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The Evolution of Group Theory: A Brief Survey

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This article gives a brief sketch of the evolution of group theory. It derives from a firm conviction that the history of mathematics can be a useful and important integrating component in the teaching of mathematics. This is not the place to elaborate on the role of history in teaching, other than perhaps to give one relevant quotation:

> Although the study of the history of mathematics has an intrinsic appeal of its own, its chief raison d'être is surely the illumination of mathematics itself. For example the gradual unfolding of the integral concept from the volume computations of Archimedes to the intuitive integrals of Newton and Leibniz and finally to the definitions of Cauchy, Riemann and Lebesgue—cannot fail to promote a more mature appreciation of modern theories of integration.
>
> —C. H. Edwards [**11**]

The presentation in one article of the evolution of so vast a subject as group theory necessitated severe selectivity and brevity. It also required omission of the broader contexts in which group theory evolved, such as wider currents in abstract algebra, and in mathematics as a whole. (We will note *some* of these interconnections shortly.) We trust that enough of the essence and main lines of development in the evolution of group theory have been retained to provide a useful beginning from which the reader can branch out in various directions. For this the list of references will prove useful.

The reader will find in this article an outline of the origins of the main concepts, results, and theories discussed in a beginning course on group theory. These include, for example, the concepts of (abstract) group, normal subgroup, quotient group, simple group, free group, isomorphism, homomorphism, automorphism, composition series, direct product; the theorems of J. L. Lagrange, A.-L. Cauchy, A. Cayley, C. Jordan-O. Hölder; the theories of permutation groups and of abelian groups. At the same time we have tried to balance the technical aspects with background information and interpretation.

Our survey of the evolution of group theory will be given in several stages, as follows:

1. Sources of group theory.
2. Development of "specialized" theories of groups.
3. Emergence of abstraction in group theory.
4. Consolidation of the abstract group *concept*; dawn of abstract group *theory*.
5. Divergence of developments in group theory.

Before dealing with each stage in turn, we wish to mention the context within mathematics as a whole, and within algebra in particular, in which group theory developed. Although our "story" concerning the evolution of group theory begins in 1770 and extends to the 20th century, the major developments occurred in the 19th century. Some of the general mathematical features of that century which had a bearing on the evolution of group theory are: (a) an increased concern for rigor; (b) the emergence of abstraction; (c) the rebirth of the axiomatic method; (d) the view of mathematics as a human activity, possible without reference to, or motivation from, physical situations. Each of these items deserves extensive elaboration, but this would go beyond the objectives (and size) of this paper.

Up to about the end of the 18th century, algebra consisted (in large part) of the study of solutions of polynomial equations. In the 20th century, algebra became a study of abstract, axiomatic systems. The transition from the so-called classical algebra of polynomial equations to the so-called modern algebra of axiomatic systems occurred in the 19th century. In addition to group theory, there emerged the structures of commutative rings, fields, noncommutative rings, and vector spaces. These developed alongside, and sometimes in conjunction with, group theory. Thus Galois theory involved both groups and fields; algebraic number theory contained elements of group theory in addition to commutative ring theory and field theory; group representation theory was a mix of group theory, noncommutative algebra, and linear algebra.

1. Sources of group theory

There are four major sources in the evolution of group theory. They are (with the names of the originators and dates of origin):

(a) Classical algebra (J. L. Lagrange, 1770)
(b) Number theory (C. F. Gauss, 1801)
(c) Geometry (F. Klein, 1874)
(d) Analysis (S. Lie, 1874; H. Poincaré and F. Klein, 1876)

We deal with each in turn.

(a) *Classical Algebra* (J. L. Lagrange, 1770)

The major problems in algebra at the time (1770) that Lagrange wrote his fundamental memoir "Réflexions sur la résolution algébrique des équations" concerned polynomial equations. There were "theoretical" questions dealing with the existence and nature of the roots (e.g., Does every equation have a root? How many roots are there? Are they real, complex, positive, negative?), and "practical" questions dealing with methods for finding the roots. In the latter instance there were exact methods and approximate methods. In what follows we mention exact methods.

The Babylonians knew how to solve quadratic equations (essentially by the method of completing the square) around 1600 B.C. Algebraic methods for solving the cubic and the quartic were given around 1540. One of the major problems for the next two centuries was the algebraic solution of the quintic. This is the task Lagrange set for himself in his paper of 1770.

In his paper Lagrange first analyzes the various known methods (devised by F. Viète, R. Descartes, L. Euler, and E. Bézout) for solving cubic and quartic equations. He shows that the common feature of these methods is the reduction of such equations to auxiliary equations—the so-called resolvent equations. The latter are one degree lower than the original equations. Next Lagrange attempts a similar analysis of polynomial equations of arbitrary degree n. With each such equation he associates a "resolvent equation" as follows: let $f(x)$ be the original equation, with roots $x_1, x_2, \ldots, x_n$. Pick a rational function $R(x_1, x_2, \ldots, x_n)$ of the roots and coefficients of $f(x)$. (Lagrange describes methods for doing so.) Consider the different values which $R(x_1, x_2, \ldots, x_n)$ assumes under all the $n!$ permutations of the roots $x_1, x_2, \ldots, x_n$ of $f(x)$. If these are denoted by $y_1, y_2, \ldots, y_k$, then the resolvent equation is given by $g(x) = (x - y_1) \cdot (x - y_2) \cdots (x - y_k)$. (Lagrange shows that k divides $n!$—the source of what we call Lagrange's theorem in group theory.) For example, if $f(x)$ is a quartic with roots x_1, x_2, x_3, x_4, then $R(x_1, x_2, x_3, x_4)$ may be taken to be $x_1x_2 + x_3x_4$, and this function assumes three distinct values under the 24 permutations of x_1, x_2, x_3, x_4. Thus the resolvent equation of a quartic is a cubic. However, in carrying over this analysis to the quintic, he finds that the resolvent equation is of degree six!

Although Lagrange did not succeed in resolving the problem of the algebraic solvability of the quintic, his work was a milestone. It was the first time that an association was made between the

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solutions of a polynomial equation and the permutations of its roots. In fact, the study of the permutations of the roots of an equation was a cornerstone of Lagrange's general theory of algebraic equations. This, he speculated, formed "the true principles for the solution of equations." (He was, of course, vindicated in this by E. Galois.) Although Lagrange speaks of permutations without considering a "calculus" of permutations (e.g., there is no consideration of their composition or closure), it can be said that the germ of the group concept (as a group of permutations) is present in his work. For details see **[12]**, **[16]**, **[19]**, **[25]**, **[33]**.

