The Graph Menagerie: Abstract Algebra and the Mad Veterinarian

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Jessica owns three adorable cats: Boo, Kodiak, and Yoshi. Yoshi, unfortunately, has a bad habit: He likes to damage Jessica's carpet. Sometimes Jessica wishes she had a machine that would magically change Yoshi into a tidier pet . . . a goldfish, perhaps. Of course, a goldfish is much smaller than a cat, so perhaps Yoshi could instead be turned into two goldfish. Or maybe two goldfish and a turtle? But goldfish and turtles aren't too cuddly; Jessica might regret the change, so she would want the machine to be able to turn two goldfish and a turtle back into a cat.

In the parlance of recreational mathematics, Jessica sometimes wishes she were a *Mad Veterinarian*. Mad Vet scenarios were originally presented by Harris [**7**], who posed questions as to which collections of animals can be transformed by Mad Vet machines into other collections. Recently, such scenarios have been used as the basis of various problem solving and Math Circle activities; see, for instance, [**13**]. In this article we take a different approach, using Mad Vet scenarios to explore the concepts of groups, semigroups, and directed graphs.

We have two main goals in analyzing Mad Vet scenarios. Corresponding to any Mad Vet scenario there is a naturally defined semigroup, which may or may not be a group. Our first main goal is to help readers gain some intuition about when a given semigroup is actually a group; to this end, we provide a number of not-so-run-of-the-mill examples involving these algebraic structures.

Our second main goal is to illustrate a practice common in mathematics: namely, answering a question in one area by recasting it in another area, answering the recast question there, and then using that result to answer the original question. There are numerous examples of such powerful cross-disciplinary pollination, including Euler's solution to the classic Königsberg Bridges Problem; see, for instance, Chapter 1 in Biggs et al. [**4**]. We provide a beautiful example of this technique, posing an abstract algebraic question and answering it using graph theory.

Along the way, we provide numerous examples and specific computations. We also present some follow-up questions and information which could be used to supplement the material in an abstract algebra course. We assume that the reader is familiar with first-semester abstract algebraic concepts such as groups and equivalence relations. A good source for these topics is Fraleigh [**5**].

1. Mad Vet scenarios

A *Mad Vet scenario* posits a Mad Veterinarian in possession of a finite number of transmogrifying machines, where

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1. Each machine transmogrifies a single animal of a given species into a finite nonempty collection of animals from any number of species;
2. Each machine can also operate in reverse; and
3. There is a one-to-one correspondence between the species with which the Mad Vet works and the transmogrifying machines; moreover, each species' corresponding machine takes as its input exactly one animal of that species.

These three requirements do not explicitly appear in the puzzles posed by Harris [**7**], but they are certainly implicit there.

Let's consider an example.

Scenario #1. Suppose a Mad Veterinarian has three machines with the following properties.

Machine 1 turns one ant into one beaver;

Machine 2 turns one beaver into one ant, one beaver and one cougar;

Machine 3 turns one cougar into one ant and one beaver.

Starting with one ant, the Mad Vet could produce infinitely many different collections of animals. For example, she could use Machine 1 to turn the ant into a beaver, and then use Machine 2 repeatedly to continually increase the number ants and cougars in her collection. Alternatively, she could use Machine 1 followed by Machine 2, and put the resulting cougar into Machine 3, yielding a collection of two ants and two beavers. Then using Machine 1 twice in reverse, she'd obtain a collection consisting of exactly four ants.

We now mathematize these Mad Vet scenarios. Given a scenario involving n distinct species of animals, we let A_i be the species of animal taken as input (in the forward direction) by Machine i, and denote by $d_{i,j}$ the number of animals of species A_j which are produced by Machine i. For example, in Scenario #1, $A_1 =$ Ant, $A_2 =$ Beaver and $A_3 =$ Cougar, and we have, for instance, $d_{1,1} = 0$, $d_{1,2} = 1$, and $d_{1,3} = 0$.

