Geometry and the Foucault Pendulum

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Nature uses only the longest thread to weave her patterns, so each small piece of fabric reveals the organization of the entire tapestry.

—Richard P. Feynman

§1. INTRODUCTION. In 1851 Jean Foucault (1819–1868) built a pendulum consisting of a heavy iron ball on a wire 200 feet long to demonstrate the rotation of the Earth (see Figure 1a and Figure 1b). Foucault observed that such rotation would cause the swing-plane of the pendulum to precess, or rotate, as time went on, eventually returning to its original direction after a period of $T = 24/\sin \upsilon_0$ hours (where υ_0 denotes the latitude where the experiment takes place).

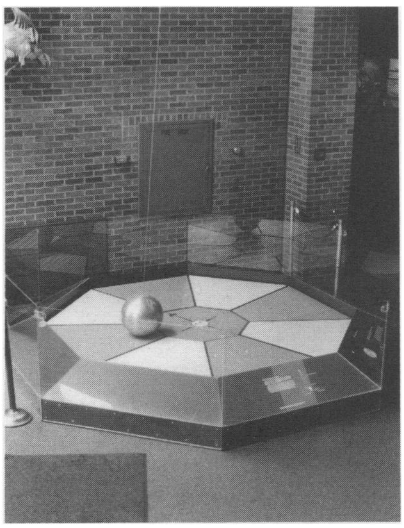

Figure 1a

In a recent *New York Times* interview **[Ang]**, the distinguished scientist and author Stephen Jay Gould proclaimed, "I've never understood why every science museum in the country feels compelled to have one of these [a Foucault pendulum]. I still don't understand how they work and I don't think most visitors do either." Gould is exactly right. Non-physicists generally have only the vaguest notion of how the behavior of the pendulum relates to the rotation of the Earth. The usual quite complicated analysis of this phenomenon of precession is in terms of rotating reference frames and the Coriolis force (see **[Sym]** and **[Arn]**). While these notions are part of elementary mechanics, they are not widely known among even mathematically aware non-physics students.

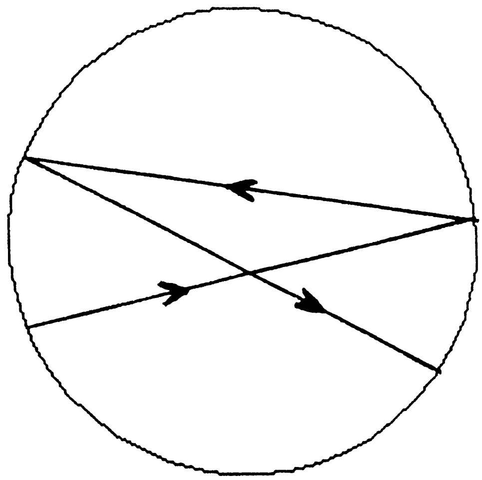

Figure 1b. Path of the Pendulum

The purpose of this article is to present the behavior of the Foucault pendulum as a simple consequence of doing Calculus on the sphere. This *holonomy* approach to the pendulum is mentioned in [**W-S**] and [**Mar** p. 16], but the details *in terms of elementary Calculus* do not seem to be well known. We believe this analysis of the Pendulum deserves a wide audience because it provides a beautiful down-to-'Earth' example of mathematical modelling in the context of Geometry and Calculus.

While we only discuss the pendulum, the geometric concept of holonomy makes its presence felt in applied mathematics from optimal control to quantum mechanics (cf. [**En1**], [**En2**] and [**W-S**]). It is hoped that the mathematical description of the Foucault pendulum presented here will spur interest in applications of Differential Geometry and will be accessible to any student acquainted with multivariable calculus and a touch of linear algebra.

§2. THE SPHERE. Our first step in analyzing Foucault's pendulum is to understand the geometry of the sphere. Consider a sphere (denoted by S^2) of radius R with patch

$$x(u,v) = (R\cos u\cos v, R\sin u\cos v, R\sin v),$$

where $0 \le u \le 2\pi$ and $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$. By 'patch' we mean a system of coordinates on the sphere, such as spherical coordinates (ρ, θ, ϕ) with a fixed radius $\rho = R$. Note however that our patch differs from spherical coordinates in that v represents the latitude on the sphere; that is, the angle *up* from the equator, *not down* from the North Pole (see Figure 2).