(b) *Number Theory* (C. F. Gauss, 1801)

In the *Disquisitiones Arithmeticae* of 1801 Gauss summarized and unified much of the number theory that preceded him. The work also suggested new directions which kept mathematicians occupied for the entire century. As for its impact on group theory, the *Disquisitiones* may be said to have initiated the theory of finite abelian groups. In fact, Gauss established many of the significant properties of these groups without using any of the terminology of group theory. The groups appear in four different guises: the additive group of integers modulo m, the multiplicative group of integers relatively prime to m, modulo m, the group of equivalence classes of binary quadratic forms, and the group of nth roots of unity. And though these examples appear in number-theoretic contexts, it is as abelian groups that Gauss treats them, using what are clear prototypes of modern algebraic proofs.

For example, considering the nonzero integers modulo $p(p$ a prime), Gauss shows that they are all powers of a single element; i.e., that the group Z_p^* of such integers is cyclic. Moreover, he determines the number of generators of this group (he shows that it is equal to $\phi(p-1)$, where ϕ is Euler's ϕ-function). Given any element of Z_p^* he defines the order of the element (without using the terminology) and shows that the order of an element is a divisor of $p-1$. He then uses this result to prove P. Fermat's "little theorem," namely, that $a^{p-1} \equiv 1 \bmod p$ if p does not divide a, thus employing group-theoretic ideas to prove number-theoretic results. Next he shows that if t is a positive integer which divides $p-1$, then there exists an element in Z_p^* whose order is t—essentially the converse of Lagrange's theorem for cyclic groups.

Concerning the nth roots of 1 (which he considers in connection with the cyclotomic equation), he shows that they too form a cyclic group. In connection with this group he raises and answers many of the same questions he raised and answered in the case of Z_p^*.

The problem of representing integers by binary quadratic forms goes back to Fermat in the early 17th century. (Recall his theorem that every prime of the form $4n+1$ can be represented as a sum of two squares x^2+y^2.) Gauss devotes a large part of the *Disquisitiones* to an exhaustive study of binary quadratic forms and the representation of integers by such forms. (A **binary quadratic form** is an expression of the form $ax^2+bxy+cy^2$, with a, b, c integers.) He defines a composition on such forms, and remarks that if K and K^1 are two such forms one may denote their composition by $K+K^1$. He then shows that this composition is associative and commutative, that there exists an identity, and that each form has an inverse, thus verifying all the properties of an abelian group.

Despite these remarkable insights one should not infer that Gauss had the concept of an abstract group, or even of a finite abelian group. Although the arguments in the *Disquisitiones* are quite general, each of the various types of "groups" he considers is dealt with separately—there is no unifying group-theoretic method which he applies to all cases. For details see [**5**], [**9**], [**25**], [**30**], [**33**].

(c) *Geometry* (F. Klein, 1872)

We are referring here to Klein's famous and influential (but see [**18**]) lecture entitled "A Comparative Review of Recent Researches in Geometry," which he delivered in 1872 on the occasion of his admission to the faculty of the University of Erlangen. The aim of this so-called Erlangen Program was the classification of geometry as the study of invariants under various groups of transformations. Here there appear groups such as the projective group, the group of rigid motions, the group of similarities, the hyperbolic group, the elliptic groups, as well as the geometries associated with them. (The affine group was not mentioned by Klein.) Now for some background leading to Klein's Erlangen Program.

The 19th century witnessed an explosive growth in geometry, both in scope and in depth. New geometries emerged: projective geometry, noneuclidean geometries, differential geometry, algebraic geometry, n-dimensional geometry, and Grassmann's geometry of extension. Various geometric methods competed for supremacy: the synthetic versus the analytic, the metric versus the projective. At mid-century, a major problem had arisen, namely, the classification of the relations and inner connections among the different geometries and geometric methods. This gave rise to the study of "geometric relations," focusing on the study of properties of figures invariant under transformations. Soon the focus shifted to a study of the transformations themselves. Thus the study of the geometric relations of figures became the study of the associated transformations. Various types of transformations (e.g., collineations, circular transformations, inversive transformations, affinities) became the objects of specialized studies. Subsequently, the logical connections among transformations were investigated, and this led to the problem of classifying transformations and eventually to Klein's group-theoretic synthesis of geometry.

Klein's use of groups in geometry was the final stage in bringing order to geometry. An intermediate stage was the founding of the first major theory of classification in geometry, beginning in the 1850's, the Cayley-Sylvester Invariant Theory. Here the objective was to study invariants of "forms" under transformations of their variables. This theory of classification, the

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precursor of Klein's Erlangen Program, can be said to be *implicitly* group-theoretic. Klein's use of groups in geometry was, of course, explicit. (For a thorough analysis of implicit group-theoretic thinking in geometry leading to Klein's Erlangen Program, see [33].) In the next section (2-(c)) we will note the significance of Klein's Erlangen Program (and his other works) for the evolution of group theory. Since the Program originated a hundred years after Lagrange's work and eighty years after Gauss' work, its importance for group theory can best be appreciated *after* a discussion of the evolution of group theory beginning with the works of Lagrange and Gauss and ending with the period around 1870.

(d) *Analysis* (S. Lie, 1874; H. Poincaré and F. Klein, 1876)

In 1874 Lie introduced his general theory of (continuous) transformation groups—essentially what we call Lie groups today. Such a group is represented by the transformations

$$x_i' = f_i(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_n), \qquad i = 1, 2, \ldots, n,$$

where the f_i are analytic functions in the x_i and a_i (the a_i are parameters, with both x_i and a_i real or complex). For example, the transformations given by

$$x' = \frac{ax+b}{cx+d}, \quad \text{where } a, b, c, d, \text{ are real numbers and } ad - bc \neq 0,$$

define a continuous transformation group.

Lie thought of himself as the successor of N. H. Abel and Galois, doing for differential equations what they had done for algebraic equations. His work was inspired by the observation

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that almost all the differential equations which had been integrated by the older methods remain invariant under continuous groups that can be easily constructed. He was then led to consider, in general, differential equations that remain invariant under a given continuous group and to investigate the possible simplifications in these equations which result from the known properties of the given group (cf. Galois theory). Although Lie did not succeed in the actual formulation of a "Galois theory of differential equations," his work was fundamental in the subsequent formulation of such a theory by E. Picard (1883/1887) and E. Vessiot (1892).

Poincaré and Klein began their work on "automorphic functions" and the groups associated with them around 1876. Automorphic functions (which are generalizations of the circular, hyperbolic, elliptic, and other functions of elementary analysis) are functions of a complex variable z, analytic in some domain D, which are invariant under the group of transformations

$$z' = \frac{az+b}{cz+d}, \quad (a, b, c, d \text{ real or complex and } ad - bc \neq 0)$$

or under some subgroup of this group. Moreover, the group in question must be "discontinuous" (i.e., any compact domain contains only finitely many transforms of any point). Examples of such groups are the modular group (in which a, b, c, d are integers and $ad - bc = 1$), which is associated with the elliptic modular functions, and Fuchsian groups (in which a, b, c, d are real and $ad - bc = 1$) associated with the Fuchsian automorphic functions. As in the case of Klein's Erlangen Program, we will explore the consequences of these works for group theory in section 2-(c).