Writing $\mathbb{N}$ for the set $\{0, 1, 2, \ldots\}$ and $\mathbf{0}$ for the trivial vector $(0, 0, \ldots, 0)$ of length n, we define a *menagerie* to be an element of the set

$$S = \mathbb{N}^n \setminus \{\mathbf{0}\}.$$

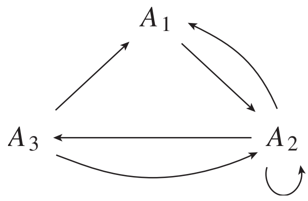
There is a natural bijective correspondence between menageries and nonempty collections of animals from species $A_1, A_2, \ldots, A_n$. For instance, in Scenario #1 a collection of two beavers and five cougars would correspond to $(0, 2, 5)$ in S.

2. Mad Vet graphs

We give here a brief introduction to some standard graph theory concepts. For a more thorough examination of the topic, see, for example, West [**11**] or Wilson and Watkins [**12**]. (Note that graph theory definitions vary widely from text to text; for instance, what we will call a *path* is what West calls a *walk* [**11**].) A *directed graph* consists of a set V of *vertices* and a set E of *edges*; the graph is *finite* if both V and E are finite. Each edge e in E has an *initial vertex*, $i(e)$, and *terminal vertex*, $t(e)$, and is represented in the graph by an arrow pointing from $i(e)$ to $t(e)$. Loops (that is, edges e for which $i(e) = t(e)$) are allowed, as are multiple edges (that is, edges that have a common initial vertex and a common terminal vertex). A vertex is a *sink* if it is not the initial vertex of any edge.

Given any Mad Vet scenario, its corresponding *Mad Vet graph* is the directed graph with $V = \{A_1, A_2, \ldots, A_n\}$, and having, for each A_i, A_j in V, exactly $d_{i,j}$ edges with initial vertex A_i and terminal vertex A_j. Note that any Mad Vet graph is sink-free, due to the third defining feature of a Mad Vet scenario.

EXAMPLE. Scenario #1 has the following Mad Vet graph.



We return to directed graphs in Section 6.

3. Menagerie equivalence classes

Now we come to the key idea. In the context of a Mad Vet scenario, there is a relationship between various menageries. Clearly, a set consisting of two ants and a cougar is different from a set consisting of an ant and three beavers. But if the vet has machines that can be used to replace the first collection of animals with the second (and vice versa), it would make sense to somehow identify the menageries (2, 0, 1) and (1, 3, 0) in S. We have here a naturally arising relation $\sim$ on S, defined formally as follows. Given $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in S, we say that a is *related to* b, and write $a \sim b$, if there is a sequence of Mad Vet machines that will transmogrify the collection of animals associated with menagerie a into the collection of animals associated with menagerie b. Using the three properties of a Mad Vet scenario, it is straightforward to show that $\sim$ is an equivalence relation on S. The equivalence class of a in S under $\sim$ is

$$[a] = \{b \in S : b \sim a\};$$

such equivalence classes partition S.

We now focus on the set

$$W = \{[a] : a \in S\}$$

of equivalence classes of S under $\sim$. Though the elements of W are actually sets themselves, we will work with them primarily as individual elements of the set W.

EXAMPLE. Suppose that our Mad Vet of Scenario #1 starts with the menagerie (1, 0, 0), that is, a collection consisting of just one ant. Then $(1, 0, 0) \sim (0, 1, 0)$ (using Machine 1); in fact, our previous discussion shows that

$$(1, 0, 0) \sim (0, 1, 0) \sim (1, 1, 1) \sim (2, 2, 0) \sim (4, 0, 0).$$

Using equivalence class notation, we've shown

$$[(1, 0, 0)] = [(0, 1, 0)] = [(1, 1, 1)] = [(2, 2, 0)] = [(4, 0, 0)],$$

that is, that these five expressions all represent same element of W.

