the maps $x \longrightarrow x^{-1}$ and $(x, y) \longrightarrow xy$ (defined on G and $G \times G$, respectively) are continuous (we are following the usual custom of writing the operation on G as multiplication and not as addition—which was the case in our three examples;

this should not be a source of confusion to the reader). In this section we shall indicate how harmonic analysis can be extended to functions defined on such groups.

On each such group G there exists a nontrivial regular measure M that, in analogy with Lebesgue measure, has the property that it is invariant with respect to translation. By this we mean that whenever A is a measurable subset of G then $m(A) = m(Ax)$ for all $x \in G$. This is equivalent to the assertion

$$\int_G f(yx)\,dm(y) = \int_G f(y)\,dm(y) \tag{5.1}$$

for all $x \in G$ whenever f is an integrable function. It is obvious that any constant multiple of m also has this property. Conversely, it can be shown that any regular measure satisfying this invariance property must be a constant multiple of m. Such measures are known as *Haar measures.*

The operation of convolution of two functions f and g in $L^1(G)$ is defined, as in the classical case, by the integral

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\,dy.$$

The four properties (i), (ii), (iii), and (iv) (see the end of the first section) hold in this case as well. In particular, $f * g \in L^1(G)$ and $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.

Moreover, we shall now show that it is possible to give a definition of the Fourier transform so that (1.18) also holds; that is, $(f * g)^\wedge = \hat{f}\hat{g}$ for all f and g in $L^1(G)$. We have seen that the Fourier transform of f is not usually defined on the domain of f. In case of the circle group, for example, Fourier transformation gave us functions defined on the integers. In order to describe the general situation we shall need the concept of a character: By a *character* of a locally compact group G we mean a continuous function, $\hat{x}$, on G such that $|\hat{x}(x)| = 1$ for all x in G and $\hat{x}(xy) = \hat{x}(x)\hat{x}(y)$ for all $x, y \in G$.

The collection of all characters of G is usually denoted by $\hat{G}$. If we define multiplication in $\hat{G}$ by letting $\hat{x}_1\hat{x}_2(x) = \hat{x}_1(x)\hat{x}_2(x)$, for all $x \in G$, whenever $\hat{x}_1, \hat{x}_2 \in \hat{G}$, $\hat{G}$ then becomes an abelian group. We introduce a topology on $\hat{G}$ by letting the sets

$$U(\epsilon, C, \hat{x}_0) = \{\hat{x} \in \hat{G};\, |\hat{x}(x) - \hat{x}_0(x)| < \epsilon, x \in C\},$$

where $\hat{x}_0 \in \hat{G}$, $\epsilon > 0$, and C is a compact subset of G, form a basis. With this topology $\hat{G}$ is then also a locally compact abelian group. $\hat{G}$ is usually called the *character group of* G or the *dual group of* G.

For example, when G is the group of real numbers we easily see that if we let a be a real number, then the mapping $\hat{x}: x \longrightarrow e^{2\pi iax}$, defined for all real x, is a character. One can show that all characters are of this type. Thus, there is a natural one-to-one correspondence between the group of real numbers and $\hat{G}$. Furthermore, this correspondence is a homeomorphism. Hence, we can identify G with $\hat{G}$ in this case.

If G is the group of reals modulo 1 the mappings $\hat{x}: x \longrightarrow e^{2\pi iax}$, $x \in G$, where a is an integer, are characters, and each character has this form. Thus, $\hat{G}$ and the integers are in a one-to-one correspondence that, in this case also, can be shown to be a homeomorphism. Therefore, we can identify $\hat{G}$ with the integers. Similarly the dual group of the integers can be identified with the group of reals modulo 1.

In general, if we fix an x in G and consider the mapping $\hat{x} \longrightarrow \hat{x}(x)$ we obtain a character on $\hat{G}$. It can be shown that every character has this form and that this correspondence between G and $(\hat{G})^\wedge$ is a homeomorphism. This result is known as the *Pontrjagin duality theorem* and it is usually stated, simply, by writing the equality $G = (\hat{G})^\wedge$. Because of this duality the functional notation $\hat{x}(x)$ is discarded and the symbol

$$\langle x, \hat{x}\rangle$$

is used instead. Thus, $\langle x, \hat{x}\rangle$ may be thought of as the value of the function x at $\hat{x}$, $x(\hat{x})$, as well as the value of $\hat{x}$ at x; these, two values are clearly equal.