2. Development of "specialized" theories of groups

In §1 we outlined four major sources in the evolution of group theory. The first source—classical algebra—led to the theory of permutation groups; the second source—number theory—led to the theory of abelian groups; the third and fourth sources—geometry and analysis—led to the theory of transformation groups. We will now outline some developments within these specialized theories.

(a) *Permutation Groups*

As noted earlier, Lagrange's work of 1770 initiated the study of permutations in connection with the study of the solution of equations. It was probably the first clear instance of implicit group-theoretic thinking in mathematics. It led directly to the works of P. Ruffini, Abel, and Galois during the first third of the 19th century, and to the concept of a permutation group.

Ruffini and Abel proved the unsolvability of the quintic by building upon the ideas of Lagrange concerning resolvents. Lagrange showed that a necessary condition for the solvability of the general polynomial equation of degree n is the existence of a resolvent of degree less than n. Ruffini and Abel showed that such resolvents do not exist for $n > 4$. In the process they developed a considerable amount of permutation theory. (See [**1**], [**9**], [**19**], [**23**], [**24**], [**25**], [**30**], [**33**] for details.) It was Galois, however, who made the fundamental conceptual advances, and who is considered by many as the founder of (permutation) group theory.

Galois' aim went well beyond finding a method for solvability of equations. He was concerned with gaining insight into general principles, dissatisfied as he was with the methods of his predecessors: "From the beginning of this century," he wrote, "computational procedures have become so complicated that any progress by those means has become impossible" [**19**, p. 92].

Galois recognized the separation between "Galois theory" (i.e., the correspondence between fields and groups) and its application to the solution of equations, for he wrote that he was presenting "the general principles and just one application" of the theory [**19**, p. 42]. "Many of the early commentators on Galois theory failed to recognize this distinction, and this led to an emphasis on applications at the expense of the theory" (Kiernan, [**19**]).

Galois was the first to use the term "group" in a technical sense—to him it signified a collection of permutations closed under multiplication: "if one has in the same group the substitutions S and T one is certain to have the substitution ST" [**33**, p. 111]. He recognized that the most important properties of an algebraic equation were reflected in certain properties of a group uniquely associated with the equation— "the group of the equation." To describe these properties he invented the fundamental notion of normal subgroup and used it to great effect. While the issue of resolvent equations preoccupied Lagrange, Ruffini, and Abel, Galois' basic idea was to bypass them, for the construction of a resolvent required great skill and was not based on a clear methodology. Galois noted instead that the existence of a resolvent was equivalent to the existence of a normal subgroup of prime index in the group of the equation. This insight shifted consideration from the resolvent equation to the group of the equation and its subgroups.

Galois defines the group of an equation as follows [**19**, p. 80]:

> Let an equation be given, whose m roots are $a, b, c, \ldots$. There will always be a group of permutations of the letters $a, b, c, \ldots$ which has the following property: 1) that every function of the roots, invariant under the substitutions of that group, is rationally known [i.e., is a rational function of the coefficients and any adjoined quantities]. 2) conversely, that every function of the roots, which can be expressed rationally, is invariant under these substitutions.

The definition says essentially that the group of the equation consists of those permutations of the roots of the equation which leave invariant all relations among the roots over the field of coefficients of the equation—basically the definition we would give today. Of course the definition does not guarantee the existence of such a group, and so Galois proceeds to demonstrate it. Galois next investigates how the group changes when new elements are adjoined to the "ground field" F. His treatment is amazingly close to the standard treatment of this matter in a modern algebra text.

Galois' work was slow in being understood and assimilated. In fact, while it was done around 1830, it was published posthumously in 1846, by J. Liouville. Beyond his technical accomplishments, Galois "challenged the development of mathematics in two ways. He discovered, but left unproved, theorems which called for proofs based on new, sophisticated concepts and calculations. Also, the task of filling the gaps in his work necessitated a fundamental clarification of his methods and their group theoretical essence" (Wussing, [**33**]). For details see [**12**], [**19**], [**23**], [**25**], [**29**], [**31**], [**33**].

The other major contributor to permutation theory in the first half of the 19th century was Cauchy. In several major papers in 1815 and 1844 Cauchy inaugurated the theory of permutation groups as an autonomous subject. (Before Cauchy, permutations were not an object of independent study but rather a useful device for the investigation of solutions of polynomial equations.) Although Cauchy was well aware of the work of Lagrange and Ruffini (Galois' work was not yet published at the time), Wussing suggests that Cauchy "was definitely not inspired directly by the contemporary group-theoretic formulation of the solution of algebraic equations" [**33**].

In these works Cauchy gives the first systematic development of the subject of permutation groups. In the 1815 papers Cauchy uses no special name for sets of permutations closed under multiplication. However, he recognizes their importance and gives a name to the number of elements in such a closed set, calling it "diviseur indicatif." In the 1844 paper he defines the concept of a group of permutations generated by certain elements [**22**, p. 65].

> Given one or more substitutions involving some or all of the elements $x, y, z, \ldots$ I call the products of these substitutions, by themselves or by any other, in any order, *derived* substitutions. The given substitutions, together with the derived ones, form what I call a *system of conjugate substitutions.*

In these works, which were very influential, Cauchy makes several lasting additions to the terminology, notation, and results of permutation theory. For example, he introduces the permutation notation $\begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$ in use today, as well as the cyclic notation for permutations; defines the product of permutations, the degree of a permutation, cyclic permutation, transposition; recognizes the identity permutation as a permutation; discusses what we would call today the direct product of two groups; and deals with the alternating groups extensively. Here is a sample of some of the results he proves.

(i) Every even permutation is a product of 3-cycles.

(ii) If p (prime) is a divisor of the order of a group, then there exists a subgroup of order p. (This is known today as "Cauchy's theorem", though it was stated without proof by Galois.)

(iii) Determined all subgroups of S_3, S_4, S_5, S_6 (making an error in S_6.)

(iv) All permutations which commute with a given one form a group (the centralizer of an element).

It should be noted that all these results were given and proved in the context of permutation groups. For details see [**6**], [**8**], [**23**], [**24**], [**25**], [**33**].

The crowning achievement of these two lines of development—a symphony on the grand themes of Galois and Cauchy—was Jordan's important and influential *Traité des substitutions et des équations algébriques* of 1870. Although the author states in the preface that "the aim of the work is to develop Galois' method and to make it a proper field of study, by showing with what facility it can solve all principal problems of the theory of equations," it is in fact group theory per se—not as an offshoot of the theory of solvability of equations—which forms the central object of study.