Now, let (a, b, c) be any menagerie in this Mad Vet scenario. We claim that (a, b, c) is equivalent to one of the menageries (1, 0, 0), (2, 0, 0), or (3, 0, 0). If $c > 0$, then using Machine 3 c times we see that $(a, b, c) \sim (a + c, b + c, 0)$; then if $b + c > 0$, we can use Machine 1 in reverse $b + c$ times to show that $(a + c, b + c, 0) \sim (a + b + 2c, 0, 0)$. By the transitivity of $\sim$, we conclude that $(a, b, c) \sim (i, 0, 0)$ for some positive integer i (namely, $i = a + b + 2c$). We noted above that $(1, 0, 0) \sim (4, 0, 0)$, which implies that $(2, 0, 0) \sim (5, 0, 0)$, $(3, 0, 0) \sim (6, 0, 0)$, and, more generally, that $(j, 0, 0) \sim (i, 0, 0)$ for any positive integers i and j that are congruent modulo 3. Thus, the only elements of W are

$$[(1, 0, 0)], [(2, 0, 0)], \text{ and } [(3, 0, 0)].$$

We now rule out any redundancy among these three elements of W. Given a menagerie $m = (a, b, c)$, define the sum $s_m = a + b + 2c$. If we apply Machine 1 to m, we obtain menagerie $x = (a - 1, b + 1, c)$; if we apply Machine 2 to m we obtain $y = (a + 1, b, c + 1)$; finally, if we apply Machine 3 to m we obtain $z = (a + c, b + c, 0)$. Since

$$s_x = (a - 1) + (b + 1) + 2c = s_m = (a + c) + (b + c) = s_y$$

and

$$s_z = (a + 1) + b + 2(c + 1) = s_m + 3,$$

we have that if menageries m and n are related under $\sim$ then s_m and s_n are congruent modulo 3. Since $s_{(1,0,0)} = 1$, $s_{(2,0,0)} = 2$ and $s_{(3,0,0)} = 3$, the equivalence classes of menageries $(1, 0, 0)$, $(2, 0, 0)$ and $(3, 0, 0)$ under $\sim$ are all distinct. Hence, for this Mad Vet scenario, W is the 3-element set

$$\{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.$$

4. Mad Vet semigroups

We can gain some understanding of a Mad Vet scenario by studying its collection, W, of menagerie equivalence classes simply as a set. But we can learn even more if we exploit a natural operation which combines menageries. We first remind the reader of some definitions.

Let S be any set, and let $*$ be a binary operation on S. Recall that $(S, *)$ is a *semigroup* if $*$ is associative; a semigroup $(S, *)$ is a *monoid* if it contains an identity element for $*$; and a monoid is a *group* if each of its elements has an inverse under $*$.

Three important types of semigroups arise in the context of Mad Vet scenarios. First, given a scenario, we have its set S of menageries, equipped with the usual addition of vectors. (Such addition is an acceptable semigroup operation on S since it is associative and since the sum of two nonzero vectors is again nonzero.) Next, we have the scenario's *Mad Vet semigroup*, which we discuss in this section. Finally, we introduce *graph semigroups* in Section 7.

To create the Mad Vet semigroup of a Mad Vet scenario, we define addition on the scenario's set W of equivalence classes of menageries by setting

$$[x] + [y] = [x + y],$$

where addition on the right-hand side of the equation takes place in S. Addition on W can be understood as follows. Suppose a Mad Vet has a collection of animals in her lab corresponding to menagerie x, and is given a new collection of animals corresponding to menagerie y. Then the sum $[x] + [y]$ in W is the equivalence class of the menagerie corresponding to the union of the animals in the two collections.

Since the elements of W are equivalence classes, we must make sure that our addition on W is well defined. But this is straightforward to see, by identifying our menageries with their associated collections of animals: If some sequence of machines transforms menagerie x into menagerie x', and some sequence transforms menagerie y into menagerie y', then these machines, used in tandem, transform menagerie $x + y$ into menagerie $x' + y'$.

