The Gravity of Hades

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Imagine being beamed into a chamber deep within the Earth, such as depicted in the Jules Verne cavern of FIGURE 1. Is the acceleration due to gravity stronger or weaker in the chamber than at the surface of the Earth? If the reader is hesitant in answering immediately, here's how Leonard Euler answered the question in 1760 [**3**, volume I, Letter L, p. 182]:

> We are certain ... that gravity ... acts with the greatest force at the surface of the earth, and is diminished in proportion as it removes from thence, whether by penetrating towards the centre or rising above the surface of the globe.

Figure 1   Frontispiece for Verne's 1871 publication of *Journey to the Center of the Earth*, 100 miles below Iceland

To answer this question, we consider various models of the Earth's structure that have been proposed over the years. We show that

- for homogeneously dense planets—Euler's implicit model—gravity weakens with descent from the surface;
- for a planet possessing a homogeneous mantle that is less than 2/3 as dense as its core, a local minimum for gravity intensity exists within a sufficiently thick mantle;
- for our own Earth, having a solid inner core, a less dense liquid outer core, and an extensive even less dense mantle, gravity intensifies with descent from -9.8 m/s^2 at the surface to an extreme of -10.8 m/s^2 where the mantle meets the core.

We also give a condition involving only surface density and mean density that determines whether gravity increases or decreases with depth from the surface of a planet; and we conclude with an analysis of a body falling through a classically envisioned hole through the Earth.

Some preliminary classical mechanics

To find the gravitational acceleration induced by the Earth on a particular point P, we follow Newton and first find the gravitational acceleration induced by each point of the Earth on P, and then take the aggregate of these accelerations as the total acceleration induced by the Earth. With this idea in mind, let the mass m of a body be condensed to the single point $Q = (x, y, z)$. We wish to determine the gravitational acceleration $a(s)$ induced by the mass at Q in the direction $\mathbf{k} = \langle 0, 0, 1 \rangle$ on point $P = (0, 0, s)$. See FIGURE 2.
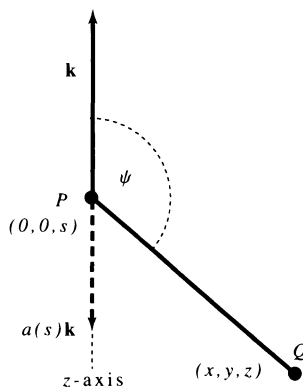

Figure 2   The pull $a(s)$ by Q on P

We start with Newton's law of gravitation, telling us that the force of gravity between two single-point masses a and b separated by r units is Gab/r^2, where $G \approx 6.67 \times 10^{-11}$ Nm^2/kg^2, the universal gravitation constant. Since the force on a mass is equal to mass times acceleration, the gravitational acceleration on the point P in the direction $\overrightarrow{PQ}$ is $Gm/(x^2 + y^2 + (z - s)^2)$. But we want only the portion of this attraction in the direction $\mathbf{k}$. Note that if ψ is the angle between $\overrightarrow{PQ}$ and $\mathbf{k}$, then

$$\cos \psi = \frac{\langle x, y, z - s \rangle \cdot \langle 0, 0, 1 \rangle}{\sqrt{x^2 + y^2 + (z - s)^2}} = \frac{z - s}{\sqrt{x^2 + y^2 + (z - s)^2}}.$$

Therefore $a(s) = \cos \psi \, Gm/(x^2 + y^2 + (z - s)^2)$, which we write as

$$a(s) = \frac{Gm(z - s)}{(x^2 + y^2 + (z - s)^2)^{\frac{3}{2}}}. \tag{1}$$

A point mass at (x,y,z)

Ideal planet models

To apply (1) to calculate the gravitational acceleration a for a heavenly body at distance s from its center of mass, we focus on *ideal planets*, those with radius R for which each of its concentric spherical shells is of constant density. Though our planet is best described as an oblate spheroid which, when compared to the best approximating

sphere, is flattened at the poles by about 14 km and bulges at the equator by about 7 km, and though its shells have only approximately constant density, the Earth is viewed as an ideal planet for the purposes of this paper. For each ideal planet we let $a(s)$ be the outward radial gravitational acceleration induced by the planet at s units from its center. For each ideal planet, we presently show that $a(s) \leq 0$ for all $s \geq 0$. Therefore, the greatest magnitude for acceleration occurs when $a(s)$ is a minimum.

The simplest ideal planet is a single nonzero dense onion layer, or a soap bubble.

A soap bubble model   Let S be a spherical shell of mass M with constant density, center $O = (0, 0, 0)$ and radius r. Then the gravitational acceleration on the point P due to the shell S is

$$a(s) = \begin{cases} 0, & \text{if } 0 \leq s < r, \\[2mm] -\dfrac{GM}{2s^2}, & \text{if } s = r, \\[2mm] -\dfrac{GM}{s^2}, & \text{if } s > r. \end{cases} \tag{2}$$