The striving for a mathematical synthesis based on key ideas is a striking characteristic of Jordan's work as well as that of a number of other mathematicians of the period (e.g., F. Klein). The concept of a (permutation) group seemed to Jordan to provide such a key idea. His approach

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enabled him to give a unified presentation of results due to Galois, Cauchy, and others. His application of the group concept to the theory of equations, algebraic geometry, transcendental functions, and theoretical mechanics was also part of the unifying and synthesizing theme. "In his book Jordan wandered through all of algebraic geometry, number theory, and function theory in search of interesting permutation groups" (Klein, [**20**]). In fact, the aim was a survey of all of mathematics by areas in which the theory of permutation groups had been applied or seemed likely to be applicable. "The work represents... a review of the whole of contemporary mathematics from the standpoint of the occurrence of group-theoretic thinking in permutation-theoretic form" (Wussing, [**33**]).

The *Traité* embodied the substance of most of Jordan's publications on groups up to that time (he wrote over 30 articles on groups during the period 1860–1880) and directed attention to a large number of difficult problems, introducing many fundamental concepts. For example, Jordan makes explicit the notions of isomorphism and homomorphism for (substitution) groups, introduces the term "solvable group" for the first time in a technical sense, introduces the concept of a composition series, and proves part of the Jordan-Hölder theorem, namely, that the indices in two composition series are the same (the concept of a quotient group was not explicitly recognized at this time); and he undertakes a very thorough study of transitivity and primitivity for permutation groups, obtaining results most of which have not since been superseded. Jordan also gives a proof that A_n is simple for $n > 4$.

An important part of the treatise is devoted to a study of the "linear group" and some of its subgroups. In modern terms these constitute the so-called classical groups, namely, the general linear group, the unimodular group, the orthogonal group, and the symplectic group. Jordan considers these groups only over finite fields, and proves their simplicity in certain cases. It should be noted, however, that he considers these groups as permutation groups rather than groups of matrices or linear transformations (see **[29]**, [33]).

Jordan's *Traité* is a landmark in the evolution of group theory. His permutation-theoretic point of view, however, was soon to be overtaken by the conception of a group as a group of transformations (see (c) below). "The *Traité* marks a pause in the evolution and application of the permutation-theoretic group concept. It was an expression of Jordan's deep desire to effect a conceptual synthesis of the mathematics of his time. That he tried to achieve such a synthesis by relying on the concept of a permutation group, which the very next phase of mathematical development would show to have been unduly restricted, makes for both the glory and the limitations of the *Traité*..." (Wussing, [33]). For details see **[9]**, [13], **[19]**, **[20]**, [22], **[24]**, **[29]**, [33].

(b) *Abelian Groups*

As noted earlier, the main source for abelian group theory was number theory, beginning with Gauss' *Disquisitiones Arithmeticae*. In contrast to permutation theory, group-theoretic modes of thought in number theory remained implicit until about the last third of the 19th century. Until that time no explicit use of the term "group" was made, and there was no link to the contemporary, flourishing theory of permutation groups. We now give a sample of some implicit group-theoretic work in number theory, especially in algebraic number theory.

Algebraic number theory arose in connection with Fermat's conjecture concerning the equation $x^n+y^n=z^n$, Gauss' theory of binary quadratic forms, and higher reciprocity laws. Algebraic number fields and their arithmetical properties were the main objects of study. In 1846 G. L. Dirichlet studied the units in an algebraic number field and established that (*in our terminology*) the group of these units is a direct product of a finite cyclic group and a free abelian group of finite rank. At about the same time E. Kummer introduced his "ideal numbers," defined an equivalence relation on them, and derived, for cyclotomic fields, certain special properties of the number of equivalence classes (the so-called class number of a cyclotomic field; in our terminology, the order of the ideal class group of the cyclotomic field). Dirichlet had earlier made similar studies of *quadratic* fields.

In 1869 E. Schering, a former student of Gauss, investigated the structure of Gauss' (group of) equivalence classes of binary quadratic forms. He found certain fundamental classes from which all classes of forms could be obtained by composition. In group-theoretic terms, Schering found a basis for the abelian group of equivalence classes of binary quadratic forms.

L. Kronecker generalized Kummer's work on cyclotomic fields to arbitrary algebraic number fields. In a paper in 1870 on algebraic number theory, entitled "Auseinandersetzung einiger Eigenschaften der Klassenzahl idealer complexer Zahlen," he began by taking a very abstract point of view: he considered a finite set of arbitrary "elements," and defined an abstract operation on them which satisfied certain laws—laws which we may take nowadays as axioms for a finite abelian group:

> Let $\theta', \theta'', \theta''', \ldots$ be finitely many elements such that with any two of them we can associate a third by means of a definite procedure. Thus, if f denotes the procedure and θ', θ'' are two (possibly equal) elements, then there exists a θ''' equal to $f(\theta', \theta'')$. Furthermore, $f(\theta', \theta'') = f(\theta'', \theta')$, $f(\theta', f(\theta'', \theta''')) = f(f(\theta', \theta''), \theta''')$ and if θ'' is different from θ''' then $f(\theta', \theta'')$ is different from $f(\theta', \theta''')$. Once this is assumed we can replace the operation $f(\theta', \theta'')$ by multiplication $\theta' \cdot \theta''$ provided that instead of equality we employ equivalence. Thus using the usual equivalence symbol " $\sim$ " we define the equivalence $\theta' \cdot \theta'' \sim \theta'''$ by means of the equation $f(\theta', \theta'') = \theta'''$.

Kronecker aimed at working out the laws of combination of "magnitudes," in the process giving an implicit definition of a finite abelian group. From the above abstract considerations

Kronecker deduces the following consequences:

(i) If θ is any "element" of the set under discussion, then $\theta^k = 1$ for some positive integer k. If k is the smallest such then θ is said to "belong to k". If θ belongs to k and $\theta^m = 1$ then k divides m.
(ii) If an element θ belongs to k, then every divisor of k has an element belonging to it.
(iii) If θ and θ' belong to k and k' respectively, and k and k' are relatively prime, then $\theta\theta'$ belongs to kk'.
(iv) There exists a "fundamental system" of elements $\theta_1, \theta_2, \theta_3, \ldots$ such that the expression $\theta_1^{h_1}\theta_2^{h_2}\theta_3^{h_3} \cdots (h_i = 1, 2, 3, \ldots, n_i)$ represents each element of the given set of elements just once. The numbers $n_1, n_2, n_3, \ldots$ to which, respectively, $\theta_1, \theta_2, \theta_3, \ldots$ belong, are such that each is divisible by its successor; the product $n_1 n_2 n_3 \cdots$ is equal to the totality of elements of the set.

The above can, of course, be interpreted as well known results on finite abelian groups; in particular (iv) can be taken as the basis theorem for such groups. Once Kronecker establishes this general framework, he applies it to the special cases of equivalence classes of binary quadratic forms and to ideal classes. He notes that when applying (iv) to the former one obtains Schering's result.