Associativity of $+$ on W is inherited from the associativity of $+$ on S. Thus, $(W, +)$ is a semigroup, called the *Mad Vet semigroup* of its corresponding Mad Vet scenario. Since addition is clearly commutative on S, every Mad Vet semigroup $(W, +)$ is commutative.

EXAMPLE. We revisit Scenario #1 and examine its Mad Vet semigroup $(W, +)$. We showed previously that in this case W is the 3-element set

$$W = \{[(1, 0, 0)], [(2, 0, 0)], [(3, 0, 0)]\}.$$

Using the operation $+$ in W, we get, for instance,

$$[(1, 0, 0)] + [(1, 0, 0)] = [(1 + 1, 0, 0)] = [(2, 0, 0)],$$

as we'd expect. But perhaps it's a bit surprising that

$$[(1, 0, 0)] + [(3, 0, 0)] = [(4, 0, 0)] = [(1, 0, 0)].$$

In other words, $[(3, 0, 0)]$ behaves like an identity element with respect to the element $[(1, 0, 0)]$ in W. In fact, $[(i, 0, 0)] + [(3, 0, 0)] = [(i, 0, 0)]$ for any $1 \leq i \leq 3$. So for this Mad Vet scenario the Mad Vet semigroup $(W, +)$ is a monoid, with identity $[(3, 0, 0)]$. Further, since

$$[(1, 0, 0)] + [(2, 0, 0)] = [(3, 0, 0)]$$

in W, every element in $(W, +)$ has an inverse. Therefore, $(W, +)$ is in fact a group; since its order is 3, it must be isomorphic to the group $\mathbb{Z}_3$.

5. Not all Mad Vet semigroups are groups

Perhaps it is not surprising that the Mad Vet semigroup of Scenario #1 is a group, in light of the explicit description of its elements. In many Mad Vet scenarios, $(W, +)$ is indeed a group; however, we will later see a Mad Vet semigroup that is not even a monoid. Notably, given any Mad Vet semigroup W, the "obvious" choice, $[\mathbf{0}]$, for an identity element of W is not even contained in W, since $\mathbf{0}$ is not in S.

Scenario #2. Suppose the same Mad Vet has replaced two of her machines with new machines.

Machine 1 still turns one ant into one beaver;
Machine 2 now turns one beaver into one ant and one cougar;
Machine 3 now turns one cougar into two cougars.

In this situation W is a monoid, but not a group. First, we claim that

$$W = \{[(i, 0, 0)] : i \in \mathbb{Z}^+\} \cup \{[(0, 0, 1)]\},$$

where $\mathbb{Z}^+$ denotes the set of positive integers. Indeed, let (a, b, c) be a menagerie for this scenario. If $a = b = 0$ (that is, there are only cougars in the menagerie) then

$c - 1$ applications of Machine 3 yields that $(0, 0, c) \sim (0, 0, 1)$. Else, suppose that at least one of a or b is nonzero. Since $(a, b, c) \sim (a + b, 0, c)$ (using Machine 1 in reverse b times), we may assume that the menagerie contains at least one ant and no beavers. If $c = 0$, then we are done. If $c \neq 0$, then we can apply Machine 3 in the appropriate direction $|a - c|$ times, obtaining a menagerie that contains a ants and a cougars; thus, $(a, 0, c) \sim (a, 0, a)$. Then applying Machine 2 in reverse a times yields $(a, 0, a) \sim (0, a, 0)$, which is equivalent to $(a, 0, 0)$ (using Machine 1).

Hence, W consists of the indicated elements. We may now use arguments similar to the argument utilized in studying Scenario #1 to show that these elements are all distinct in W. This establishes our claim.