A spherical shell

The derivation of (2) is probably in every classical mechanics text, such as Fowles and Cassiday [**5**, pp. 207–209]. For completeness and because the derivation herein is a bit simpler than the ones in the mechanics texts surveyed by this author, we derive (2) by integrating (1) over the shell S. Let $\mathbf{r} = \langle r \cos\theta \sin\phi, \ r \sin\theta \sin\phi, \ r \cos\phi \rangle$ be a parameterization of S, where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. Then the area element for S is $\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} \| \, d\theta \, d\phi$, which simplifies to $r^2 \sin\phi \, d\theta \, d\phi$. Since surface area of a sphere of radius r is $4\pi r^2$, the point-mass m of (1) can be taken as $\frac{M}{4\pi r^2} r^2 \sin\phi \, d\theta \, d\phi = \frac{M}{4\pi} \sin\phi \, d\theta \, d\phi$. Therefore from (1), acceleration $a(s)$ by S on point P is

$$a(s) = \int_0^\pi \int_0^{2\pi} \frac{G\frac{M}{4\pi}(r\cos\phi - s)\sin\phi}{(r^2 + s^2 - 2rs\cos\phi)^{\frac{3}{2}}} \, d\theta \, d\phi = \int_0^\pi \frac{G\frac{M}{2}(r\cos\phi - s)\sin\phi}{(r^2 + s^2 - 2rs\cos\phi)^{\frac{3}{2}}} \, d\phi. \tag{3}$$

The change of variable $u = \cos\phi$ gives

$$a(s) = \frac{GM}{2} \int_{-1}^1 \frac{ru - s}{(r^2 + s^2 - 2rsu)^{\frac{3}{2}}} \, du, \tag{4}$$

which, via integration by parts and simplifying, gives

$$a(s) = \frac{GM}{2s^2} \left(\frac{r - s}{\sqrt{(r - s)^2}} - \frac{r + s}{\sqrt{(r + s)^2}} \right). \tag{5}$$

If $0 < s < r$ then (5) gives 0; if $s > r$ then (5) gives $-GM/s^2$. To obtain the anomalous result for $s = r$, first simplify (4) before integrating.

Since a general ideal planet is simply a series of concentric soap bubbles, when s exceeds R, (2) is the familiar principle that when dealing with forces exerted by heavenly bodies, one can treat their masses as being concentrated at their centers. Furthermore, when $0 \leq s < R$, (2) is a well-known result of electrostatics, which has features in common with gravitation; that is, for a hollow, metal sphere carrying a surface charge, the electrical field within the sphere is 0. Similarly, for any void within

a charged metal object, the field is zero as well [4, chapter 5], which is *not* the case for the gravitational field within an arbitrary void of a planet.

As a simple example showing that neither of these two properties necessarily holds if a heavenly body is not an ideal planet, let S be a hollow, right circular cylinder with mass M, height 2, radius 1, center at the origin, and central axis aligned with the z-axis. Since the surface area of S is 6π, assume that S has uniform density of $M/6\pi$. To calculate the gravitational acceleration induced by S at the point $(0, 0, s)$ where $s \geq 0$, use (1) and integrate over the top, the bottom, and the sides of S, giving $a(s)$ as

$$\frac{GM}{6\pi} \left(\int_0^1 \int_0^{2\pi} \frac{(1-s)r}{(r^2 + (1-s)^2)^{\frac{3}{2}}} \, d\theta \, dr - \int_0^1 \int_0^{2\pi} \frac{(1+s)r}{(r^2 + (1+s)^2)^{\frac{3}{2}}} \, d\theta \, dr \right.$$

$$\left. + \int_{-1}^1 \int_0^{2\pi} \frac{(z-s)}{(1 + (z-s)^2)^{\frac{3}{2}}} \, d\theta \, dz \right),$$

which simplifies to

$$a(s) = \begin{cases} \dfrac{GM}{3} \left(\dfrac{s+2}{\sqrt{1 + (1+s)^2}} + \dfrac{s-2}{\sqrt{1 + (1-s)^2}} \right), & \text{if } 0 \leq s < 1, \\[3ex] \dfrac{GM(3 - 2\sqrt{5})}{15}, & \text{if } s = 1, \\[3ex] \dfrac{GM}{3} \left(\dfrac{s+2}{\sqrt{1 + (1+s)^2}} + \dfrac{s-2}{\sqrt{1 + (1-s)^2}} - 2 \right), & \text{if } s > 1. \end{cases}$$

Observe that for $0 < s < 1$, $a(s) > 0$, which means that if a particle is on the z-axis between 0 and 1, then it is attracted towards the top of the cylinder. To see that this formula conforms asymptotically to an inverse square law, observe that $\lim_{s\to\infty} s^2 a(s)$ is a constant.

In general, when calculating the gravitational acceleration induced by a heavenly body at a certain point, any nonspherical symmetry of the body promulgates nightmarish integrals. But fortunately for astrophysics, planets and stars are more or less ideal planets; furthermore, many of the central problems of celestial mechanics involve forces of gravitation at many radii from the heavenly bodies in question, so that the inverse square law can be used with confidence and relief.

To find the cumulative gravitational acceleration induced by the point-masses of a nontrivial ideal planet, it is convenient to sum them layer by collective spherical layer. With this idea in mind, we integrate (2) appropriately to find $a(s)$.

The general density model   Let S be a ball of radius R, mass M, and center O whose density at s units from O is given by $\delta(s)$. That is, S is spherically symmetric with respect to its density. Then

$$a(s) = \begin{cases} -\dfrac{4\pi G}{s^2} \displaystyle\int_0^s \rho^2 \delta(\rho) \, d\rho, & \text{if } 0 \leq s \leq R, \\[3ex] -\dfrac{GM}{s^2}, & \text{if } s \geq R. \end{cases} \tag{6}$$

An ideal planet with general density

To derive (6) from (2) note that the mass of the shell at radius ρ is $4\pi\rho^2 \delta(\rho)$. Therefore,

$$a(s) = \begin{cases} -\displaystyle\int_0^s \frac{4G\pi\rho^2\delta(\rho)}{s^2}\,d\rho, & \text{if } 0 \le s \le R, \\[2ex] -\displaystyle\int_0^R \frac{4G\pi\rho^2\delta(\rho)}{s^2}\,d\rho & \text{if } s \ge R, \end{cases}$$

which simplifies to (6), since $\int_0^R 4\pi\rho^2\delta(\rho)\,d\rho = M$. Furthermore, note that the two formulas of (6) agree at $s = R$.

