NOTES

Dinner, Dancing, and Tennis, Anyone?

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Introduction In August 1996 the United States Tennis Association (USTA) had a problem. They had just conducted the men's draw for the annual U.S. Open tournament, but, contrary to previous years' procedure, the draw was made before the seedings were determined for the top 16 of the 128 players. Worse yet, the seedings announced did not follow the standard computer rankings for the top players, as had long been the policy. Americans Michael Chang and André Agassi, for example, were ranked third and eighth, but were seeded second and sixth, respectively. Amid allegations that the seedings were rigged to promote late-round match-ups between high-visibility players, the USTA conceded that this unusual procedure had created at least the appearance of impropriety, and agreed to redo the draw, this time after the seedings had been announced.

This unprecedented re-draw did not satisfy all of the critics, particularly those who felt that the seedings favored Americans, but it did quiet talk of a boycott from several players. It also raised some interesting questions about probability. In the first draw, Andrei Medvedev of the Ukraine was pitted against Jean-Philippe Fleurian of France. In the re-draw, these same two unseeded players were scheduled to meet in the first round. A USTA official found this a remarkable coincidence, and contacted the Department of Mathematics and Computer Science at nearby St. John's University to inquire about the probability of such an event.

I happened to field the phone call. It took some time to determine just what the official was asking. For example, consider the difference between the following two questions:

The Easy U.S. Open Question: What is the probability that Medvedev and Fleurian would be drawn as first-round opponents in both draws?

The Hard U.S. Open Question: What is the probability that at least one pair of players would be drawn as first-round opponents in both draws?

The official seemed to be more interested in the first question, whose answer is much smaller and much easier to obtain than the answer to the second question. The second question turned out to be subtler than expected. It also brought to mind several related questions that were also interesting in their own right—two problems that I refer to as the dinner problem and the dancing problem. The dinner problem is a generalization of the classical problem of coincidences, first discussed by Montmort in 1708. The dancing problem is related to another classical problem, the *problème des ménages*, in which n married couples are seated at a round table. Both problems can be defined recursively, and solutions are readily obtained for finite values by direct computation. Solutions to the dinner and dancing problems will be used to answer the Hard U.S. Open Problem.

The dinner problem I first give the simple version of the problem, and then throw in a twist.

> *The Dinner Problem* (*simple version*): Suppose n people are invited to a dinner party. Seats are assigned and a name card made for each guest. However, floral arrangements on the tables unexpectedly obscure the name cards. When the n guests arrive, they seat themselves randomly. What is the probability that no guest sits in his or her assigned seat?

Another equivalent formulation of this problem is the well known hat-check problem, in which a careless attendant loses all of the slips and returns the hats at random. The probability that nobody gets his or her own hat is equal to the probability in the simple version of the dinner problem.

We will generalize the dinner problem, for two reasons. First, we will need the answer to the more general problem to answer the Hard U.S. Open question. Second, and perhaps surprisingly, the recursion formulas are more easily developed with the more general problem.

> *The Dinner Problem* (*with party crashers*): Suppose n people are invited to a dinner party as before, with the same confusion about the seating arrangement. This time k of the n diners are party crashers, where $0 \le k \le n$. (No name cards exist, of course, for the party crashers.) Once again, when the n diners arrive, they seat themselves randomly at the tables. What is the probability $p_{n,k}$ that no invited guest sits in his or her assigned seat?

The simple version of the problem, with no party crashers, asks for $p_{n,0}$.

Recursive formulas for the dinner problem We give the first few cases for small values of n and k, and derive the recursive equations. If there is one guest ($n = 1$), then she is either invited ($k = 0$), or not ($k = 1$). In the first case she must sit in her own seat and $p_{1,0} = 0$; otherwise, she cannot sit in her own seat, and $p_{1,1} = 1$.