The same sorts of computations as before show that $[(0, 0, 1)]$ is an identity element for this Mad Vet semigroup, and hence W in this case is a monoid. But W is not a group, because, for instance, there is no element $[x]$ in W for which $[(1, 0, 0)] + [x] = [(0, 0, 1)]$.

Given a Mad Vet scenario, we can pose a variety of questions regarding the structure of its Mad Vet semigroup. For instance, is its semigroup finite or infinite? Is it a monoid? If it is a monoid, is it a group? Note that if it is a group, then that group is necessarily abelian (since all Mad Vet semigroups are commutative)—but is it necessarily cyclic?

To give some sense of just how diverse Mad Vet semigroups can be, we provide below five additional Mad Vet scenarios (Scenarios #3–7) which include, in some order, a scenario for which (1) W is an infinite group; (2) W is a finite noncyclic group; (3) W is a finite nonmonoid; (4) W is a finite cyclic group, not isomorphic to $\mathbb{Z}_3$; and (5) W is an infinite nonmonoid.

In fact, these five different structures even arise in scenarios where the Mad Vet has just three species in her lab. Our readers are encouraged to try their hands at matching the above-described scenarios with those of Scenarios #3–7. Teachers can also find a sample Mad Vet homework assignment, appropriate for a first-semester abstract algebra course, at the MAGAZINE website. Descriptions of the semigroups arising in the following five Mad Vet scenarios are provided at the end of the article, so that readers can check their work.

Scenario #3.

Machine 1 turns one ant into one beaver and one cougar;

Machine 2 turns one beaver into one ant and one cougar;

Machine 3 turns one cougar into one ant and one beaver.

Scenario #4.

Machine 1 turns one ant into two ants;

Machine 2 turns one beaver into two beavers;

Machine 3 turns one cougar two cougars.

Scenario #5.

Machine 1 turns one ant into one beaver and one cougar;

Machine 2 turns one beaver into one ant and one beaver;

Machine 3 turns one cougar into one ant and one cougar.

Scenario #6.

Machine 1 turns one ant into one beaver;
Machine 2 turns one beaver into one cougar;
Machine 3 turns one cougar into one cougar.

Scenario #7.

Machine 1 turns one ant into one ant, one beaver and one cougar;
Machine 2 turns one beaver into one ant and one cougar;
Machine 3 turns one cougar into one ant and one beaver.

Given the varied properties of Mad Vet semigroups displayed thus far, one may wonder how one can possibly identify when Mad Vet semigroups are groups. In the next section, we translate this algebraic question into a comparable graph-theoretical question, whose solution is used to obtain an answer in the algebraic realm.

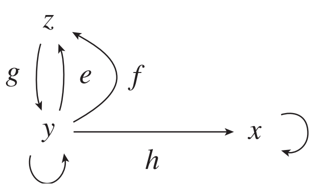
6. The Mad Vet Group Test

In this section, we answer the question: **Given a Mad Vet scenario, when is its Mad Vet semigroup *W* actually a group?**

We need a bit more (standard) graph theory terminology. A *path* in a directed graph Γ is a sequence $P = e_1 e_2 \cdots e_m$ of one or more edges in Γ for which $t(e_j) = i(e_{j+1})$ for each $1 \leq j \leq m-1$; we say that P is a path *from* $i(e_1)$ *to* $t(e_m)$. If v and w are vertices in Γ, we say v *connects to* w in case either $v = w$ or there is a path in Γ from v to w. More generally, if $P = e_1 e_2 \cdots e_m$ is any path in Γ and v is any vertex in Γ, we say v *connects to* P in case v connects to $i(e_j)$ for some edge e_j of P, $1 \leq j \leq m$. For a vertex v in V, a *cycle based at* v is a path $e_1 e_2 \cdots e_m$ from v to v for which the vertices $i(e_1), i(e_2), \ldots, i(e_m)$ are distinct. A loop at a vertex is therefore a cycle, with $m = 1$.