Constant acceleration: an old implicit model

When reading of mythical heroes venturing off into the underworld, such as Orpheus seeking to reclaim his lost love or Hercules throttling Cerberus, the watchdog of Hades with three heads, one forms the impression that no matter how deep the heroes descend into the Earth, gravity remains constant. To catch a bit of the flavor of these tales, here's a snippet of Ovid's *Metamorphoses* [11, Book X], describing Orpheus and his wife's ascent from Hades:

> *Now thro' the noiseless throng their way they bend,*
> *And both with pain the rugged road ascend;*
> *Dark was the path, and difficult, and steep,*
> *And thick with vapours from the smoaky deep.*

The drawings in FIGURE 3, based on two old Grecian urns, show these two heroes in Hades; Orpheus is serenading the lords of the underworld and Hercules is about to chain Cerberus, depicted with only two heads. Turner [15] enumerates other literary characters who frequented the underworld.
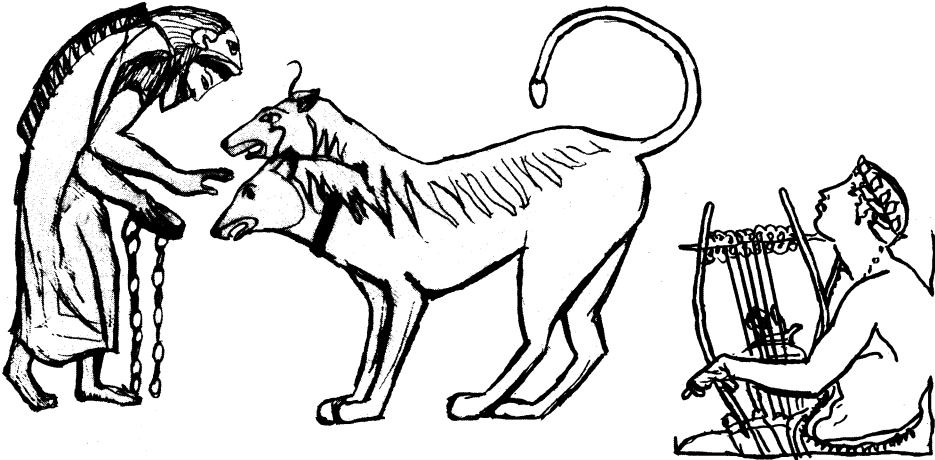

Figure 3   Hercules and Orpheus in Hades (renditions by Jason Challas)

Such an impression is especially vivid when reading Dante (in *The Divine Comedy* written about 1300 C.E.), who journeys with his ghost guide Virgil from the Earth's surface, down through 24 levels of Hell, down through the very center of the Earth. FIGURE 4 [2, p. 84] shows one artist's rendition of Dante's model of the Earth. When Dante scrambles over a broken bridge deep within Hell it is with the same effort he would have used along a road to Rome. Arrows and other projectiles lofted in Hell
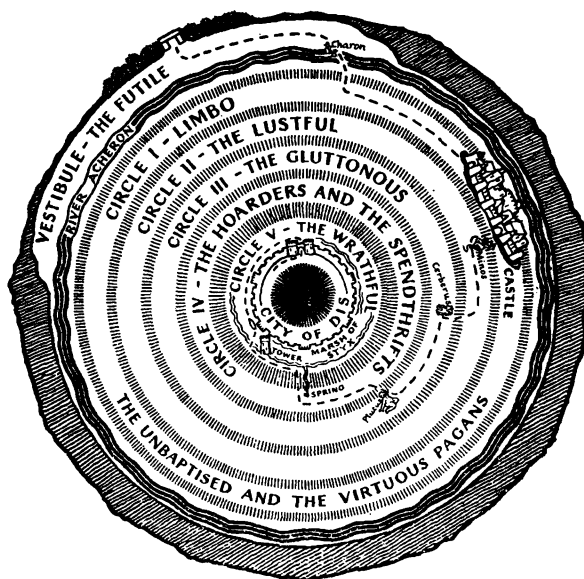
Figure 4   Levels in Dante's Earth

appear to follow the same trajectories as at the surface of the Earth. When he clambers about the hip of a somewhat immobilized, gargantuan, and hair-covered Satan at the literal center of the Earth on his way to the antipodes, Dante changes directions, maneuvering with a degree of exertion akin to that of reversing one's bodily position while clinging to a root-covered, vertical rockface in the Apennines.

> *And when we had come to where the huge thigh-bone*
> *Rides in its socket at the haunch's swell,*
> *My guide, with labour and great exertion,*
>
> *Turned head to where his feet had been, and fell*
> *To hoisting himself up upon the hair,*
> *So that I thought us mounting back to Hell.*
>   Canto xxxiv, lines 76–81 [**2**, p. 287]

Up and down reverse dramatically. Dante's Satan, more precisely, Satan's crotch, is thus a veritable singularity. After climbing on a bit more, Dante takes a rest, looks back, and sees Lucifer's legs sticking up.