If $n > 1$, we can express the probability $p_{n,k}$ in terms of probabilities involving one fewer guests, in one of two ways. If k is positive, then there is at least one party crasher. Ironically, it is easier to establish an equation in this case if we abandon all pretense towards social fairness and seat a party crasher first. There is no possibility that the party crasher will sit in her assigned seat; the probability is $\frac{k}{n}$ that she will sit at a place designated for one of the k absent invitees. If she sits in one of these k seats, then the probability that none of the remaining $n-1$ guests will sit in his or her assigned seat is, by definition, $p_{n-1,k-1}$. On the other hand, the probability that she will sit in one of the $n-k$ seats assigned to a guest who is present is $\frac{n-k}{n}$. In this event, then $n-1$ guests will remain to be seated, of whom $k-1$ are party crashers and one was invited—but just had his seat taken by the party crasher who was seated first! This invitee no longer has any chance of sitting in his own seat, and so becomes indistinguishable from the remaining $k-1$ party crashers. Therefore, the probability that none of the remaining $n-1$ guests will sit in his or her assigned seat is $p_{n-1,k}$. In summary we have the following reduction formula for $n > 1$ and $k > 0$:

$$p_{n,k} = \frac{k}{n} p_{n-1,k-1} + \frac{n-k}{n} p_{n-1,k} \tag{1}$$

If there are no party crashers ($k = 0$), we seat one invited guest at random. The probability is $\frac{n-1}{n}$ that she will not sit in her own seat, in which case one of the remaining $n - 1$ guests can no longer sit in his own seat. For the problem of seating the remaining $n - 1$ guests, this displaced soul becomes, in essence, a party crasher himself. The probability that none of the remaining $n - 1$ guests will sit in his or her assigned seat is therefore $p_{n-1,1}$. Thus, the reduction equation for the case where there are no party crashers is

$$p_{n,0} = \frac{n-1}{n} p_{n-1,1}. \qquad (2)$$

Equations 1 and 2, together with the base cases described above, allow us to generate as many of the probability numbers, where $0 \le k \le n$, as patience or computer speed will allow. Table 1 gives values for $n \le 10$.

TABLE 1. The Dinner Problem Probabilities $p_{n,k}$, $0 \le k \le n \le 10$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1.0000										
1	0.0000	1.0000									
2	0.5000	0.5000	1.0000								
3	0.3333	0.5000	0.6667	1.0000							
4	0.3750	0.4583	0.5833	0.7500	1.0000						
5	0.3667	0.4417	0.5333	0.6500	0.8000	1.0000					
6	0.3681	0.4292	0.5028	0.5917	0.7000	0.8333	1.0000				
7	0.3679	0.4204	0.4817	0.5536	0.6381	0.7381	0.8571	1.0000			
8	0.3679	0.4139	0.4664	0.5266	0.5958	0.6756	0.7679	0.8750	1.0000		
9	0.3679	0.4088	0.4547	0.5066	0.5651	0.6313	0.7063	0.7917	0.8889	1.0000	
10	0.3679	0.4047	0.4455	0.4910	0.5417	0.5982	0.6613	0.7319	0.8111	0.9000	1.0000

Notice the rapid convergence of the numbers $p_{n,0}$ in the first column. Indeed, there is a well-known non-recursive formula for these simple dinner problem probabilities (see **[2]**, **[4]**, or **[7]**):

$$p_{n,0} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

so that the limit is $e^{-1} \approx 0.3679$.

The dancing problem In order to solve our Hard U.S. Open Question, we need to double the fun we have had with the dinner problem and consider the dancing problem. Again, there will be both a simple version and one with a twist, but this time we will have twice as many people to work with.

> *The Dancing Problem* (*simple version*): Suppose n married couples ($2n$ people) are invited to a party. Dance partners are chosen at random, without regard to gender. What is the probability that nobody will be paired with his or her spouse?

Just as dance partners are paired without regard to gender, we do not assume, as current laws in most states do, that a married couple must consist of a male and a female.

Again, we answer a simple question by posing a harder, more general one. This time, we introduce single guests into the problem.

The Dancing Problem (with single people): Suppose $n - k$ married couples and $2k$ single people are invited to a party. (That's still $2n$ people.) Dance partners are chosen at random, without regard to gender. What is the probability $d_{n,k}$ that nobody will be paired with his or her spouse?

The simple version of the problem, with no single people, is to determine $d_{n,0}$.

Recursive formulas for the dancing problem We consider the base cases first. If there are no guests, then no spouses can be paired, so $d_{0,0} = 1$. If there are only two guests ($n = 1$), then either they are married, in which case $d_{1,0} = 0$; or they are both single, in which case $d_{1,1} = 1$.

