# How Inge Lehmann Discovered the Inner Core of the Earth

*Christiane Rousseau*

**Christiane Rousseau** (rousseac@DMS.UMontreal.CA) is a professor at the University of Montreal, past President of the Canadian Mathematical Society (2002–2004), and currently Vice-President of the International Mathematical Union (until 2014). Her research is in dynamical systems. A passionate popularizer of mathematics, she is the initiator and coordinator of the special year *Mathematics of Planet Earth 2013*. Her interest in pre-service teacher education led her to write the book *Mathematics and Technology*, jointly with Yvan Saint-Aubin. She writes, "One of my pleasures when working for MPE2013 is that I learn new applications on a regular basis. I knew that Inge Lehmann discovered that the inner core of the Earth was solid, and I wanted to understand the idea, so I read her paper. It is not necessarily easy to read a paper in another discipline, here geophysics. Fortunately, near the end, Lehmann sketches a simple model of the Earth, which she used to sell her discovery. I was stimulated to fill in the mathematical details and missing calculations. I found them sufficiently interesting to share."

It is not easy to study the interior of the Earth. We cannot dig and see with our eyes. All observations are indirect, and what we cannot see directly, we must see with *mathematical eyes*. As an example, the Greek mathematician Erastosthenes knew that the Earth was a sphere and could estimate its radius by measuring the angle of the sun from the vertical at different cities (see, for instance, [**2**]). We can estimate the approximate mass of the Earth by measuring the gravitational force at the surface of the Earth as described by Newton's gravitational law, and learn from this that the inner part of the Earth has a much higher density than the surface. Indeed, let $M$ be the mass of the Earth, and $R$ its radius. Newton's law implies that a body of mass $m$ located at the surface of the Earth is attracted by the Earth with a gravitational force of size

$$F = GmM\frac{1}{R^2},$$

where $G$ is the known gravitational constant. On the other hand, the size of this force is $F = mg$, where $g$ is the acceleration at the surface of the Earth, which can be measured to be approximately 9.8 m/s. Putting $GmM\frac{1}{R^2} = mg$ and simplifying $m$, we get $M = \frac{gR^2}{G}$. The mass obtained is much too large for the average density of 3,000 $\text{kg/m}^3$ observed near the surface of the Earth, implying that denser materials exist at deeper levels.

http://dx.doi.org/10.4169/college.math.j.44.5.399
MSC: 00-01, 00A69, 01A60

Exploring the inner structure of the Earth by indirect methods is part of the field of *remote sensing*, on the basis of which there is now a consensus among scientists on the structure of the Earth. The theory of a liquid interior dates to Richard Dixon Oldham at the turn of the 20th century, but it was only in 1936 that Inge Lehmann discovered the inner core. More than 30 years later, Freeman Gilbert and Adam M. Dziewonski established that the inner core is solid by other indirect types of arguments.

Inge Lehmann (1888–1993) had a long, productive working life at the Royal Danish Geodetic Institute. Her scientific education was at the University of Copenhagen, but as a child she went to a co-educational school run by an aunt of Niels Bohr. "No difference between the intellect of boys and girls was recognized," she wrote, (as quoted in [**1**]), "a fact that brought some disappointment later in life when I had to recognize that this was not the general attitude." Denmark, which has few earthquakes, was actually a good place for seismic research, being very far from the South Pacific, with its numerous powerful earthquakes.

To explain her analysis of the internal structure of the Earth, we introduce a model of the Earth and refine it in several steps.

## A spherical model of the Earth

Let us approximate the Earth by a sphere of radius $R$, and suppose that its interior has spherical symmetry, i.e., it has the same structure along any ray through the center. An earthquake occurring somewhere on the surface of the Earth sends seismic waves in all directions. Each tangent line to a seismic wave at the epicenter, together with the center of the Earth, determines a plane. By spherical symmetry, the seismic wave propagates inside that plane, leading to the planar model in Figure 1(a). If the earthquake occurs at the point $(R, 0)$, then we have a family of seismic waves starting at $(R, 0)$ and directed towards the points $(R\cos\theta, R\sin\theta)$, where $R = 6360\,\text{km}$ is the radius of the Earth and $\theta \in [0, 2\pi]$.