The patch x has two special families of curves associated to it: the *longitudes* $\beta(v) = x(u_0, v)$ obtained by setting u equal to a constant and the *latitudes* $\alpha(u) = x(u, v_0)$ obtained by setting v equal to a constant. Since these curves are in $\mathbb{R}^3$, their tangent vectors α' and β' are given by differentiating each coordinate of their expressions. For latitude and longitude tangent vectors respectively, we have

$$\alpha' = (-R\sin u\cos v_0, R\cos u\cos v_0, 0),$$

$$\beta' = (-R\cos u_0\sin v, -R\sin u_0\sin v, R\cos v).$$

Note that the dot product $\alpha' \cdot \beta'$ is zero, so that α' and β' are perpendicular (or

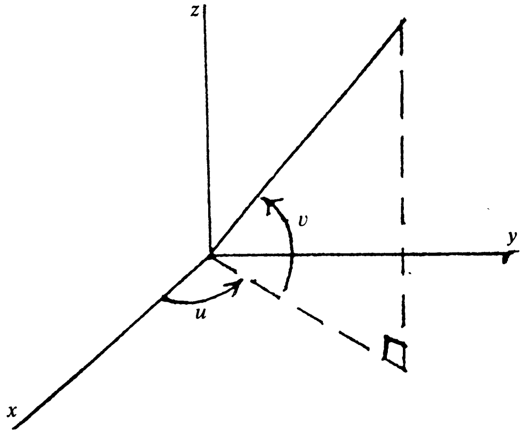

Figure 2

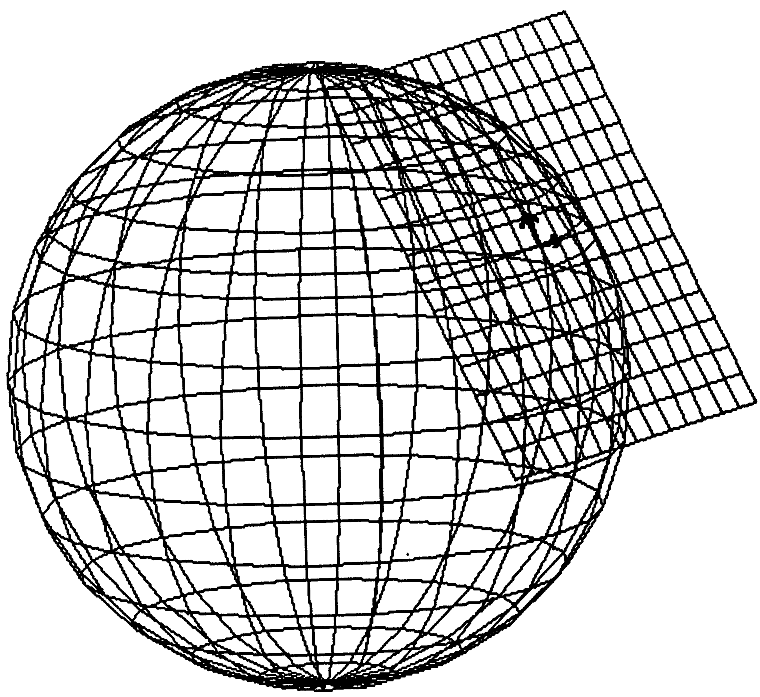

Figure 3. Tangent Plane with Basis E1, E2

orthogonal) for all u and v. In particular, α' and β' form a *basis* for the tangent plane T_pS^2 where $p = x(u_0, v_0)$. That is, every tangent vector w at $x(u, v)$ may be written in a unique way as $w = A\alpha' + B\beta'$ for some real numbers A and B (see Figure 3).

This basis for the tangent plane may be extended to a basis for $\mathbb{R}^3$ itself by taking a vector perpendicular to both α' and β'; namely, the cross product $\alpha' \times \beta'$.