To obtain recursive formulas for the dancing problem, we assume that there are at least two people invited ($n \geq 1$), and then express $d_{n,k}$ in terms of probabilities with strictly smaller values of n. If there are any single people ($k > 0$), we pair one of them first. We select a single person, whom we'll call Sam. Of the remaining $2n - 1$ people, $2k - 1$ are single and $2n - 2k$ are married. The probability that Sam will be paired with another single person is $\frac{2k-1}{2n-1}$. In that case, the probability of pairing off the remaining $2n - 2$ people, of whom $2k - 2$ are single, so that no spouses are paired together is $d_{n-1,k-1}$. On the other hand, the probability that Sam will be paired with a married person is $\frac{2n-2k}{2n-1}$. In this case, the remaining $2n - 2$ people comprise $2k - 1$ single people, $n - k - 1$ married couples, and one leftover married person whose spouse was paired with Sam. For the purposes of pairing off the remaining $2n - 2$ people into dance partners, this leftover spouse has become indistinguishable from the other single people, since he or she cannot be paired with his or her spouse. Sam has, in effect, broken up this marriage! Thus, the probability that none of the remaining $2n - 2$ people will be paired with a spouse is $d_{n-k,k}$, since there are still $2k$ effectively single people. Putting this together gives the following reduction formula, provided $n > 1$ and $k > 0$:

$$d_{n,k} = \frac{2k-1}{2n-1} d_{n-1,k-1} + \frac{2n-2k}{2n-1} d_{n-1,k}. \tag{3}$$

If there are no single people ($k = 0$), then we must pair a married person first. The probability that this married person will not be paired with his or her spouse is $\frac{2n-1}{2n-2}$. In this case, that leaves $2n - 2$ people to be paired as dance partners, including two lone people whose spouses were just paired together. Since neither of these people can be paired with a spouse, they are considered single for the problem of pairing the remaining $2n - 2$ people. The probability that none of the remaining $2n - 2$ people will be paired with a spouse is thus $d_{n-1,1}$, yielding the following formula, when $n > 1$:

$$d_{n,0} = \frac{2n-2}{2n-1} d_{n-1,1}. \tag{4}$$

Once again, we can generate all desired values of $d_{n,k}$, for $0 \leq k \leq n$, using equations (3) and (4), and the base cases derived above. The values for $0 \leq k \leq n \leq 10$ and several values of $d_{n,0}$ for higher values of n appear in Table 2.

Notice that the sequence of numbers in the first column of Table 2, corresponding to the simple version of the dancing problem, does not converge nearly as rapidly as in the simple dinner problem. Nonetheless, it does seem to converge to a number strictly less than 1. Can you guess the limit?

TABLE 2 The Dancing Problem Probabilities $d_{n,k}$, $0 \le k \le n \le 10$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1.0000										
1	0.0000	1.0000									
2	0.6667	0.6667	1.0000								
3	0.5333	0.6667	0.8000	1.0000							
4	0.5714	0.6476	0.7429	0.8571	1.0000						
5	0.5757	0.6392	0.7111	0.7937	0.8889	1.0000					
6	0.5810	0.6334	0.6915	0.7561	0.8283	0.9091	1.0000				
7	0.5847	0.6294	0.6781	0.7313	0.7894	0.8531	0.9231	1.0000			
8	0.5874	0.6264	0.6683	0.7135	0.7623	0.8149	0.8718	0.9333	1.0000		
9	0.5895	0.6241	0.6609	0.7002	0.7422	0.7871	0.8350	0.8863	0.9412	1.0000	
10	0.5912	0.6223	0.6551	0.6899	0.7268	0.7658	0.8072	0.8512	0.8978	0.9474	1.0000
100	0.6050124905										
1000	0.6063790072										
10,000	0.6065154929										
100,000	0.6065291359										

Why the Hard U.S. Open Question is hard We now have almost all of the necessary tools to answer the Hard U.S. Open Question. We recall that in each draw there are 128 players, of whom 16 are seeded; seeded players cannot meet in the first round.

If there were no seedings at all, then we could answer this question directly using the dancing numbers $d_{n,k}$. Imagine pairing up the 128 players into 64 married couples according to the first-round match-ups in the first draw. Then we pair up dance partners according to the first-round match-ups in the second draw. Having at least one pair of players drawn as first-round opponents in both draws would be equivalent to having at least one married couple paired as dance partners. This is the complement of having no spouses paired as dance partners, so the answer to the Hard U.S. Open Problem, *if there were no seedings*, would be

$$1 - d_{64,0} \approx 1 - 0.604157 = 0.395843.$$

But then it wouldn't be a very hard problem, would it?