> *And if I stood dumbfounded and aghast,*
> *Let those thick-witted gentry judge and say,*
> *Who do not see what point it was I'd passed.*
>   Canto xxxiv, lines 91–93 [**2**, p. 287]

Dante may have borrowed a similar image from the prolific, first-century writer, Plutarch, who mentions this same singularity, although less graphically so, conjecturing

> *... if a man should so coalesce with the earth that its centre is at his navel, the*
> *same person at the same time has his head up and his feet up too.* [**12**, p. 67]

Such literary evidence suggests that the *implicit* density model of the Earth used in these imaginings is that density function $\delta(s)$ which generates a constant gravitational acceleration. For $0 < s < R$, such a density function satisfies

$$-\frac{4\pi G}{s^2} \int_0^s \rho^2 \delta(\rho)\, d\rho = C,$$

where C is some constant. Multiplying the above equation by s^2 and then differentiating, and simplifying yields $\delta(s) = k/s$, where k is some constant. That is, the ancient model of the Earth appears to be one wherein density and distance from the Earth's center vary inversely! Note that even in such a model, where the density at the center is unbounded, the mass of the Earth is yet finite.

A common-notion constant density model

Insisting that gravity remain constant on descent into the Earth requires a singularity in its density. Much simpler and more natural is to take this density as constant. In this author's informal survey of colleagues both in mathematics and physics, their almost universal and emphatic answer, "*Gravity decreases with descent!*," subsumes this model. Perhaps one of the reasons for such ardent responses from physicists and mathematicians is that almost all of us in our undergraduate differential equation days solved an exercise very much like this one, (which we solve in general at the end of this paper):

> *Inside the earth, the force of gravity is proportional to the distance from the center. If a hole is drilled through the earth from pole to pole, and a rock is dropped in the hole, with what velocity will it reach the center?* [**13**, p. 24]

A tempting reason to guess that gravity wanes with depth is the following, incorrect, intuitive argument: "*As we go down, the portion of the Earth above will exert an upward force on us, hence lessening the downward force on us exerted by the remainder of the Earth.*" But as (2) makes abundantly clear, at s units from the center, only the mass within s units of the center determines $a(s)$.

Taking a ball S of uniform density δ, mass M, and radius R, gives by way of (6),

$$a(s) = \begin{cases} -\dfrac{4\pi G \delta s}{3}, & \text{if } 0 \le s \le R, \\[2mm] -\dfrac{GM}{s^2} & \text{if } s \ge R. \end{cases} \tag{7}$$

A planet of constant density

That is, gravity and distance from the center are directly proportional up to the surface of the Earth. See FIGURE 5a.

The reader may enjoy contrasting (7) with the acceleration functions of nonideal planets of constant density. For example, let S be a right circular cylinder of radius 1, height 2, mass M, center at the origin, and central axis aligned with the z-axis. Integrating (1) over S using cylindrical coordinates gives the gravitational acceleration $a(s)$ induced by S at the point $(0, 0, s)$ for $s \ge 0$ as

$$a(s) = MG\left(\sqrt{1 + (1+s)^2} - \sqrt{1 + (1-s)^2} + |1 - s| - |1 + s|\right),$$

a. Constant density

b. Hollow Earth

c. Two-tier Earth

d. Homogeneous Core, Linear Mantle

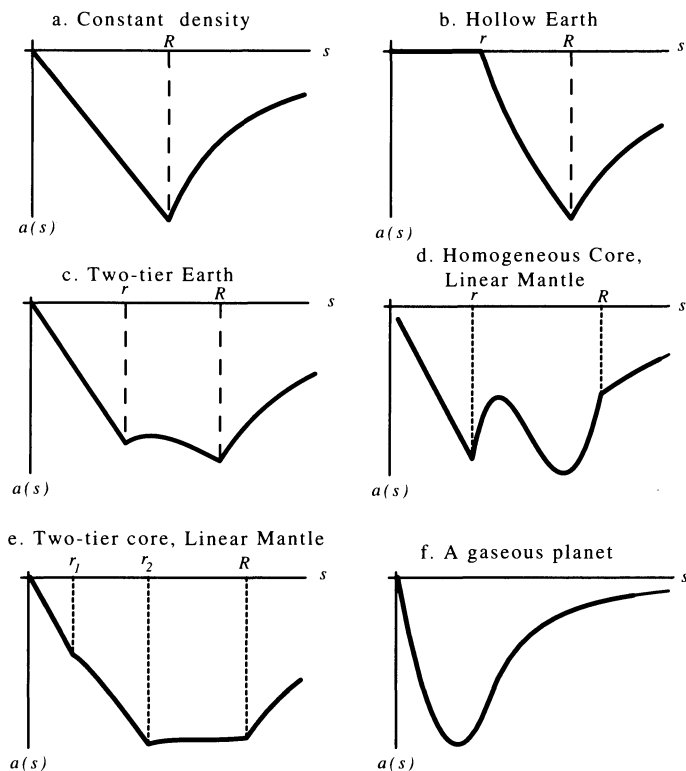e. Two-tier core, Linear Mantle

f. A gaseous planet

Figure 5   Simple earth models (R = radius of Earth)

whose graph qualitatively looks much like FIGURE 5a. This particular application injects life into the surprising, but otherwise artificial limit $\lim_{x\to\infty} \sqrt{1 + (1+x)^2} - \sqrt{1 + (1-x)^2} = 2$.