In fact, things become simpler if we take unit vectors in the directions of α', β' and $\alpha' \times \beta'$ obtained by dividing these vectors by their lengths $|\alpha'|$, $|\beta'|$ and $|\alpha' \times \beta'|$. The vectors of our basis are now,

$$E_1 = \frac{\alpha'}{|\alpha'|} = (-\sin u, \cos u, 0) \qquad E_2 = \frac{\beta'}{|\beta'|} = (-\cos u \sin v, -\sin u \sin v, \cos v)$$

and

$$U = \frac{\alpha' \times \beta'}{|\alpha' \times \beta'|} = (\cos u \cos v, \sin u \cos v, \sin v).$$

The basis $\{E_1, E_2, U\}$ provides a framework for comparing Euclidean geometry of $\mathbb{R}^3$ to geometry seen from the perspective of a 2-dimensional resident of the sphere. Because the perceptions of such a person are restricted to the 2-dimensional space spanned by E_1 and E_2, any event or object in $\mathbb{R}^3$ is 'seen' by the resident of the sphere only through its projection onto the tangent plane. In particular, a vector w in $\mathbb{R}^3$ may be written uniquely as

$$w = aE_1 + bE_2 + cU$$

but the resident of the sphere only sees $aE_1 + bE_2$. The viewpoint described here is useful in forming analogies between Euclidean geometry and curved geometry. For example, in $\mathbb{R}^3$ we know that lines, which may be parametrized by $\gamma(t) = p + tv$ for fixed p and v, are shortest paths between points. Further, from the parametrization, it is clear that lines are characterized by having zero acceleration vectors. By analogy, 'shortest paths' (or *geodesics*) on the sphere are characterized by having zero acceleration vectors *as perceived by residents of the sphere*. That is, any curve on the sphere with an acceleration vector entirely in the U-direction is a geodesic. Such curves on the sphere turn out to be the great circles. In the next section we carry this viewpoint further.

§3. PARALLEL VECTORS ON THE SPHERE. What does it mean to say that two tangent vectors on the sphere in different tangent planes are parallel? It definitely cannot mean, in general, that the two vectors are parallel in $\mathbb{R}^3$. For consider a latitude circle on the sphere S^2 at latitude v_0

$$\alpha(u) = (R \cos u \cos v_0, R \sin u \cos v_0, R \sin v_0).$$

It is easy to compute that, in $\mathbb{R}^3$, $\alpha'(0)$ may be written as

$$\alpha'(0) = -R \sin v_0 \cos v_0 \ E_2\left(\frac{\pi}{2}, v_0\right) + R \cos^2 v_0 \ U\left(\frac{\pi}{2}, v_0\right)$$

with respect to the basis $\{E_1, E_2, U\}$ at $\alpha(\frac{\pi}{2})$. The non-zero U-component shows that no vector of the tangent plane at $\alpha(\frac{\pi}{2})$ is $\mathbb{R}^3$-parallel to $\alpha'(0)$.

One way to compare vectors along a curve $\gamma(t)$ in $\mathbb{R}^3$ is to start with a tangent vector V_0 at $\gamma(0)$ and create a *field* of tangent vectors $V(t)$ at $\gamma(t)$ which is differentiable in t. The rate of change in vectors along γ may then be computed as $(d/dt)V(t)$. Further, we may say that a vector field V is *parallel along* γ if $(d/dt)V(t) = 0$ for all t. Of course this then implies that $V(t) = V_0$, a constant, and this fits with our notion of parallelism in $\mathbb{R}^3$.

We may extend this idea in a simple way to a tangent vector field $V(u)$ along a latitude circle $\alpha(u)$ in S^2 by saying that V is *parallel along* α if $(d/du)V(u)$ has no $E_1(u)$ or $E_2(u)$ components. This means that $(d/du)V(u) = C(u)U(u)$ for all u or, equivalently, that the projection of $(d/du)V(u)$ onto the tangent plane at $\alpha(u)$,

$\mathrm{proj}_{TS^2}(d/du)V(u)$, is *zero*. We may think of this as saying that residents of the sphere see no change in vectors along α. (For readers versed in differential geometry, note that we may avoid the covariant derivative here because α is a constant-length u-parameter curve and $V(u)$ is given in terms of u. Thus, covariant differentiation in $\mathbb{R}^3$, which is coordinatewise directional differentiation, reduces to ordinary differentiation d/du.)

To return to our latitude circle, let $V(u)$ be a parallel vector field along the latitude $\alpha(u)$. (We always assume that vectors are tangent to S^2.) Then we may write $V(u) = A(u)E_1(u) + B(u)E_2(u)$. The first thing we notice is

Lemma. *V has constant length.*

Proof: Because V is parallel, $(d/du)V(u) = C(u)U(u)$ and therefore,

$$\begin{aligned}\frac{d}{du}(V(u)\cdot V(u)) &= 2\frac{d}{du}V(u)\cdot V(u)\\ &= C(u)U(u)\cdot V(u)\\ &= 0.\end{aligned}$$