Although Kronecker did not relate his implicit definition of a finite abelian group to the (by that time) well established concept of a permutation group, of which he was well aware, he clearly recognized the advantages of the abstract point of view which he adopted:

> The very simple principles... are applied not only in the context indicated but also frequently, elsewhere—even in the elementary parts of number theory. This shows, and it is otherwise easy to see, that these principles belong to a more general and more abstract realm of ideas. It is therefore proper to free their development from all inessential restrictions, thus making it unnecessary to repeat the same argument when applying it in different cases. ... Also, when stated with all admissible generality, the presentation gains in simplicity and, since only the truly essential features are thrown into relief, in transparency.

The above lines of development were capped in 1879 by an important paper of G. Frobenius and L. Stickelberger entitled "On groups of commuting elements." Although Frobenius and Stickelberger built on Kronecker's work, they used the concept of an abelian group explicitly and, moreover, made the important advance of recognizing that the abstract group concept embraces congruences and Gauss' composition of forms as well as the substitution groups of Galois. (They also mention, in footnotes, groups of infinite order, namely groups of units of number fields and the group of all roots of unity.) One of their main results is a proof of the basis theorem for finite abelian groups, including a proof of the uniqueness of decomposition. It is interesting to compare their explicit, "modern," formulation of the theorem to that of Kronecker ((iv) above):

> A group that is not irreducible [indecomposable] can be decomposed into purely irreducible factors. As a rule, such a decomposition can be accomplished in many ways. However, regardless of the way in which it is carried out, the number of irreducible factors is always the same and the factors in the two decompositions can be so paired off that the corresponding factors have the same order [**33**, p. 235].

They go on to identify the "irreducible factors" as cyclic groups of prime power orders. They then apply their results to groups of integers modulo m, binary quadratic forms, and ideal classes in algebraic number fields.

The paper by Frobenius and Stickelberger is "a remarkable piece of work, building up an independent theory of finite abelian groups on its own foundation in a way close to modern views" (Fuchs, [**30**]). For details on this section (b), see [**5**], [**9**], [**24**[, [**30**], [**33**].

(c) *Transformation Groups*

As in number theory, so in geometry and analysis, group-theoretic ideas remained implicit until the last third of the 19th century. Moreover, Klein's (and Lie's) explicit use of groups in geometry influenced conceptually rather than technically the evolution of group theory, for it signified a

Georg Frobenius

genuine shift in the development of that theory from a preoccupation with permutation groups to the study of groups of transformations. (That is not to imply, of course, that permutation groups were no longer studied.) This transition was also notable in that it pointed to a turn from finite groups to infinite groups.

Klein noted the connection of his work with permutation groups but also realized the departure he was making. He stated that what Galois theory and his own program have in common is the investigation of "groups of changes," but added that "to be sure, the objects the changes apply to are different: there [Galois theory] one deals with a finite number of discrete elements, whereas here one deals with an infinite number of elements of a continuous manifold" [**33**, p. 191]. To continue the analogy, Klein notes that just as there is a theory of permutation groups, "we insist on a *theory of transformations*, a study of groups generated by transformations of a given type" [**33**, p. 191].

Klein shunned the abstract point of view in group theory, and even his technical definition of a (transformation) group is deficient: "Now let there be given a sequence of transformations $A, B, C, \dots$ If this sequence has the property that the composite of any two of its transformations yields a transformation that again belongs to the sequence, then the latter will be called a group of transformations" [**33**, p. 185]. His work, however, broadened considerably the conception of a group and its applicability in other fields of mathematics. Klein did much to promote the view that group-theoretic ideas are fundamental in mathematics: "Group theory appears as a distinct discipline throughout the whole of modern mathematics. It permeates the most varied areas as an ordering and classifying principle" [**33**, p. 228].

There was another context in which groups were associated with geometry, namely, "motion-geometry;" i.e., the use of motions or transformations of geometric objects as group elements. Already in 1856 W. R. Hamilton considered (implicitly) "groups" of the regular solids. Jordan, in 1868, dealt with the classification of all subgroups of the group of motions of Euclidean 3-space. And Klein in his *Lectures on the Icosahedron* of 1884 "solved" the quintic equation by means of the symmetry group of the icosahedron. He thus discovered a deep connection between the groups of rotations of the regular solids, polynomial equations, and complex function theory. (In these *Lectures* there also appears the "Klein 4-group".)

Already in the late 1860's Klein and Lie had undertaken, jointly, "to investigate geometric or analytic objects that are transformed into themselves by *groups of changes*." (This is Klein's retrospective description, in 1894, of their program.) While Klein concentrated on discrete groups, Lie studied continuous transformation groups. Lie realized that the theory of continuous transformation groups was a very powerful tool in geometry and differential equations and he set himself the task of "determining all groups of...[continuous] transformations" [**33**, p. 214]. He achieved his objective by the early 1880's with the classification of these groups (see [**33**] for details). A classification of discontinuous transformation groups was obtained by Poincaré and Klein a few years earlier.

Beyond the technical accomplishments in the areas of discontinuous and continuous transformation groups (extensive theories developed in both areas and both are still nowadays active fields of research), what is important for us in the founding of these theories is that

(i) They provided a major extension of the scope of the concept of a group—from permutation groups and abelian groups to transformation groups;

(ii) They provided important examples of infinite groups—previously the only objects of study were finite groups;

(iii) They greatly extended the range of applications of the group concept to include number theory, the theory of algebraic equations, geometry, the theory of differential equations (both ordinary and partial), and function theory (automorphic functions, complex functions).

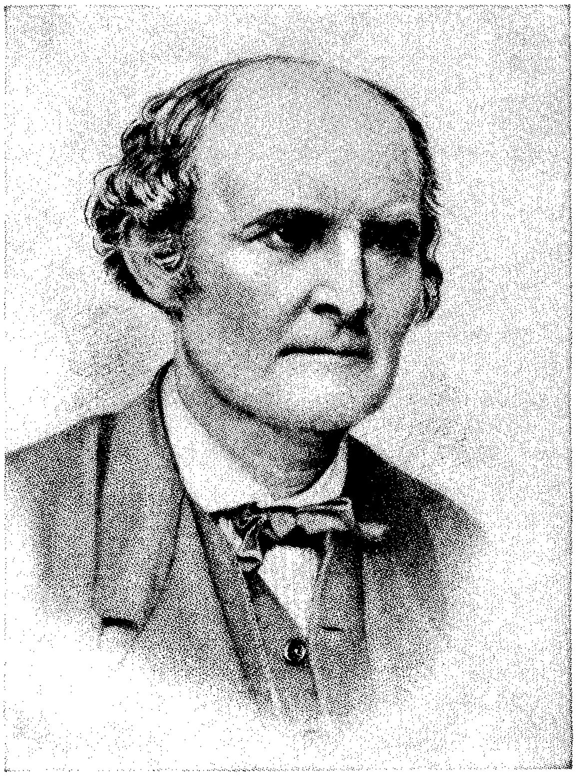
All this occurred prior to the emergence of the abstract group concept. In fact, these developments were instrumental in the emergence of the concept of an abstract group, which we describe next. For further details on this section (c), see [**5**], [**7**], [**9**], [**17**], [**18**], [**20**], [**24**], [**29**], [**33**].