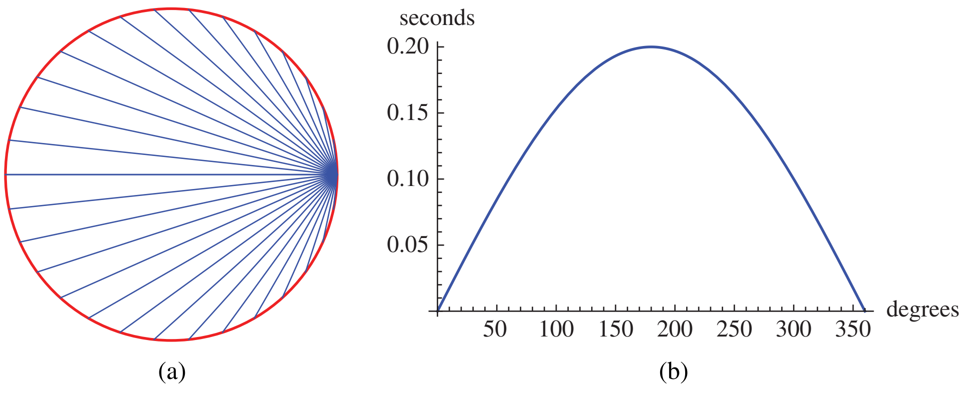



**Figure 1.** The paths and travel times of seismic waves in a homogeneous Earth. (a) The paths of the waves. (b) The travel time of the waves in (a) depending on the angular coordinate of the end point of the wave.

If the interior of the Earth were homogeneous, the seismic waves would propagate along straight lines at constant speed, as in Figure 1(a). We can check that the length of a path starting at $(R, 0)$ and ending at $(R\cos\theta, R\sin\theta)$ is $2R\sin\frac{\theta}{2}$. Near the crust, the speed of seismic waves is approximately 10 km/s, which yields the paths and travel times of Figure 1. *Any deviation from these observations tells us that the hypothesis of a uniform Earth is not valid*, and we need to refine the model.

Geophysics is a complex field, and there are a lot of phenomena that need to be taken into account. For instance, several types of seismic waves are generated by an earthquake. Two types of waves propagate inside the Earth: The $P$-waves are longitudinal *pressure waves*, and the $S$-waves are transversal *shear waves*. The $S$-waves do not travel in liquids. Since they are not detected far from the epicenter of an earthquake, Richard Dixon Oldham concluded that they are stopped by a liquid interior of the Earth.

We now come to the simple model proposed by Inge Lehmann. The Earth is a flattened ellipsoid of revolution. Nonetheless, we limit ourselves to the spherical approximation. In Lehmann's model, the interior of the Earth has three main strata: the mantle to a depth of 2890 km, the outer core to a depth of 5150 km, and finally, the inner core. There are more strata in practice, but these do not significantly change this rough picture.

## Propagation of seismic waves inside the Earth

In a nonuniform Earth, the speed of propagation of a seismic wave varies. This also changes the direction of the wave, following the Snell–Descartes law of refraction.

**Laws of refraction and reflection.** Consider a beam of light, as it travels through a uniform material with speed $v_1$ and enters another uniform material, where it travels with speed $v_2$. Let $\theta_1$ be the angle of the beam of light through the first material measured from the perpendicular of the interface between the materials. Similarly, let $\theta_2$ be the angle of the beam of light through the second material measured from the same perpendicular (see Figure 2(a)). The *law of refraction* states that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