Since $V \cdot V$ is constant, so is $|V|$. □

From our expression for $V(u)$ we see that we must have $A(u)^2 + B(u)^2 = |V|^2 = L^2$ where L is a constant. Therefore we may write $A(u) = L\cos\theta(u)$, $B(u) = L\sin\theta(u)$ where $\theta(u)$ is the angle from $V(u)$ to $E_1(u)$. We then have

$$V(u) = L\cos\theta(u)E_1(u) + L\sin\theta(u)E_2(u).$$

From this expression it is clear that, in order to compute $(d/du)V(u)$, we must first compute $(d/du)E_1(u)$ and $(d/du)E_2(u)$. We do this coordinatewise.

$$\frac{d}{du}E_1 = (-\cos u, -\sin u, 0) \qquad \frac{d}{du}E_2 = (\sin u \sin v_0, -\cos u \sin v_0, 0).$$

The reader may check that, in terms of the basis $\{E_1, E_2, U\}$ we have

Proposition.

$$\frac{d}{du}E_1 = \sin v_0 E_2 - \cos v_0 U \qquad \frac{d}{du}E_2 = -\sin v_0 E_1.$$

Remark. Note that the Proposition says that neither E_1 nor E_2 are parallel along α.

The second thing we notice is that parallel vector fields always exist. In fact, the proof of this standard (but essential) result tells us precisely how vectors rotate to maintain parallelism.

Theorem. *Let V_0 be a tangent vector at $\alpha(0)$. Then there exists a parallel vector field V along α with $V(0) = V_0$.*

Proof: The expression above for $V(u)$ shows that a prospective parallel vector field V is determined by the angle $\theta(u)$. The condition that V be parallel will translate below into a complete determination of $\theta(u)$, thus constructing the desired V. The product and chain rule give

$$\frac{d}{du}V(u) = -\sin\theta\frac{d\theta}{du}E_1 + \cos\theta\frac{d}{du}E_1 + \cos\theta\frac{d\theta}{du}E_2 + \sin\theta\frac{d}{du}E_2.$$

Using our previous calculations of the derivatives of E_1 and E_2 along α, we obtain

$$\frac{d}{du}V(u) = -\sin\theta\left[\sin v_0 + \frac{d\theta}{du}\right]E_1 + \cos\theta\left[\sin v_0 + \frac{d\theta}{du}\right]E_2 - \cos\theta\cos v_0 U.$$

Because a parallel V cannot have E_1 or E_2 components, and since $\sin\theta$ and $\cos\theta$ cannot be zero simultaneously, we must have $d\theta/du = -\sin v_0$ or

$$\theta(u) = \theta(0) - \int \sin v_0\, du$$
$$= \theta(0) - u\sin v_0.$$

This formula then defines θ and, hence, the parallel vector field V. □

Definition-Proposition. *The angle of rotation as u varies from* 0 *to* 2π *is called the* holonomy *along* α. *By the proof of the Theorem above, the holonomy along* α *is given by*

$$-2\pi\sin v_0.$$

Remark. Of course, all of this may be done in complete generality. Standard (and very good) references on Differential Geometry are **[O'N]**, **[Spi]** and **[DoC]**; general results on parallelism and the *covariant derivative* may be found there.

The calculation of holonomy above says that parallel tangent vectors rotate by $-2\pi\sin v_0$ as they move completely around a latitude circle. Of course, as the