Halley's hollow Earth model

Edmond Halley, the "Father of Geophysics" and the financier behind Newton's publication of the *Principia*, viewed Newton's (erroneous) calculation in 1687 of the relative densities of the Earth and the Moon being 5 to 9, as one of the more significant of the discoveries presented in the *Principia*. Believing Newton to be correct and yet wanting to conclude that solid chunks of the Earth and Moon should be equally dense, Halley proposed that the Earth must be 4/9 hollow. Since his study of magnetic compass variations suggested that the Earth had four separate north poles, he went on to claim that the Earth's outer shell was 500 miles thick; inside were three more concentric shells, being the radii of Venus, Mars, and Mercury. FIGURE 6 shows Halley in a 1736 portrait, holding his hollow Earth sketch. Furthermore, just as the aurora borealis phenomenon illuminates the northern skies, he postulated that the atmospheres over each of these shells were alive with magnetic lightning-like flashes; and he went on to speculate that life existed on these inner surfaces [**7, 8**].

Such musings prompted inane and romantic science fiction tales in years to come. One spurious legacy of Newton's error is a Hollow Earth Society, which apparently loves to speak of UFOs and superior creatures haunting the inner circles of our planet. On their web page [**14**], you may view a purported map of this inner world, shown in an

Figure 6   Halley at 80 with sketch of hollow Earth model

artist's rendition on the cover of this issue. A well-known science fiction tale inspired by Halley's model is by Tarzan's creator, Edgar Rice Burroughs; in *At the Earth's Core*, written in 1908, two heroes within an out-of-control tunneling machine burrow from the surface of the earth, and emerge three days later and 500 miles down in the inner world of *Pellucidar*. The MAGAZINE cover also portrays one charming depiction of this *earth ship*, looking not too distantly different from the great earth-eating machines that tunneled out the Chunnel under the English Channel. At the halfway point, 250 miles deep, up and down switch for Burroughs in the same manner as when Dante climbed down Satan's torso at the center of the Earth. But is such a scenario possible, walking upright on the inside of the Earth's outer layer?

That is, let S be a shell of inner radius r and outer radius R, uniform density δ, mass M, then (6) gives,

$$a(s) = \begin{cases} 0, & \text{if } 0 \le s \le r \\ -\dfrac{4}{3}\pi G\delta \left(s - \dfrac{r^3}{s^2} \right), & \text{if } r \le s \le R, \\ -\dfrac{GM}{s^2} & \text{if } s \ge R. \end{cases} \qquad (8)$$

A planet with a hollow core

FIGURE 5b illustrates this acceleration. Note that the entire empty core is a zero-gravity haven!

But on Pellucidar, worse than having zero gravity on the surface, or a marginal attraction due to the spin of the Earth, the situation is grim, for its sky is illuminated by both a moon and a central sun, so any object not clinging to the surface of Pellucidar would fall into this sun. Lest we be overly critical of Burroughs's model, he does have his character sense "a certain airy lightness of step" on Pellucidar [1, *Freedom* chapter], and explains, "The force of gravity is less upon ... the inner world ... due

... to the counter-attraction of that portion of the earth ... directly opposite the spot ... at which one's calculations are ... made," which is probably a good rendition of how the typical man-in-the-street might reason.

Hooke's onion model

Robert Hooke is credited with positing an Earth of multiple layers, one reason for which was to explain why the magnetic north pole appears to wander. His idea was that the magnetic source might be embedded in a separate layer, which rotated at a rate slightly different than the surface layer. As a simple representative example of this model, suppose that the Earth consists of two homogeneous layers, the inner one called the *core* and the outer one called the *mantle*, a term coined by E. Wiechert in 1897. Let S be a ball consisting of an inner core of density δ_1, mass M_1, radius r, and of an outer mantle, density δ_2, mass M_2, inner radius r and outer radius R, then two applications of (6) yield

$$
a(s) = \begin{cases}
-\dfrac{4\pi G \delta_1 s}{3}, & \text{if } 0 \le s \le r, \\[2mm]
-\dfrac{GM_1}{s^2} - \dfrac{4}{3}\pi G \delta_2 \left(s - \dfrac{r^3}{s^2}\right), & \text{if } r \le s \le R \\[2mm]
-\dfrac{G(M_1 + M_2)}{s^2}, & \text{if } s \ge R.
\end{cases} \tag{9}
$$

An onion model with two layers

In this case, the mantle contains a local extremum for gravitational acceleration at $s_c = r\sqrt[3]{2(\delta_1 - \delta_2)/\delta_2}$, provided $r < s_c < R$, which means that a necessary condition for the existence of a local extremum is for $\delta_2/\delta_1 < 2/3$. That is, a planet's mantle must be no more than $2/3$ as dense as its core in order for an extreme to exist. See FIGURE 5c where $\delta_1 = 2\delta_2$ and $R = 2r$.