The following graph-theoretic definitions might be more unfamiliar to a reader. A finite graph Γ is *cofinal* in case every vertex v of Γ connects to every cycle and to every sink in Γ. Next, if $C = f_1 f_2 \cdots f_m$ is a cycle in Γ, then an edge e is called an *exit for* C if $i(e) = i(f_j)$ for some $1 \leq j \leq m$, and $e \neq f_j$. (Intuitively, an exit for C is an edge e, not included in C, which provides a way to momentarily "step away" from C.)

EXAMPLE. Consider the following graph.



The cycle eg based at y has three different exits: f, h and the loop at y. These same three edges are also exits for the cycle ge based at z. Similarly, the loop at y has exits e, f and h. On the other hand, the loop at x has no exit. Also, notice that this graph is not cofinal, since, for example, vertex x does not connect to the cycle eg.

Now we are ready to answer the main question of this section.

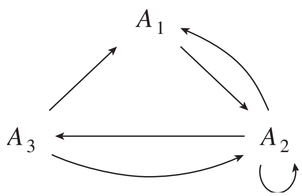
MAD VET GROUP TEST. *The Mad Vet semigroup W of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph Γ has the following two properties.*

(1) *Γ is cofinal; and*

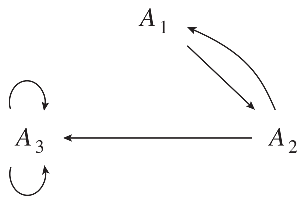
(2) *Every cycle in Γ has an exit.*

The proof of this test is too long for this article; however, in Section 7 we will show how the result follows from a more general theorem (whose complete proof is provided in a supplement at the MAGAZINE website). Here, we see how this test applies to some Mad Vet scenarios.

EXAMPLES. Consider again the Mad Vet graph Δ associated with Scenario #1.



By inspection we see that Δ is cofinal (there are no sinks in Δ and every vertex connects to each of the cycles in Δ) and that every cycle in Δ has an exit. Thus the Mad Vet Group Test reconfirms that the Mad Vet Semigroup for this scenario is indeed a group, a fact we established directly in Section 4. On the other hand, recall the Mad Vet graph Θ of Scenario #2.



We see that Θ is not cofinal, since vertex A_3 does not connect to the cycle $A_1A_2A_1$. So the Mad Vet Group Test reconfirms that the Mad Vet semigroup of Scenario #2 is not a group, as we saw in Section 5.

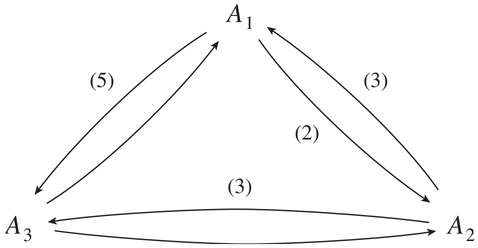
Scenario #8. Consider the Mad Vet scenario described by Harris [**7**], in which the Mad Vet has three machines with the following properties.

Machine 1 turns one cat into two dogs and five mice;

Machine 2 turns one dog into three cats and three mice;

Machine 3 turns one mouse into a cat and a dog.

This scenario has the following Mad Vet graph, where $A_1 = \text{Cat}$, $A_2 = \text{Dog}$, and $A_3 = \text{Mouse}$. The label (d) on an edge e indicates that there are actually d edges in the graph from $i(e)$ to $t(e)$.



It is straightforward to see that this graph satisfies the two properties enumerated in the Mad Vet Group Test; thus, the Mad Vet semigroup in this case is a group, which we identify in Section 8.

You may now want to draw the Mad Vet graphs of Scenarios #3–7, and use the Mad Vet Group Test to determine (or confirm) which three of those Mad Vet scenarios produce Mad Vet groups. Here's one additional observation about the Mad Vet graphs of the remaining two scenarios: One of the graphs is cofinal but contains a cycle without an exit, and the other is not cofinal, though each of its cycles has an exit.