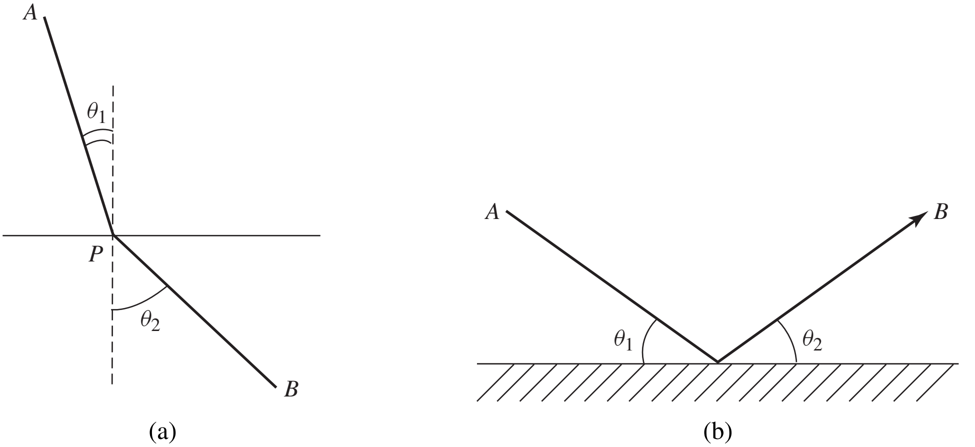



**Figure 2.** The laws of refraction and reflection. (a) The law of refraction. (b) The law of reflection.

If $\theta_1$, $v_1$, and $v_2$ are known, then we can calculate $\theta_2$ by means of

$$\sin\theta_2 = \frac{v_2}{v_1}\sin\theta_1. \tag{1}$$

But if $v_2$ is larger than $v_1$, i.e., the beam of light is coming from the slow side, and $\theta_1$ is sufficiently large, then $\frac{v_2}{v_1}\sin\theta_1 > 1$ and (1) has no solution. What then? The beam of light is reflected as in the *law of reflection* (see Figure 2(b)). Fermat's principle unifies these two laws, and explains why a beam of light that cannot be refracted is reflected:

**Fermat's principle.** *The path followed by a beam of light traveling from a point A to a point B minimizes the travel time from A to B.*

This principle allows us to compute the travel path of a beam of light in nonhomogeneous media. The mathematical techniques are part of the beautiful field of *calculus of variations*, but we won't go into these details. The curious reader can consult the relevant chapter in [**4**].

## An Earth with a mantle and a core

In Inge Lehmann's time, it was admitted that a mantle approximately 2,890 km thick would surround a core. Since the radius of the Earth is approximately 6,360 km, this gives a core with radius 5/9 of that of the Earth. Following Lehmann, we take a speed of 10 km/h for the seismic waves in the mantle and of 8 km/h in the core, and we compute the travel paths of the seismic waves and their travel time.

Consider an earthquake occurring at one point on the surface of the Earth and sending seismic waves in all directions. As before, we limit ourselves to a planar model (see Figure 3). Suppose that the earthquake occurs at the point $(R, 0)$, and consider a seismic wave starting at $(R, 0)$ and directed toward the point $(R\cos\theta, R\sin\theta)$. For the travel time, we only plot the values of $\theta \in [0, \pi]$. Let $\tau$ be the value of $\theta$ in degrees, and $\phi(\tau)$ be the angular coordinate where the wave reaches the surface of the Earth. The wave starting at $(R, 0)$ is tangent to the core if $\tau = 112°$. So the waves follow