A molten core, variable density mantle model

To account for volcanic activity, geophysicists of the nineteenth century postulated that the Earth's central region was molten, a thick pea soup of convective rock currents, which in turn suggested that the core might be homogeneous. Being solid, the mantle was presumed to have a density that increased with depth. As a simple representative member of this model, let S be an ideal planet with a homogeneous core of density δ, mass M_1, radius r, and a variable mantle of density $\delta(s) = \mu + \lambda s$, where $r < s < R$, mass M_2, and where μ and λ are constants. Then

$$
a(s) = \begin{cases}
-\dfrac{4}{3}\pi G \delta s, & \text{if } 0 \le s < r, \\[2mm]
-\dfrac{GM_1}{s^2} - 4\pi G \left(\dfrac{\mu s}{3} + \dfrac{\lambda s^2}{4} - \mu \dfrac{r^3}{3s^2} - \lambda \dfrac{r^4}{4s^2}\right), & \text{if } r \le s \le R, \\[2mm]
-\dfrac{G(M_1 + M_2)}{s^2}, & \text{if } s \ge R.
\end{cases} \tag{10}
$$

Homogeneous core with mantle of variable density

FIGURE 5d shows the case when $\delta = 1$, $\mu = 2$, $\lambda = -1$, $r = .82$, and $R = 1.2$. Note that for these values, there are two local extrema within the mantle, the reason being that solving the equation $a'(s) = 0$ for $r < s < R$ involves finding the roots of a quartic polynomial. Nevertheless, it is a bit surprising that $a(s)$ can display such exotic behavior when $\delta(s)$ is so tamely monotonic.

A seismic model

From earthquake analysis and from the understanding that seismic waves are transmitted differently in liquids versus solids, today's thinking suggests that our Earth consists essentially of three layers. We'll simplify matters, and assume that it consists of a solid central core of uniform density $\delta_1 = 13$ units (where a unit is 1000 kg/m^3), radius 1275 km; a liquid outer core of uniform density $\delta_2 = 10$, thickness 2225 km; and an affinely dense mantle of thickness 2900 km, with density varying from 6 to 3.3 at the surface. We ignore the thin crust and tenuous atmosphere. Let $r_1 = 1.275$ units (where a unit is 1000 km), $r_2 = 3.5$, and $R = 6.4$; let M_1, M_2, and M_3 be the masses of the inner core, the outer core, and the mantle respectively. Then within the mantle, $\delta(s) = \mu + \lambda s \approx 9.26 - 0.931s$, where s is in thousands of km, and

$$
a(s) = \begin{cases}
-\dfrac{4}{3}\pi G \delta_1 s, & \text{if } 0 \le s < r_1, \\[2ex]
-\dfrac{GM_1}{s^2} - \dfrac{4}{3}\pi G \delta_2 \left(s - \dfrac{r_1^3}{s^2} \right), & \text{if } r_1 \le s \le r_2, \\[3ex]
-\dfrac{G(M_1 + M_2)}{s^2} - \dfrac{4}{3}\pi G \left(\mu s + 0.75\lambda s^2 - \mu \dfrac{r_2^3}{s^2} - 0.75\lambda \dfrac{r_2^4}{s^2} \right), & \text{if } r_2 \le s \le R, \\[3ex]
-\dfrac{G(M_1 + M_2 + M_3)}{s^2}, & \text{if } s \ge R.
\end{cases}
$$

$$\textit{Solid center, molten outer core, variable mantle} \tag{11}$$

FIGURE 5e gives a graph of $a(s)$. Note that the Earth has no local extremum within the outer core because $\delta_2/\delta_1 = 10/13 > 2/3$, a violation of the necessary condition given in Hooke's model. The overall extreme value of gravitational acceleration for this model of the Earth is approximately -10.2 m/s^2 at the juncture between the outer core and the mantle. In retrospect, the graph of FIGURE 5e also supports the intuition of those who spun the Greek legends, for Hades could not have been more than 1000 km below Greece; and the acceleration of gravity within this level of the Earth is approximately constant!

Studies from 1989 on the interplay between data and models give a density model as in FIGURE 7, which in turn gives an acceleration curve qualitatively much like FIGURE 5e; the current best guess of gravity's extreme within the Earth is -10.8 m/s^2 at the core-mantle boundary [9, p. 155].

A gaseous model

To complete a list of simple planetary models, imagine a gas ball of uniform temperature and uniform composition; the density of such a ball is exponential [17, p. 68], that is, $\delta(s) = \delta_o e^{-ks}$, where δ_o is the density at the center and k is a positive constant.
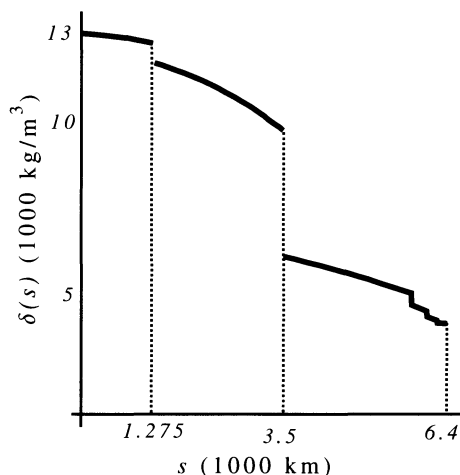
Figure 7 Earth's density

To be faithful to the definition of an ideal planet, we say that $R = \infty$. Then by (6) for positive s,

$$a(s) = -4\pi G\delta_o \frac{e^{-ks}(k^2s^2 + 2ks + 2) - 2}{k^3s^2}. \tag{12}$$

Gaseous model

See FIGURE 5f. The extreme for gravity occurs at $s_c \approx 1.45123/k$, at which point the mass of the ball of radius s_c is $4.50\delta_o/k^3$. (Over 98% of the mass is contained within a ball of radius $3s_c$.) More realistic density models for gas balls are complex and lead to elaborately grand models for the stars.