3. Emergence of abstraction in group theory

The abstract point of view in group theory emerged slowly. It took over one hundred years from the time of Lagrange's implicit group-theoretic work of 1770 for the abstract group concept to evolve. E. T. Bell discerns several stages in this process of evolution towards abstraction and axiomatization:

> The entire development required about a century. Its progress is typical of the evolution of any major mathematical discipline of the recent period; first, the discovery of isolated phenomena, then the recognition of certain features common to all, next the search for further instances, their detailed calculation and classification; then the emergence of general principles making further calculations, unless needed for some definite application, superfluous; and last, the formulation of postulates crystallizing in abstract form the structure of the system investigated [**2**].

Although somewhat oversimplified (as all such generalizations tend to be), this is nevertheless a useful framework. Indeed, in the case of group theory, first came the "isolated phenomena"—e.g., permutations, binary quadratic forms, roots of unity; then the recognition of "common features"—the concept of a finite group, encompassing both permutation groups and finite abelian groups (cf. the paper of Frobenius and Stickelberger cited in section 2(b)); next the search for "other instances"—in our case transformation groups (see section 2(c)); and finally the formulation of "postulates"—in this case the postulates of a group, encompassing both the finite and infinite



Arthur Cayley

cases. We now consider when and how the intermediate and final stages of abstraction occurred.

In 1854 Cayley, in a paper entitled "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$," gave the first abstract definition of a finite group. (In 1858 R. Dedekind, in lectures on Galois theory at Göttingen, gave another.) Here is Cayley's definition:

> A set of symbols 1, $\alpha, \beta, \ldots$ all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a *group*.

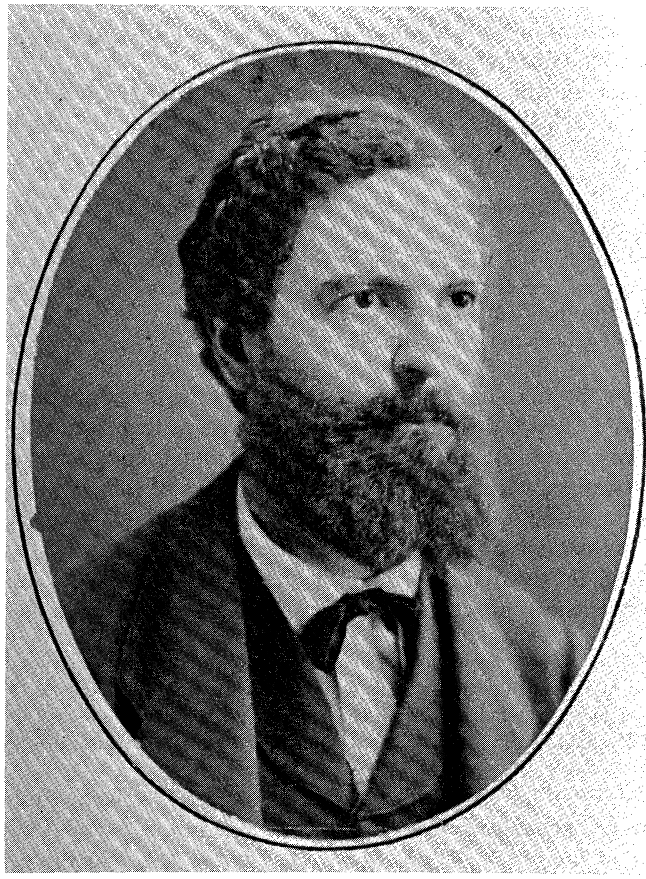
Cayley goes on to say that

> These symbols are not in general convertible [commutative], but are associative,

and

> it follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor [i.e., on the left or on the right], the effect is simply to reproduce the group.

Cayley then presents several examples of groups, such as the quaternions (under addition), invertible matrices (under multiplication), permutations, Gauss' quadratic forms, and groups arising in elliptic function theory. Next he shows that every abstract group is (in our terminology) isomorphic to a permutation group, a result now known as "Cayley's theorem." He seems to have been well aware of the concept of isomorphic groups, although he does not define it explicitly. He introduces, however, the multiplication table of a (finite) group and asserts that an abstract group



Heinrich Weber

is determined by its multiplication table. He then goes on to determine all the groups of orders four and six, showing there are two of each by displaying multiplication tables. Moreoever, he notes that the cyclic group of order n "is in every respect analogous to the system of the roots of the ordinary equation $x^n - 1 = 0$," and that there exists only one group of a given prime order.

Cayley's orientation towards an abstract view of groups—a remarkable accomplishment at this time of the evolution of group theory—was due, at least in part, to his contact with the abstract work of G. Boole. The concern with the abstract foundations of mathematics was characteristic of the circles around Boole, Cayley, and Sylvester already in the 1840's. Cayley's achievement was, however, only a personal triumph. His abstract definition of a group attracted no attention at the time, even though Cayley was already well known. The mathematical community was apparently not ready for such abstraction: permutation groups were the only groups under serious investigation, and more generally, the formal approach to mathematics was still in its infancy. As M. Kline put it in his inimitable way [**21**]: "Premature abstraction falls on deaf ears, whether they belong to mathematicians or to students." For details see [**22**], [**23**], [**24**], [**25**], [**29**], [**33**].

It was only a quarter of a century later that the abstract group concept began to take hold. And it was Cayley again who in four short papers on group theory written in 1878 returned to the abstract point of view he adopted in 1854. Here he stated the general problem of finding all groups of a given order and showed that any (finite) group is isomorphic to a group of permutations. But, as he remarked, this "does not in any wise show that the best or easiest mode of treating the general problem is thus to regard it as a problem of substitutions; and it seems clear that the better course is to consider the general problem in itself, and to deduce from it the theory of groups of substitutions" [**22**, p. 141]. These papers of Cayley, unlike those of 1854, inspired a number of fundamental group-theoretic works.

Another mathematician who advanced the abstract point of view in group theory (and more generally in algebra) was H. Weber. It is of interest to see his "modern" definition of an abstract (finite) group given in a paper of 1882 on quadratic forms [**23**, p. 113]:

> A system G of h arbitrary elements $\theta_1, \theta_2, \ldots, \theta_h$ is called a group of degree h if it satisfies the following conditions:
> I. By some rule which is designated as composition or multiplication, from any two elements of the same system one derives a new element of the same system. In symbols $\theta_r\theta_s = \theta_t$.
> II. It is always true that $(\theta_r\theta_s)\theta_t = \theta_r(\theta_s\theta_t) = \theta_r\theta_s\theta_t$.
> III. From $\theta\theta_r = \theta\theta_s$ or from $\theta_r\theta = \theta_s\theta$ it follows that $\theta_r = \theta_s$.