In addition to the dinner and dancing probability numbers $p_{n,k}$ and $d_{n,k}$, we will use a simpler, non-recursive combinatorial formula. Suppose a sample of m people is picked at random from a group of n couples. (That's $2n$ people, with no singles.) We denote by $r(n, m, k)$ the probability that exactly k of the n couples will be included in the sample of m people, where $0 \le k \le \lfloor m/2 \rfloor$.

We leave as an exercise to the interested reader to show that

$$r(n,m,k) = \frac{\binom{n}{k}\binom{n-k}{m-2k}2^{m-2k}}{\binom{2n}{m}}.$$

Answering the Hard U.S. Open Question The principal effect of the seedings on our Question is that no two seeded players can meet as first-round opponents. Notice that there are two different ways in which players might be paired as first-round opponents in both draws. An unseeded player might draw the same seeded opponent twice, or he might draw the same unseeded opponent twice. We use the dinner problem probabilities to address the first issue, and the dancing number probabilities for the second. Of the 128 players, 16 are seeded and 112 are unseeded. Of the 112 unseeded players, 96 are initially lucky because they drew unseeded opponents in the first draw. The answer to the Hard U.S. Open Question will depend on how many of these 96 "lucky" unseeded players happen to draw a seeded opponent in the second draw. We let m denote the number of unseeded players who drew an unseeded opponent in the first draw, but a seeded opponent in the second draw. These players were thus rather unlucky, and most likely none too pleased with the decision to remake the draw.

This integer m can range in value from 0 to 16. Let E_m denote the event described above, that exactly m unseeded players drew an unseeded opponent in the first draw, but a seeded opponent in the second draw. The probability that the event E_m occurs is given by

$$P(E_m) = \frac{\binom{96}{m}\binom{16}{16-m}}{\binom{112}{16}} = \frac{\binom{96}{m}\binom{16}{m}}{\binom{112}{16}}.$$

The denominator counts all possible ways of picking 16 unseeded players to play against seeded players in the second draw. In the numerator, of the 96 unseeded players who drew unseeded opponents in the first draw, we choose m of them to have seeded opponents in the second draw. Then, of the 16 unseeded players who drew seeded opponents in the first draw, we choose $16 - m$ of them to have seeded opponents again in the second draw. (These are the doubly unlucky unseeded players!)

Now, suppose that the event E_m occurs. The probability that, of the $16 - m$ players who drew seeded opponents twice, none of them drew the same seeded opponent in both draws is exactly the dinner problem probability number $p_{16,\,m}$. To see this, think of the seeded players as the place settings in the dinner problem. The 16 unseeded players who drew seeded opponents in the first draw are the invited guests, whose (hidden) name cards are at the place settings corresponding to their seeded opponents in the first draw. The m unseeded players who drew unseeded opponents in the first draw, and seeded opponents in the second, are the m party crashers, since they cannot be matched with the same opponent twice. The $16 - m$ players who drew seeded opponents in both draws are the invited guests that actually made it to the dinner party.

Now we consider the possibility that two unseeded players were chosen as first-round opponents in both draws. Continue supposing that event E_m occurs. Of the m players who drew an unseeded opponent in the first draw and a seeded opponent in the second draw, suppose that k pairs of them were scheduled against each other as opponents in the first draw. The probability of this event is given by the number

$$r(48, m, k) = \frac{\binom{48}{k}\binom{48-k}{m-2k}2^{m-2k}}{\binom{96}{m}}$$

discussed earlier, where $0 \le k \le \lfloor m/2 \rfloor$.

To summarize, we are supposing that event E_m occurs; that is, that there are m unseeded players who drew an unseeded opponent in the first draw and a seeded opponent in the second draw; and that of those m players, k pairs of them were scheduled as opponents in the first draw. Given those conditions, what is the probability that, of the $96 - m$ unseeded players who drew unseeded opponents in both draws, no two were scheduled against each other in both draws? The answer, of course, is the dancing problem probability number $d_{48,\,m-k}$.