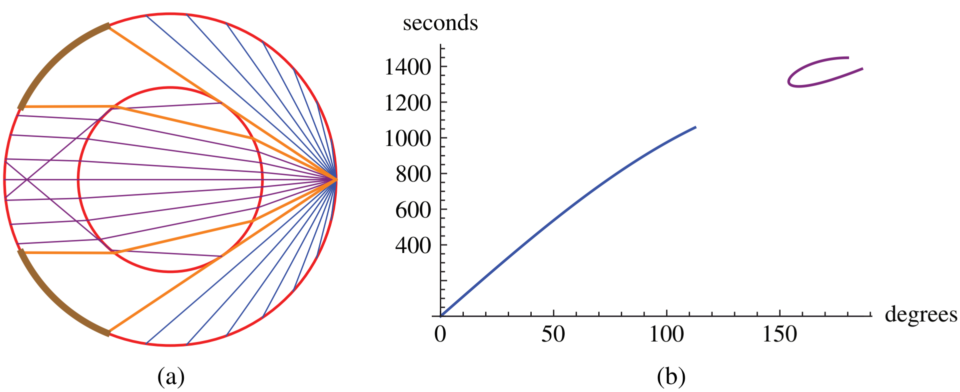



**Figure 3.** The paths and travel times of the seismic waves in the Earth, for a speed of 10 km/s inside the mantle and of 8 km/h inside the core. Note that refracted waves can intersect. (a) The paths of the waves. (b) The travel time of the waves in (a) depending on the angular coordinate of the end point of the wave.

straight lines for $\tau \in [0, 112°]$. For a higher value of $\tau$, the wave is refracted when entering the core. The refracted wave corresponding to $\tau = 112°$ reaches the surface of the Earth at $\phi(\tau) = 186°$. The refracted waves may intersect each other, and reach the Earth's surface at points corresponding to $\phi(\tau) \in [154°, 186°]$. No wave is detected satisfying $\phi(\tau) \in [112°, 154°]$, and there are some values of $\phi(\tau) \in [154°, 180°]$ that correspond to distinct waves with different travel times.

If you are intrigued by the special form of the right part of the curve in Figure 3(b), the explanation is found in Figure 4. We consider a wave starting at $(R, 0)$ and directed toward $R(\cos\tau, \sin\tau)$, where the angle $\tau$ (in degrees) satisfies $\tau \in [112°, 180°]$, i.e., the wave is refracted. Figure 4(a) graphs the travel time $T(\tau)$ of this wave until it reaches the surface of the Earth at a point $R(\cos\phi(\tau), \sin\phi(\tau))$. Figure 4(b) graphs $\phi(\tau)$, and Figure 4(c) is the parametric plot $(\phi(\tau), T(\tau))$.

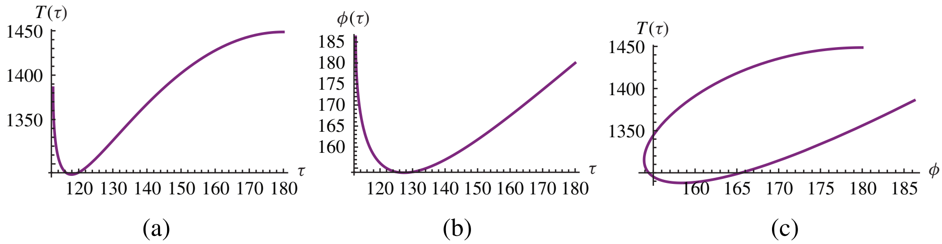



**Figure 4.** Travel time and exit angles of refracted waves. (a) Travel time $T(\tau)$ as a function of $\tau$. (b) Exit angle $\phi(\tau)$ as a function of $\tau$. (c) Parametric plot $(\phi(\tau), T(\tau))$.

Can some waves be reflected off the inner sphere? No. A wave arriving from the earthquake makes an angle $\theta_1$ with the normal to the inner sphere, as in Figure 5. Since it is on the fast side, it is necessarily refracted, and the refracted wave makes an angle $\theta_2$ with the normal to the sphere. It will again intersect the inner sphere with the same angle $\theta_2$ with the normal at a second intersection point, and hence be refracted outside, making the same angle $\theta_1$ with the normal.