Concluding remarks

As we have seen, given the global density function for an ideal planet, integration tells us how gravity changes with descent. However, even without knowing the global density function, we can determine whether gravity increases or decreases with descent from a planet's surface; all that is needed is the mean density of the planet and its surface density. Surface density of the Earth is easy enough to compute—just measure some representative surface rock. But in Euler's day determining the mean density of the Earth was not yet possible; the product GM was known, but not its component parts; it was not until 1798, fifteen years after Euler's death, that Henry Cavendish experimentally determined G, which then gave the scientific community M and the mean density of the Earth. The following theorem, foreshadowed by the watershed 2/3 ratio for extrema in the simple Hooke model, shows how local density data can answer the question of how gravity changes with descent. Let planet P have radius R and density $\delta(s), 0 \le s \le R$. Let $m(s)$ be the mass of P up to radial distance s; that is, $m(s) = 4\pi \int_0^s \rho^2\delta(\rho)\,d\rho$. Define the *mean density* $\hat{\delta}(s)$ of P at s as $m(s)$ divided by the volume of a sphere of radius s; that is,

$$\hat{\delta}(s) = \frac{3}{s^3} \int_0^s \rho^2\delta(\rho)\,d\rho.$$

Define the *normalized density* $\Delta(s)$ at s as $\Delta(s) = \delta(s)/\hat{\delta}(s)$. For example, if $\delta(s)$ is constant, then P's normalized density is $\Delta(s) = 1$. If as in the constant acceleration model $\delta(s) = k/s$, where k is some constant, then the normalized density of P is $\Delta(s) = 2/3$.

THEOREM: THE CRITICAL CONDITION. *If $a(s)$ is differentiable, then gravity intensifies with descent at s whenever $\Delta(s) < 2/3$ and wanes with descent at s whenever $\Delta(s) > 2/3$. If $\Delta(s) = 2/3$, gravity remains constant with descent at s.*

To verify this condition, note that (6) for $0 \le s \le R$ can be written as

$$-4\pi G \int_0^s \rho^2 \delta(\rho)\, d\rho = s^2 a(s).$$

By the fundamental theorem of calculus,

$$-4\pi G s^2 \delta(s) = 2sa(s) + s^2 a'(s),$$

which means that

$$a'(s) = -4\pi G \delta(s) - \frac{2a(s)}{s}. \tag{13}$$

Note that $a(s)$ intensifies with descent precisely when $a'(s) > 0$; that is, recall that $a(s)$ is negative, so that the phrase "intensifying with descent" translates into the phrase "a is increasing at s." This condition is equivalent, via (6) and (13), to

$$\frac{-4\pi G}{s^2} \int_0^s \rho^2 \delta(\rho)\, d\rho = a(s) < -2\pi G s \delta(s),$$

which is equivalent to $\Delta(s) < 2/3$. Similarly, the alternate conditions follow.

For our Earth, since its mean density $\hat{\delta}(R)$ is about 5515 kg/m^3 and since crustal rock density $\delta(R)$ is about 2800 kg/m^3, then $\Delta(R) \approx 0.51 < 2/3$. So by the theorem, gravity intensifies with descent from the Earth's surface. If the several thousand Moon rock specimens brought back to Earth are accurate indicators, the surface density of the Moon is comparable to that of Earth; since the Moon's mean density is about 3330 kg/m^3, then $\Delta(R) \approx 0.84 > 2/3$, where R is the Moon's radius, which means that gravity weakens with descent from the Moon's surface.

As promised, we now solve the generalized problem of falling through a hole in the Earth:

If a hole is drilled through the Earth from pole to pole, and a rock is dropped in the hole, with what velocity will it reach the center, and how long will it take?

Plutarch posed a version of this problem when he has a debater catalog various reasons as to why some folks chose not to believe in a spherical Earth [12, p. 65]:

Not that ... masses ... falling through the ... earth stop when they arrive at the centre, though nothing encounter or support them; and, if in their downward motion the impetus should carry them past the centre, they swing back again and return of themselves?

Euler played with this problem as well, explaining in his popular letters to a German princess [3, p. 178] that

You will remember how Voltaire used to laugh at the idea of a hole reaching to the centre of the earth, ... but there is no harm in supposing it, in order to discover what would be the end result.

To solve this problem, we neglect air resistance and any relativistic concerns. Let s, v, a, and t be the distance of the rock from the Earth's center, the velocity of the rock, the acceleration of the rock, and time, respectively, where at $t = 0$, $s = R$, $v = 0$, and $a = -g$, where R is the radius of the Earth and $g \approx 9.8$ m/s^2. Note that acceleration times velocity can be written two ways as

$$v\frac{dv}{dt} = a\frac{ds}{dt}.$$

Integrating this expression as time t goes from 0 to τ gives

$$\int_0^\tau v\frac{dv}{dt}\,dt = \int_0^\tau a\frac{ds}{dt}\,dt.$$

Since both s and v are monotonic functions in terms of t as the rock falls from the Earth's surface to its center and since the initial velocity of the rock is 0, then the above equation simplifies to

$$\frac{v^2(s)}{2} = \int_R^s a\,dr,$$

where v is a function in terms of s, and r is a dummy variable representing distance from the Earth's center. Since the rock's velocity during fall is negative,

$$v(s) = -\sqrt{2\int_R^s a\,dr}. \tag{14}$$

Thus the speed of the rock as it passes through the Earth's center is $v(0 \text{ km})$. To find a formula for the time T it takes for the rock to reach the center, obtain a uniform partition of the interval $[0, R]$: $0 = s_0 < s_1 < \cdots < s_n = R$, so that $\Delta s = R/n$, for each positive integer n. Approximate the time for the rock to fall from s_{i+1} to s_i by $-\Delta s/v_i$, where $v_i = v(s_i)$ for each i, $0 \le i \le n - 1$. The approximate value of time T is therefore $\sum_{i=0}^{n-1} -\Delta s/v_i$. That is,