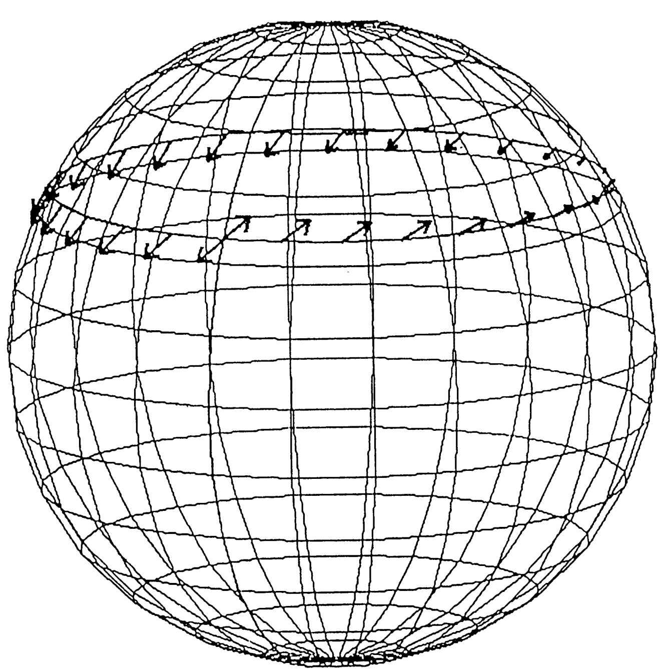

Figure 4. A Parallel Vector Field on the Sphere

terminology 'parallel' signifies, 2-dimensional residents of the sphere see the vectors as parallel—so, from their viewpoint, not rotating at all. This may seem contradictory since the angle between $V(u)$ and $E_1(u)$ is changing with u, but it must be remembered that the vector field E_1 along α *is not parallel*, so any angle change may be attributed to the direction change of E_1. In fact, the product rule guarantees that two *parallel* vector fields along a curve maintain the same angle between their constituent vectors.

Exercise. What happens at the Equator and why is the Equator special among the circles of latitude?

§4. THE FOUCAULT PENDULUM. In order to analyze the Foucault pendulum from the viewpoint of geometry, assume the Earth to be non-rotating and the pendulum to be situated at latitude v_0. Instead of the Earth rotating to move the pendulum, we move the pendulum once around the latitude circle in 24 hours at constant speed on this stationary Earth. This is clearly equivalent to the standard situation. The long cable of the pendulum and the slow progression around the latitude circle have two consequences (which are the usual physics arguments).

First, the long cable provides a relatively small swing for the pendulum which is then approximately flat. Hence, we may consider each swing as a tangent vector to the sphere. By orienting these vectors consistently, we obtain a vector field of *pendulum swing plane directions* V. At each moment of time t there is such a swing direction vector $V(t)$ and all these vectors may be placed along the latitude circle $\alpha(u)$ by associating a given moment of time t with the unique point describing the pendulum's movement along $\alpha(u)$. Hence we write $V(u)$ for the swing plane vector field.

Secondly, because we move around the latitude circle slowly, the consequent centripetal force on the pendulum is negligible ($\simeq 1/290$) compared with the downward force mg. That says that the only force F felt by the pendulum is in the normal direction U. Thus, the vertical swing plane of the pendulum experiences no tangential force and so appears unchanging to a 2-dimensional resident of the sphere. That is, *projected to the tangent plane* TS^2,

$$\mathrm{proj}_{TS^2}\frac{dV(u)}{du} = 0,$$

where the covariant derivative again reduces to the ordinary derivative due to our special parametrization. By our earlier discussion, we then have

Theorem. *The vector field V associated to the Foucault pendulum is parallel along a latitude circle.*

Of course, as we transport the Foucault pendulum once around the latitude circle α, *holonomy* rotates the parallel vector field V by $-2\pi \sin v_0$ radians. In particular, the angular speed of this vector rotation is then $\omega = (2\pi \sin v_0 \text{ rads}/24 \text{ hours})$. The equivalence of our geometric situation with the physical one then gives

Theorem. *The period of the Foucault pendulum's precession is*

$$\frac{2\pi \text{ rads}}{\omega} = \frac{24}{\sin v_0} \text{ hours}.$$

Of course, this is precisely the period obtained in physics. Here, however, the precession of the swing-plane of the Foucault pendulum results from the holonomy along α induced by the curvature of the Earth. Further, since we view the whole pendulum apparatus as stationary relative to the Earth, what can explain the observed precession of the swing-plane? As Foucault argued, we must have

Corollary. *The Earth rotates along its latitude circles.*

Exercise. Suppose a Foucault pendulum is transported around a latitude circle on a torus. (You should still assume the only force is normal to the torus.) Compute the holonomy and explain whether this experiment alone can tell you whether we live on a sphere or torus.

Remark. While we have treated the pendulum because of its relative simplicity, a similar type of analysis can be made for one of the most useful of optimal control devices, the *gyroscope*. Indeed, in 1852 Foucault built a very refined gyroscope whose precession also demonstrated the Earth's revolution. Foucault, in fact, coined the term gyroscope from the Greek *gyros* meaning 'circle' and *skopein* meaning 'to view' because his gyroscope allowed him to see the rotation of the Earth. For more on gyroscopes see [**Sca**] for example.

In its own simple way, this mathematical analysis of the Foucault pendulum epitomizes the physics of the 20^{th} century—a physics which takes a decidedly geometric view of Nature.

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