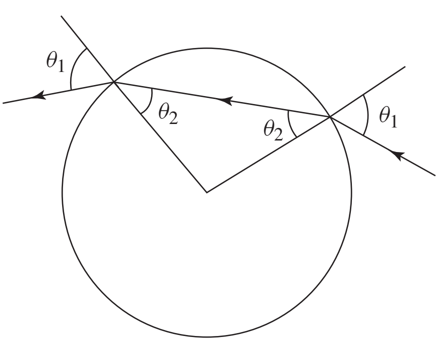


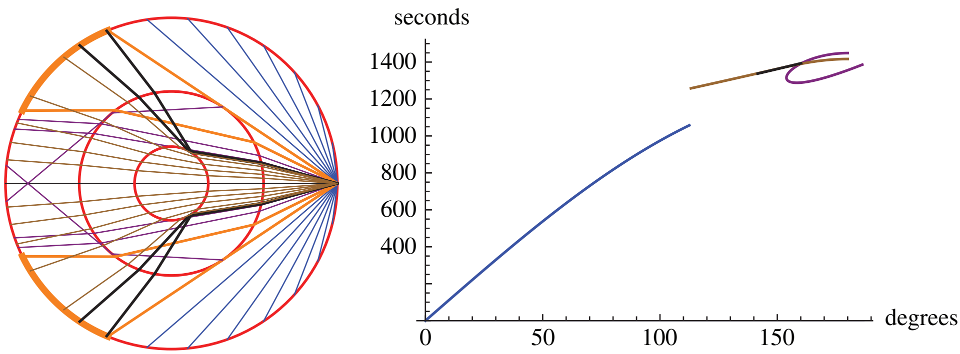

**Figure 5.** The symmetry of the travel path of a refracted wave through the inner sphere.

## The inner core

But what if we detect waves for some values of $\tau \in [112°, 154°]$ and measure their travel time? Then our model has a flaw, and we must correct it so that it fits the observed data. We now face an *inverse problem*: understanding the inner structure of the

Earth from the travel times of waves at different locations of the Earth. Inge Lehman deduced that the core was not homogeneous, and that there is a smaller inner core surrounded by an outer core. The wave travels faster in the inner core. Hence, the wave can be reflected off the inner core if it arrives too tangentially.

Here, we limit ourselves to the direct problem and complete Inge Lehmann's model. To Figure 3, we add an inner core whose radius is approximately $2/9$ that of the Earth, and suppose that the travel speed of the waves in the inner core is 8.8 km/h (see Figure 6). These parameters are the result of Lehmann's inverse analysis. All we do is compute explicitly the travel paths of the different waves depending on their departure angle. These waves can be reflected *or* refracted when they change layer. Each step is easily computed with elementary Euclidean and analytic geometry, but putting all the pieces together for several waves requires some computational assistance. Our figures were prepared using *Mathematica*. Instead of presenting all details, we just show the elementary steps so you can program it yourself.




(a) (b)

**Figure 6.** Completing Lehmann's model. (a) The paths of the waves: The paths between the two black lines are reflected on the inner core, the others are refracted. (b) The travel time of the waves in (a) depending on the angular coordinate of the end point of the wave.

If you think that it is difficult to draw Figure 6, you will be surprised at how elementary the first steps are. After a few of them, you won't need further explanations.

## Computing the waves and their intersection with the circles

We start with three circles of radius $R$, $\frac{5}{9}R$, and $\frac{2}{9}R$, which we call the large circle, middle circle, and small circle, respectively. To simplify, we can of course suppose that $R = 1$. We also suppose that the epicenter of the earthquake is at $(1, 0)$.

We will need to work with the equation of lines supporting segments of travel paths. It is not a good idea to work with the standard form, $y = ax + b$, of the equation of a line, since some waves will be reflected or refracted vertically. A better choice is to use parametric equations. For instance, waves from the epicenter travel along the lines $\{(1 + t\cos\gamma, t\sin\gamma) \mid t \in \mathbb{R}\}$. One such line intersects the middle circle at $t$ such that $x(t)^2 + y(t)^2 = \frac{25}{81}$. This yields a quadratic equation in $t$ with two positive solutions, $0 < t_1 \le t_2$, when the line intersects the circle, i.e., $\gamma \in [\theta_0, 2\pi - \theta_0]$, where $\theta_0 \approx 1.96353$, corresponding to $\tau_0 = 112°$. We need the smallest solution, $t_1$, corresponding to the intersection point closest to $(1, 0)$. This gives the point $B$ in Figure 7, and its angular coordinate $\epsilon$. The next step is to find the equation of the refracted wave through $B$.