It is now clear, if we let ourselves be motivated by our three classical examples of locally compact abelian groups, that a natural definition of the Fourier transform for $f \in L^1(G)$, when G is a general locally compact abelian group, is to let it be the function $\hat{f}$ on $\hat{G}$ given by

$$\hat{f}(\hat{x}) = \int_G f(x)\,\overline{\langle x, \hat{x}\rangle}\,dm(x).$$

Many of the results we presented in the previous section hold in this case as well. For example, $\hat{f}$ is a continuous function on $\hat{G}$; when $\hat{G}$ is not compact the Riemann-Lebesgue theorem holds:

(5.2) *If $\hat{G}$ is not compact, $f \in L^1(G)$, and $\epsilon > 0$, then there exists a compact set $C \subset \hat{G}$ such that $|\hat{f}(\hat{x})| < \epsilon$ if $\hat{x}$ is outside of C.*

The basic relation (1.18) between convolution and Fourier transformation is true in general:

(5.3) *If f and g belong to $L^1(G)$ then $(f * g)^\wedge = \hat{f}\hat{g}$.*

Wiener's theorem (3.15) is still valid:

(5.4) *If $f \in L^1(G)$ and $\hat{f}(\hat{x})$ is never 0 then any $g \in L^1(G)$ can be approximated arbitrarily closely in the L^1-norm by functions of the form*

$$\Sigma\, a_k f(xt_k),$$

where the a_k's are a finite collection of complex numbers and the t_k's belong to G.

The Plancherel theorem also has an analog to this general case:

(5.5) *If we restrict the transformation $f \longrightarrow \hat{f}$ to $L^1(G) \cap L^2(G)$ then the L^2 norms are preserved; that is, $\hat{f} \in L^2(\hat{G})$ and Parseval's formula holds.*

$$||f||_2 = ||\hat{f}||_2.$$

Furthermore, this transformation can be extended to a norm preserving transformation of $L^2(G)$ onto $L^2(G)$.

Harmonic analysis can be generalized still further. For example, locally compact groups that are not abelian are associated with important versions of harmonic analysis (the theory of spherical harmonics is associated with the group of rotations in 3-space). We will not, however, pursue this topic further.

6. A SHORT GUIDE TO THE LITERATURE

So many books and papers have been written in harmonic analysis that no attempt will be made here to give anything like a comprehensive bibliography. Rather, our intention is to give

some *very* brief suggestions to the reader who would like to pursue the subject further.

All that has been discussed here concerning Fourier series is contained in A. Zygmund's two-volume *Trigonometric Series* [10]. This scholarly book contains essentially all the important work that has been done on the subject. Anyone seriously interested in classical (or, for that matter, modern) harmonic analysis would do well to become acquainted with it. It is often worthwhile, however, to read a short treatment of a subject when learning it. R. R. Goldberg's *Fourier transforms* [3] does an excellent job of presenting that part of Fourier integral theory that generalizes to locally compact abelian groups. In this book the reader will find a proof of Wiener's theorem and a more thorough discussion of the problem of spectral synthesis. For more comprehensive treatments of Fourier integral theory we refer the reader to S. Bochner's *Lectures on Fourier Integrals* [2] and E. C. Titchmarsh's *The Theory of the Fourier Integral* [8].

The literature dealing with the more abstract forms of harmonic analysis is also very large. Pontrjagin's classic *Topological Groups* [6] is still highly recommended reading. The same is true of A. Weil's *L'intégration dans les groupes topologiques et ses applications* [9]. Two very readable modern works that treat the subject of harmonic analysis on groups are Rudin's *Fourier Analysis on Groups* [7] and *Abstract Harmonic Analysis* by Hewitt and Ross [5]. We also recommend an excellent survey on this subject by J. Braconnier [1].

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TOEPLITZ MATRICES

Harold Widom

1. INTRODUCTION

Otto Toeplitz is one of the few mathematicians who has had his name attached to two distinct mathematical objects. What is especially unusual in the case of Toeplitz is that these objects have exactly the same name: *Toeplitz matrix.*

The more famous Toeplitz matrices are associated with procedures for attaching "sums" to divergent series. We shall not mention them again. As far as we are concerned a Toeplitz matrix is an array of complex numbers

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c_0 & c_{-1} & c_{-2} & c_{-3} & \cdot \\ \cdot & c_1 & c_0 & c_{-1} & c_{-2} & \cdot \\ \cdot & c_2 & c_1 & c_0 & c_{-1} & \cdot \\ \cdot & c_3 & c_2 & c_1 & c_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

What distinguishes such a matrix is that each diagonal has equal