How is that? Marital status is determined by how the players were matched up in the first draw, while dancing partners are chosen by the match-ups in the second draw. There are 96 unseeded players who drew unseeded opponents in the first draw. These 96 people represent 48 originally married couples in the dancing problem, paired by their scheduled opponents in the first draw. Of these 96 players, m drew seeded opponents in the second draw. These m players represent some, but not all, of the single guests, since they cannot be matched with their "spouses" in the second draw. Now, since we are assuming that k pairs of those m players were paired up in the first draw, $m-2k$ of these m players' first-draw opponents (spouses) must have drawn unseeded opponents in the second draw. These $m-2k$ spouses are effectively single as well, since they cannot be matched with their first-draw opponents in the second draw. This gives a total of $m+(m-2k)=2(m-k)$ single people out of $2\cdot 48=96$ guests. The probability that no two of the 96 players will be paired together in both draws is thus $d_{48,\, m-k}$.

The answer to the Hard U.S. Open Question To answer the Hard U.S. Open Question at last, we add up the probabilities for each of the possible values of k, $0 \le k \le \lfloor m/2 \rfloor$, for each of the possible values of m, $0 \le m \le 16$. Thus, the probability that no two players would be drawn as first round opponents in both draws is

$$\sum_{m=0}^{16} P(E_m)\, p_{16,\, m}\left(\sum_{k=0}^{\lfloor m/2 \rfloor} r(48, m, k)\, d_{48,\, m-k} \right) \approx 0.5983933573.$$

A computer algebra system can give the exact answer in rational form. Curiously, in this case it is a (reduced) fraction of integers, each with exactly 100 digits.

The probability that at least one pair of players would be drawn as first-round opponents in both draws is, accurate to ten decimal places, $1-0.5983933573 = 0.4016066427$.

Recall that if there were no seedings the answer would have been $1-d_{64,\,0} \approx 0.395843$, which is fairly close to the actual answer.

How does our answer to the Hard U.S. Open Question compare to that of the Easy U.S. Open Question: "What is the probability that Medvedev and Fleurian would be drawn as first-round opponents in both draws?" We leave it to the interested reader to show that the answer is

$$\left(\frac{2}{259} \right)^2 = \frac{4}{67{,}081} = \frac{1}{16{,}770.25} \approx 0.00005963.$$

Conclusions It is indeed quite improbable that any two particular players would be drawn as first-round opponents in both draws. This is in dramatic contrast to the answer to the Hard U.S. Open Question, where we saw that there is a greater than 40% chance that at least one pair of players will be drawn as first-round opponents in both draws. What I find most surprising is not the disparity in the two answers, but the subtlety required to answer the Hard U.S. Open Question. Along the way, we explored two related problems and developed formulas for each of them. It is remarkable that in the simple versions of both of these problems, the probabilities do not converge to either of the extreme values, 0 or 1, as the number of guests increases without bound. In the simple version of the dinner problem, the probability approaches $e^{-1} \approx 0.3679$, and the convergence is quite rapid. In the simple version of the dancing problem, we conjecture that the probability converges to $\frac{1}{\sqrt{e}} \approx 0.606530659$. Finally, we leave the interested reader with a few problems to explore.

1. Determine the probability that *exactly* k pairs of players will be selected as first-round opponents in both draws, for $0 \leq k \leq 64$; then calculate the expected number of repeated pairs.
2. Find a non-recursive formula for the simple dancing problem probabilities, $d_{n,0}$.
3. Prove (or disprove) that $\lim_{n \to \infty} d_{n,0} = \frac{1}{\sqrt{e}}$.

Acknowledgment. I thank the referees for many valuable comments and suggestions.

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Variations on a Theme: A_4 Definitely Has No Subgroup of Order Six!

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Introduction To obtain *one* valid proof of a theorem is an achievement, but there may be many different proofs of the same theorem. For example, there are said to be over 370 of Pythagoras's theorem. Once a result has been proved, the story seldom ends. Instead the search begins for refined, reduced, or simplified proofs.

It is just as important to have a collection of different approaches to proving a given result as it is to have a collection of different results that can be derived using a given technique. An advantage of this attitude is that if one has already proved a result using a certain technique, then a different method of proving the same result may sometimes yield a generalization of the original result which may *not* be possible with the original technique of proof. We illustrate this phenomenon by examining various proofs of the fact that A_4, the alternating group on four symbols, has no subgroup of order six.

Preliminaries One of the cornerstones of theory of finite groups is the following theorem of the Italian mathematician J. L. Lagrange (1736–1813):