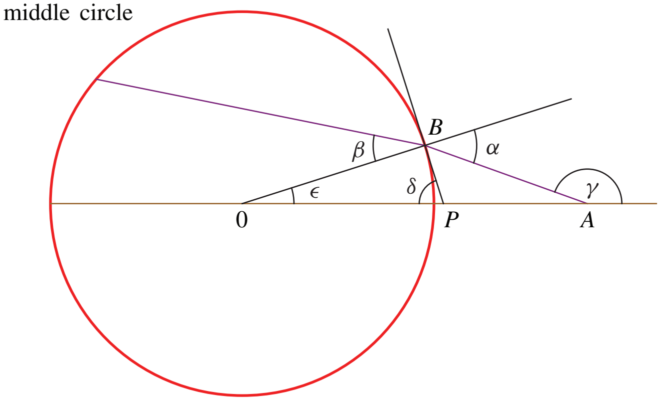



**Figure 7.** Computing the path of a refracted wave.

## Computing a refracted wave

For this purpose, look at Figure 7. Our given angle is $\gamma$, and we calculated $\epsilon$, having found $B$. Now, since $PB$ is orthogonal to $OB$, $\delta = \frac{\pi}{2} - \epsilon$. We need to calculate $\alpha = \frac{\pi}{2} - \angle PBA$. In the triangle $BPA$, the two other angles are $\pi - \delta$ and $\pi - \gamma$. Hence, $\angle PBA = \delta + \gamma - \pi$, yielding $\alpha = \pi + \epsilon - \gamma$. Using the law of refraction, we calculate $\beta$ given that $\sin \beta = \frac{v_2}{v_1} \sin \alpha$, where $v_1$ (resp. $v_2$) is the speed of the wave outside (resp. inside) the middle circle. We can verify that the reflected wave makes an angle of $\pi + \epsilon - \beta$ with the horizontal right semi-axis, allowing us to obtain its parametric equation.

Once inside the middle circle, there are three possibilities for the refracted wave:

1. It can exit the middle circle without touching the small circle. We must calculate its intersection with the middle circle as above, and find the parametric equation of the refracted wave, using the symmetries of Figure 5.
2. It can be refracted when entering the small circle: The parametric equation of the refracted wave can be calculated as above.
3. It can be reflected on the inner circle: We make the computation for this case below.

## Computing a reflected wave

Consider Figure 8. We are given $\gamma$, and $\epsilon$ has been calculated. As before, $\delta = \frac{\pi}{2} - \epsilon$. Check that $\alpha = \gamma - \epsilon - \frac{\pi}{2}$, and that the reflected wave makes an angle of $\frac{\pi}{2} + \epsilon - \alpha$ with the horizontal right semi-axis.

Of course this is not the end of the game. We must iterate these steps. There is no special difficulty, but the calculations are a bit tedious.

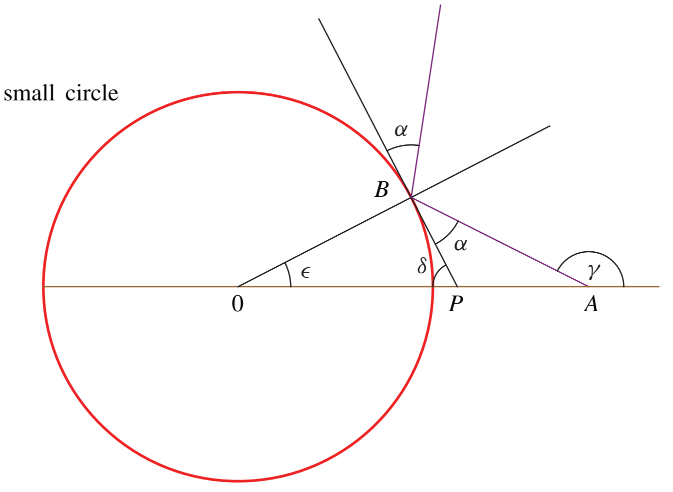



**Figure 8.** Computing the path of a reflected wave.

## Discussion of the model

In Figure 6(b), the travel times of the waves reflected by the inner core are very similar to some travel times of waves refracted inside the inner core. This is a coincidence, arising from the particular traveling speeds in this case. Hence, we cannot distinguish the two types of waves from their travel time alone. This does not exclude that other criteria (for instance, intensity of the waves) or more sophisticated methods of signal analysis might distinguish them.