7. Explanation of the Mad Vet Group Test

With the Mad Vet Group Test in hand, we have achieved the second main goal of our article: that is, answering an algebraic question using graph theory. But we have not proven the Mad Vet Group Test. We omit its lengthy proof, but note that the result follows from a theorem about *graph semigroups*. In Section 2, we described a natural connection between Mad Vet scenarios and directed graphs. In fact, a tighter connection can be forged. Any directed graph Γ has an associated commutative *graph monoid*, $(M_\Gamma, +)$. (The interested reader can find the specifics of this construction on p. 163 of Ara et al. in [**2**].) It turns out that if $x, y \in M_\Gamma$ with $x + y = 0$, then $x = y = 0$. Thus, the set $W_\Gamma = M_\Gamma \setminus \{0\}$ is closed under $+$, and so $(W_\Gamma, +)$ is a semigroup, called the *graph semigroup* of Γ.

It follows directly from these constructions that given a Mad Vet scenario with Mad Vet semigroup W and Mad Vet graph Γ, the semigroups W and W_Γ are isomorphic. Thus, information about graph semigroups may be brought to bear in a Mad Vet context. In particular, the main question of the previous section can be answered if we can answer the related question: **Given a directed graph Γ, when is its graph semigroup W_Γ actually a group?**

As it turns out, this question about graph semigroups has recently received significant attention in various mathematical research circles. Some of the related research ideas are described in Section 9. Though in this article we are interested only in sink-free graphs, we do not limit ourselves to such graphs in stating the following result.

GRAPH SEMIGROUP GROUP TEST. *Let Γ be a finite directed graph. Then the graph semigroup W_Γ is a group if and only if Γ has the following three properties.*

(1) Γ *is cofinal;*

(2) *Every cycle in* Γ *has an exit; and*

(3) Γ *contains no sinks.*

Since Mad Vet graphs are sink-free, this test immediately implies the Mad Vet Group Test. The interested reader can find Enrique Pardo's proof of this result at the MAGAZINE website. While Pardo's proof is too long to include here, we note that the Mad Vet Group Test can be proven using only undergraduate-level graph theory and abstract algebra tools.

8. Classification of Mad Vet groups

Though we have achieved our two main goals, another natural question remains: **When a Mad Vet semigroup is a group, just exactly what group is it?** We turn to another area of mathematics—namely, linear algebra—for an algorithmic way of finding the structure of any Mad Vet group. Note that a Mad Vet semigroup must be a *group* in order for this method to apply.

Let Γ be the Mad Vet graph of a Mad Vet scenario whose Mad Vet semigroup is a group. The graph Γ has an associated *incidence matrix* A_Γ, defined as follows: Suppose Γ has n vertices, $v_1, v_2, \ldots, v_n$. Then A_Γ is the $n \times n$ matrix (d_{ij}), where d_{ij} is the number of edges with initial vertex v_i and terminal vertex v_j (for all $1 \leq i, j \leq n$). For example, if Δ is the graph of Scenario #1, then

$$A_\Delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

First, we form the matrix $I_n - A_\Gamma$, where I_n is the $n \times n$ identity matrix. For instance, using the above matrix A_Δ, we have

$$I_3 - A_\Delta = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Then we put the (square) matrix $I_n - A_\Gamma$ in *Smith normal form*. The Smith normal form of an $n \times n$ matrix having integer entries is a diagonal $n \times n$ matrix whose diagonal entries are nonnegative integers

$$\alpha_1, \alpha_2, \ldots, \alpha_q, 0, 0, \ldots, 0$$

such that α_i divides α_{i+1} for each $1 \leq i \leq q-1$. The Smith normal form of a matrix A can be obtained by performing on A a combination of these matrix operations: interchanging rows or columns, or adding an integer multiple of a row [column] to another row [column]. The resulting Smith normal form of matrix A is thus of the form *PAQ*, where P and Q are integer-valued matrices with determinants equal to ± 1. Many computer algebra systems have a built-in Smith normal form function.† For more information about the Smith normal form of a matrix, see, for example, Stein [**10**] or Chapter 23 in Hogben [**8**].