$$T = \int_0^R \frac{-1}{v}\,ds. \tag{15}$$

Case 1. The homogeneous Earth model   By (7), $a(s) = ks$, for $0 \le s \le R$, where k is a constant. Since $a(R) = -g$, then $k = -g/R$. Therefore by Equation (14)

$$v(s) = -\sqrt{2\int_R^s \frac{-g}{R}r\,dr} = -\sqrt{\frac{g}{R}(R^2 - s^2)}.$$

Thus $v(0) = -\sqrt{gR} \approx -7.9$ km/s. By Equation (15),

$$T = \int_0^R \frac{1}{-v}\,ds = \int_0^R \sqrt{\frac{R}{g}}\frac{1}{\sqrt{R^2 - s^2}}\,ds = \frac{\pi}{2}\sqrt{\frac{R}{g}} \approx 21.2 \text{ minutes.}$$

Case 2. The constant acceleration model   If $a(s) = -g$ for $0 \le s \le R$ then

$$v(s) = -\sqrt{2 \int_R^s -g \, dr} = -\sqrt{2g(R - s)},$$

so that $v(0) = -\sqrt{2gR} \approx -11.2$ km/s (which also happens to be the escape velocity of the Earth from its surface), and

$$T = \int_0^R \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{R - s}} \, ds = \sqrt{2} \sqrt{\frac{R}{g}} \approx 19.0 \text{ minutes}.$$

Case 3. An extreme black hole model   We now assume all of the Earth's mass is concentrated at its center, so that $a(s) = k/s^2$ by Newton's law of gravitation, for $0 < s < R$ and where k is some constant. As before, since $a(R) = -g$, then $k = -gR^2$. Thus

$$v(s) = -\sqrt{2 \int_R^s \frac{-gR^2}{r^2} \, dr} = -\sqrt{2gR \left(\frac{R}{s} - 1 \right)},$$

so that $v(0) = -\infty$. Note that $v(1 \text{ cm})$ is just shy of the speed of light. The time it takes for the rock to reach the Earth's center, barring relativistic considerations, is therefore

$$T = \int_0^R \frac{1}{\sqrt{2gR}} \sqrt{\frac{s}{R - s}} \, ds = \frac{\pi}{2\sqrt{2}} \sqrt{\frac{R}{g}} \approx 15.0 \text{ minutes}.$$

To see that no other ideal planet model either yields a faster falling speed or a less fall time, observe that the fastest falling speed in least time will surely occur if the acceleration on the rock is always as extreme as is possible, which certainly happens if all the mass of the Earth is always nearer to the center of the Earth than is the rock. Contrariwise the slowest falling speed and the greatest falling time occur when all the mass is on a shell of radius R, in which case the rock is stationary.

Case 4. Our Earth   Using (11), (14), and (15), and wading through all the approximating integrals gives a speed of $v(0) \approx -9.8$ km/s and a time of $T \approx 19.2$ minutes. Modifying the density function to mirror the data of FIGURE 7 or whatever new data geophysicists generate in years to come should produce more accurate acceleration functions, which in turn will influence these values of $v(0)$ and T slightly.

Lucifer cast down   We close with a problem, the details of which the reader may enjoy completing. In John Milton's *Paradise Lost*, published in 1667, Satan is cast from Heaven and falls for nine days before landing in Hell.

> *Him (Satan) the Almighty Power*
> *Hurled headlong flaming from th' ethereal sky,*
> *With hideous ruin and combustion, down*
> *To bottomless perdition ...*
>
> Book I, Lines 44–47, [**10**, p. 9].

Although Milton used a Ptolemaic model of the universe, it is a fun exercise to recast this problem as dropping a rock through a galaxy. We shall allow this rock to follow the dynamics of (14) and (15) no matter how large v becomes, since conceivably Satan's speed shouldn't be constrained by that of light. Let's take the galaxy as the

Milky Way, which has a mass M of about 3×10^{41} kg, and whose radius R is about 9.3×10^{20} meters. For the sake of simplicity, assume as a first approximating model that the Milky Way is a flat, homogeneous pancake, which is positioned on the x-y plane with its center at the origin. Assume that the rock falls from rest at $(0, 0, R)$. Integrating (1) over the Milky Way via polar coordinates gives

$$a(s) = \frac{GM}{\pi R^2} \int_0^R \int_0^{2\pi} \frac{-rs}{(r^2 + s^2)^{\frac{3}{2}}} \, d\theta \, dr = \frac{2GM}{R^2} \left(\frac{s}{\sqrt{R^2 + s^2}} - 1 \right).$$

By (14),

$$v(s) = -\frac{2\sqrt{GM}}{R} \sqrt{\sqrt{R^2 + s^2} - s + R(1 - \sqrt{2})}.$$

Note that $v(0)$ is approximately 225 km/s. By (15), the rock will reach the center of the Milky Way in about 340 million years; light would make this journey in about one hundred thousand years. Since the episode of Satan being *hurled* out of Heaven took place when the galaxy was in chaos, possibly before the Big Bang, the reader may wish to toy with the pancake model and the rock's initial velocity, so as to estimate the Miltonian distance between Heaven and Hell.

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