The rough model of the inner structure of the Earth is now presented in Figure 9. According to Inge Lehmann, she built this model to illustrate that an inner core, in which the waves traveled faster, would explain the waves detected in the forbidden region. The idea was accepted by Beno Gutenberg and Charles Francis Richter (the creator of the Richter magnitude scale), seismologists at the California Institute of Technology. They placed a small inner core inside the Earth and adjusted its radius and the traveling speed of waves inside the inner core until the calculated time curves agreed with the data observed. Lehmann's model also led to the observation of the

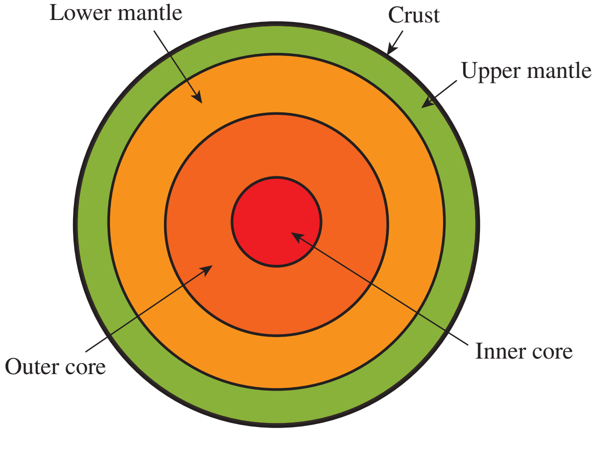



**Figure 9.** The inner structure of the Earth.

upper branch of refracted waves in Figure 3(b). Indeed, the intensity of waves in this upper branch is small, and the branch first went unnoticed.

According to this description, analyzing the inner structure of the Earth seems fairly simple. When dealing with real data, however, things were not so simple. There was a reasonable network of seismic stations in Europe, but not everywhere. Additionally, the work we have presented relies on careful location of the epicenter of the earthquake, but that also requires precise observations well distributed around the epicenter, which was rarely the case. Indeed, how do you locate the epicenter?

## Locating the epicenter

We have a system of four equations to solve. The four unknowns are the three coordinates, $(x, y, z)$, of the position of the earthquake and the time of occurrence, $t$, of the earthquake. Observe the times when the seismic waves are registered at $n$ different stations located not too far apart, so that we can assume the speed of the waves, $v$, to be constant along straight line travel paths. From these times, we derive a system of $n$ equations in the four unknowns $(x, y, z, t)$. If the $i$th station is located at $(a_i, b_i, c_i)$ and registered the earthquake at time $t_i$, then the distance $d_i$ from the epicenter to the station is

$$d_i = \sqrt{(x-a_i)^2 + (y-b_i)^2 + (z-c_i)^2}.$$

It is also equal to the speed multiplied by the travel time of the wave, which is $t_i - t$, namely $d_i = v(t_i - t)$. This yields a system of $n$ quadratic equations:

$$(x-a_i)^2 + (y-b_i)^2 + (z-c_i)^2 = v^2(t_i - t)^2, \quad i = 1, \dots, n.$$

A minimum of four equations is necessary to obtain a finite number of solutions, in this case two solutions. More equations are needed if there is no obvious way to discard the wrong solution. Details on how to solve such systems can be found in [**4**], which discusses the functioning of a GPS. Of course, an additional hypothesis is that the clocks of the stations are well synchronized! Returning to Inge Lehmann, strong earthquakes usually do not occur near the European seismic stations and, according to her [**3**], more subtle adjustments had to be made.