Here's a way of answering the "just exactly what group is it?" question.

MAD VET GROUP IDENTIFICATION THEOREM. *Given a Mad Vet scenario whose Mad Vet semigroup, W, is a group, let Γ be its associated Mad Vet graph. Then*

$$W \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \cdots \oplus \mathbb{Z}_{\alpha_q} \oplus \mathbb{Z}^{n-q},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_q$ are the nonzero diagonal entries of the Smith normal form of the matrix $I_n - A_\Gamma$.

The justification of this theorem is beyond the scope of this article, but the very enthusiastic reader can find a similar justification in Section 3 of Abrams et al. [**1**].

†For instance, to use Maple to compute the Smith normal form of a matrix B, define B in Maple, load the package *LinearAlgebra*, and use the command *SmithForm(B)*. A word of caution: the Smith normal form function in some computer algebra systems will not find the Smith normal form of a matrix of determinant 0, even though such a Smith normal form always exists in this case. A matrix of that type may arise in some Mad Vet scenarios; indeed, it arises in one of our eight numbered Mad Vet scenarios.

EXAMPLE. Letting Δ be the Mad Vet graph of Scenario #1, the Smith normal form of the matrix $I_3 - A_\Delta$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Because we already know that Scenario #1's semigroup is a group, the Mad Vet Group Identification Theorem implies that it is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_3 \cong \{0\} \oplus \{0\} \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3$, as expected.

See if you can now use this method to identify the three groups which arise among Scenarios #3–7. Finally, try applying this method to Scenario #8; you should get that the Mad Vet group in that case is isomorphic to $\mathbb{Z}_{34}$.

9. Beyond the Mad Vet

By this point, you may be wondering: **Who really cares about Mad Vet semigroups anyway?** Good question! In case you are not convinced that Mad Vet semigroups are of interest in their own right, we present the following theorem. Although this result is rather technical, our point in stating it is to emphasize the fact that Mad Vet semigroups do indeed play a central role in current, active lines of mathematical research. Not only that, but this theorem actually bridges two apparently different branches of mathematics (algebra and analysis) and the Graph Semigroup Group Test is exactly the link between them.

PURELY INFINITE SIMPLICITY THEOREM. *For a finite directed sink-free graph Γ, the following are equivalent:*

(1) *The Leavitt path algebra $L_{\mathbb{C}}(\Gamma)$ is purely infinite and simple. (This is a statement about an algebraic structure.)*
(2) *The graph C^*-algebra $C^*(\Gamma)$ is purely infinite and simple. (This is a statement about an analytic structure.)*
(3) *Γ satisfies the conditions of the Graph Semigroup Group Test.*
(4) *The graph semigroup W_Γ is a group.*

In the interest of brevity, we have not stated the most general form of this result. Pardo's direct proof of the equivalence of (3) and (4), which involves only undergraduate-level graph- and group-theoretic ideas, is new; the only published proof of this equivalence of which the authors are aware involves showing that both (3) and (4) are equivalent to (1). The *very* energetic reader may wish to consult Arando Pino et al. [**3**].

Finally, as promised earlier, here is a description of the Mad Vet semigroups arising in Scenarios #3–7. In order, these scenarios' semigroups are (up to isomorphism) the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, a 7-element nonmonoid, the group $\mathbb{Z}$, the monoid $\mathbb{Z}^+$, and the group $\mathbb{Z}_4$. For details, see our Analyses of Mad Vet Scenarios #3–7, available at the MAGAZINE website.

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Summary In this paper, we explore Mad Veterinarian scenarios. We show how these recreational puzzles naturally give rise to semigroups (which are sometimes groups), and we point out a beautiful, striking connection between abstract algebra and graph theory. Linear algebra also plays a role in our analysis.

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