Weber's and other definitions of abstract groups given at the time applied to *finite* groups only. They thus encompassed the two theories of permutation groups and (finite) abelian groups, which derived from the two sources of classical algebra (polynomial equations) and number theory, respectively. Infinite groups, which derived from the theories of (discontinuous and continuous) transformation groups, were not subsumed under those definitions. It was W. von Dyck who, in an important and influential paper in 1882 entitled "Group-theoretic studies," consciously included and combined, for the first time, all of the major historical roots of abstract group theory —the algebraic, number theoretic, geometric, and analytic. In von Dyck's own words:

> The aim of the following investigations is to continue the study of the properties of a group in its abstract formulation. In particular, this will pose the question of the extent to which these properties have an invariant character present in all the different realizations of the group, and the question of what leads to the exact determination of their essential group-theoretic content.

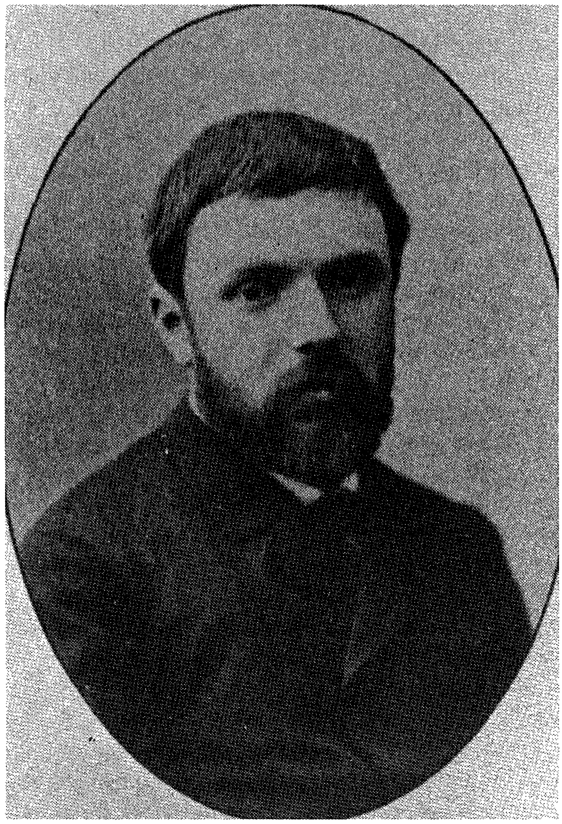
Von Dyck's definition of an abstract group, which included both the finite and infinite cases, was given in terms of generators (he calls them "operations") and defining relations (the definition is somewhat long—see [**7**, pp. 5, 6]). He stresses that "in this way all... *isomorphic* groups are included in *a single* group," and that "the *essence* of a group is no longer expressed by a particular presentation form of its operations but rather by their mutual relations." He then goes on to construct the free group on n generators, and shows (essentially, without using the terminology) that every finitely generated group is a quotient group of a free group of finite rank. What is important from the point of view of postulates for group theory is that von Dyck was the first to require explicitly the existence of an inverse in his definition of a group: "We require for our considerations that a group which contains the operation T_k must also contain its inverse T_k^{-1}." In a second paper (in 1883) von Dyck applied his abstract development of group theory to permutation groups, finite rotation groups (symmetries of polyhedra), number theoretic groups, and transformation groups.

Although various postulates for groups appeared in the mathematical literature for the next twenty years, the abstract point of view in group theory was not universally applauded. In particular, Klein, one of the major contributors to the development of group theory, thought that the "abstract formulation is excellent for the working out of proofs but it does not help one find new ideas and methods," adding that "in general, the disadvantage of the [abstract] method is that it fails to encourage thought" [**33**, p. 228].

Despite Klein's reservations, the mathematical community was at this time (early 1880's) receptive to the abstract formulations (cf. the response to Cayley's definition of 1854). The major reasons for this receptivity were:

(i) There were now several major "concrete" theories of groups—permutation groups, abelian groups, discontinuous transformation groups (the finite and infinite cases), and continuous transformation groups, and this warranted abstracting their essential features.

(ii) Groups came to play a central role in diverse fields of mathematics, such as different parts of algebra, geometry, number theory and several areas of analysis, and the abstract view of



Henri Poincaré

groups was thought to clarify what was essential for such applications and to offer opportunities for further applications.

(iii) The formal approach, aided by the penetration into mathematics of set theory and mathematical logic, became prevalent in other fields of mathematics, for example, various areas of geometry and analysis.

In the next section we will follow, very briefly, the evolution of that abstract point of view in group theory.

4. Consolidation of the abstract group *concept*; dawn of abstract group *theory*

The abstract group concept spread rapidly during the 1880's and 1890's, although there still appeared a great many papers in the areas of permutation and transformation groups. The abstract viewpoint was manifested in two ways:

(a) Concepts and results introduced and proved in the setting of "concrete" groups were now reformulated and reproved in an abstract setting;

(b) Studies originating in, and based on, an abstract setting began to appear.

An interesting example of the former case is a reproving by Frobenius, in an abstract setting, of Sylow's theorem, which was proved by Sylow in 1872 for permutation groups. This was done in 1887, in a paper entitled "Neuer Beweis Sylowschen Satzes." Although Frobenius admits that the fact that every finite group can be represented by a group of permutations proves that Sylow's theorem must hold for all finite groups, he nevertheless wishes to establish the theorem abstractly:

"Since the symmetric group, which is introduced in all these proofs, is totally alien to the context of Sylow's theorem, I have tried to find a new derivation of it. . . ." (For a case study of the evolution of abstraction in group theory in connection with Sylow's theorem see [**28**] and [**32**].)

Hölder was an important contributor to abstract group theory, and was responsible for introducing a number of group-theoretic concepts abstractly. For example, in 1889 he introduced the abstract notion of a quotient group (the "quotient group" was first seen as the Galois group of the "auxiliary equation", later as a homomorphic image and only in Hölder's time as a group of cosets), and "completed" the proof of the Jordan-Hölder theorem, namely, that the quotient groups in a composition series are invariant up to isomorphism (see section 2(a) for Jordan's contribution). In 1893, in a paper on groups of order p^3, pq^2, pqr, and p^4, he introduced abstractly the concept of an automorphism of a group. Hölder was also the first to study simple groups abstractly. (Previously they were considered in concrete cases—as permutation groups, transformation groups, and so on.) As he says [**29**, p. 338]. "It would be of the greatest interest if a survey of all simple groups with a finite number of operations could be known." (By "operations" Hölder meant elements.) He then goes on to determine the simple groups of order up to 200.

Other typical examples of studies in an abstract setting are the papers by Dedekind and G. A. Miller in 1897/1898 on Hamiltonian groups–i.e., nonabelian groups in which all subgroups are normal. They (independently) characterize such groups abstractly, and introduce in the process the notions of the commutator of two elements and the commutator subgroup (Jordan had previously introduced the notion of commutator of two permutations).