## Final words

It took a few years before the idea of the inner core was finally accepted by the community of seismologists. According to Nils Groes (quoted in [**1**, p. 106]), Lehmann once said, “You should know how many incompetent men I had to compete with—in vain.” In a milder vein, we cite the last sentences of [**3**]: “The first results for the properties of the inner core were naturally approximate. Much has been written about it, but the last word has probably not yet been said.”

**Acknowledgment.** This work was supported by NSERC in Canada.

**Summary.** The mathematics behind Inge Lehmann’s discovery that the inner core of the Earth is solid is explained using data collected around the Earth on seismic waves and their travel time through the Earth.

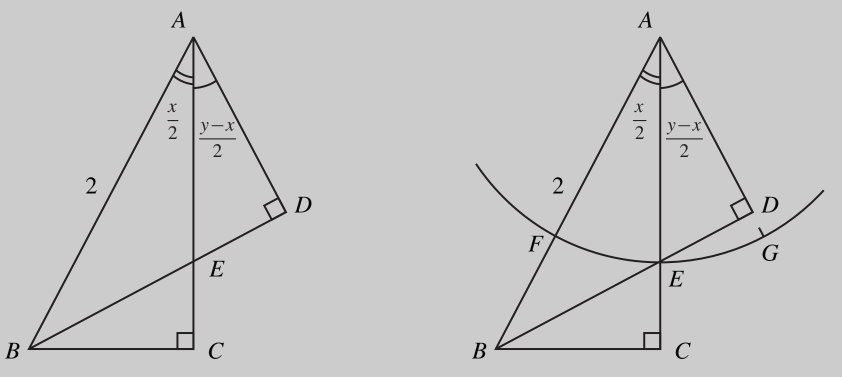
## References

1. B. A. Bolt, Inge Lehmann (1888–1993) in *Out of the Shadows: Contributions of Twentieth Century Women to Physics*, N. Byers and G. Williams, eds., Cambridge University Press, 2006; available at http://www.physics.ucla.edu/~cwp/articles/bolt.html.
2. Wikipedia contributors, Erastosthenes, *Wikipedia, The Free Encyclopedia*; available at http://en.wikipedia.org/wiki/Eratosthenes.
3. Inge Lehmann, Seismology in the Days of Old, *Eos, Transactions American Geophysical Union*, **68** (3), 1987, 33–35; available at http://www.geus.dk/geuspage-dk.htm.
4. C. Rousseau and Y. Saint-Aubin, *Mathematics and Technology*, Springer, New York, 2008.


## Proof Without Words: Monotonicity of $\frac{\sin x}{x}$ on $(0, \pi/2)$


Xiaoxue H. Li, Department of Mathematics, Emory & Henry College, Emory, VA 24327, xli@ehc.edu


$$0 < x < y < \pi/2 \Rightarrow \frac{\sin x}{x} > \frac{\sin y}{y}.$$




$|\triangle ABC| = \sin x, \quad |\triangle ABD| = \sin y$

Let $0 < x < y < \pi/2$.

$$\left.\begin{array}{l} \dfrac{\sin y}{\sin x} = \dfrac{|\triangle ABD|}{|\triangle ABC|} < \dfrac{|\triangle ABD|}{|\triangle ABE|} = 1 + \dfrac{|\triangle AED|}{|\triangle ABE|} \\[2ex] \dfrac{|\triangle AED|}{|\triangle ABE|} < \dfrac{|\widehat{AEG}|}{|\widehat{AFE}|} = \dfrac{(y-x)/2}{x/2} = \dfrac{y}{x} - 1 \end{array}\right\} \Rightarrow \frac{\sin y}{\sin x} < \frac{y}{x} \Rightarrow \frac{\sin x}{x} > \frac{\sin y}{y}.$$


**Summary.** A visual proof that $\sin x/x$ is monotonically increasing on $(0, \pi/2)$. For $\tan x/x$, see p. 420.



http://dx.doi.org/10.4169/college.math.j.44.5.408
MSC: 26A48, 33B10