The theory of group characters and the representation theory for finite groups (created at the end of the 19th century by Frobenius and Burnside/Frobenius/Molien, respectively) also belong to the area of abstract group theory, as they were used to prove important results about abstract groups. See [**17**] for details.

Although the abstract group *concept* was well established by the end of the 19th century, "this was not accompanied by a general acceptance of the associated method of presentation in papers, textbooks, monographs, and lectures. Group-theoretic monographs based on the abstract group concept did not appear until the beginning of the 20th century. Their appearance marked the birth of abstract group *theory*" (Wussing, [**33**]).

The earliest monograph devoted entirely to abstract group theory was the book by J. A. de Séguier of 1904 entitled *Elements of the Theory of Abstract Groups* [**27**]. At the very beginning of the book there is a set-theoretic introduction based on the work of Cantor: "De Séguier may have been the first algebraist to take note of Cantor's discovery of uncountable cardinalities" (B. Chandler and W. Magnus, [**7**]). Next is the introduction of the concept of a semigroup with two-sided cancellation law and a proof that a finite semigroup is a group. There is also a proof, by means of counterexamples, of the independence of the group postulates. De Séguier's book also includes a discussion of isomorphisms, homomorphisms, automorphisms, decomposition of groups into direct products, the Jordan-Hölder theorem, the first isomorphism theorem, abelian groups including the basis theorem, Hamiltonian groups, and finally, the theory of p-groups. All this is done in the abstract, with "concrete" groups relegated to an appendix. "The style of de Séguier is in sharp contrast to that of Dyck. There are no intuitive considerations. . . and there is a tendency to be as abstract and as general as possible. . . " (Chandler and Magnus, [**7**]).

De Séguier's book was devoted largely to finite groups. The first abstract monograph on group theory which dealt with groups in general, relegating finite groups to special chapters, was O. Schmidt's *Abstract Theory of Groups* of 1916 [**26**]. Schmidt, founder of the Russian school of group theory, devotes the first four chapters of his book to group properties common to finite and infinite groups. Discussion of finite groups is postponed to chapter 5, there being ten chapters in all. See [**7**], [**10**], [**33**].

5. Divergence of developments in group theory

Group theory evolved from several different sources, giving rise to various concrete theories. These theories developed independently, some for over one hundred years (beginning in 1770) before they converged (early 1880's) within the abstract group concept. Abstract group theory emerged and was consolidated in the next thirty to forty years. At the end of that period (around 1920) one can discern the divergence of group theory into several distinct "theories." Here is the barest indication of *some* of these advances and new directions in group theory, beginning in the 1920's (with contributors and approximate dates):

(a) Finite group theory. The major problem here, already formulated by Cayley (1870's) and studied by Jordan and Hölder, was to find all finite groups of a given order. The problem proved too difficult and mathematicians turned to special cases (suggested especially by Galois theory): to find all simple or all solvable groups (cf. the Feit-Thompson theorem of 1963, and the classification of all finite simple groups in 1981). See [**14**], [**15**], [**30**].

b) Extensions of certain results from finite group theory to infinite groups with finiteness conditions; e.g., O. J. Schmidt's proof, in 1928, of the Remak-Krull-Schmidt theorem. See [**5**].

c) Group presentations (Combinatorial Group Theory), begun by von Dyck in 1882, and continued in the 20th century by M. Dehn, H. Tietze, J. Nielsen, E. Artin, O. Schreier, et al. For a full account, see [**7**].

d) Infinite abelian group theory (H. Prüfer, R. Baer, H. Ulm et al.—1920's to 1930's). See [**30**].

e) Schreier's theory of group extensions (1926), leading later to the cohomology of groups.

f) Algebraic groups (A. Borel, C. Chevalley et al.—1940's).

g) Topological groups, including the extension of group representation theory to continuous groups (Schreier, E. Cartan. L. Pontrjagin, I. Gelfand, J. von Neumann et al.—1920's and 1930's). See [**4**].

FIGURE 1 gives a diagrammatic sketch of the evolution of group theory as outlined in the various sections and as summarized at the beginning of this section.

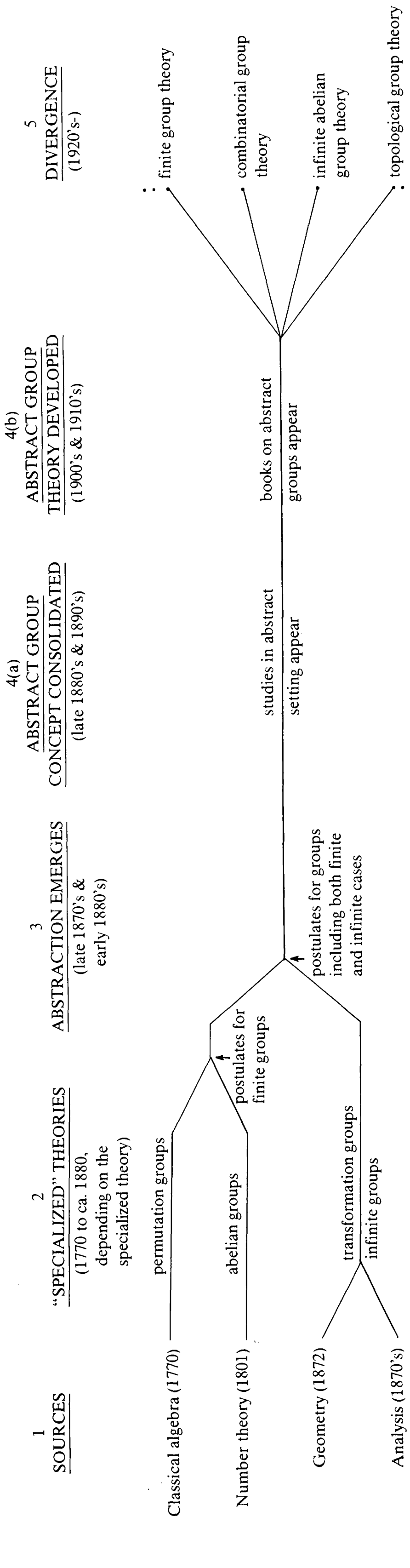

1
SOURCES
2
"SPECIALIZED" THEORIES
(1770 to ca. 1880, depending on the specialized theory)
3
ABSTRACTION EMERGES
(late 1870's & early 1880's)
4(a)
ABSTRACT GROUP CONCEPT CONSOLIDATED
(late 1880's & 1890's)
4(b)
ABSTRACT GROUP THEORY DEVELOPED
(1900's & 1910's)
5
DIVERGENCE
(1920's-)
Classical algebra (1770)
Number theory (1801)
Geometry (1872)
Analysis (1870's)
permutation groups
abelian groups
transformation groups
infinite groups
postulates for finite groups
postulates for groups including both finite and infinite cases
studies in abstract setting appear
books on abstract groups appear
finite group theory
combinatorial group theory
infinite abelian group theory
topological group theory

FIGURE 1

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