Consider the sum $2+6+4+7$. When people say that "order does not matter" when computing such a sum they actually mean two things: the order of the individual terms of the sum can be rearranged without affecting the final result (for instance, $7+4+6+2$ and the original sum are sure to each give the same answer, 19) and, moreover, the order in which one chooses to compute the individual addition operations is unimportant (for instance, $((2+6)+4)+7$ and $2+((6+4)+7)$ both yield the same final result of 19). This conclusion relies on the three fundamental beliefs of integer arithmetic:

- Integer addition is **closed**; that is, $a+b$ is itself an integer for all integers a and b.
- Integer addition is **commutative**; that is, $a+b=b+a$ for all integers a and b.
- Integer addition is **associative**; that is, $(a+b)+c=a+(b+c)$ for all integers a, b, and c.

1. Consider the set of all rotations about the origin of the plane.
   [Recall that transformations (e.g., rotations) are functions. As such, for rotations r_α and r_β on $\mathbb{R}^2$, the *composition* of r_α followed by r_β, $r_\beta \circ r_\alpha$ is defined by $r_\beta \circ r_\alpha(P) = r_\beta(r_\alpha(P))$ where $P \in \mathbb{R}^2$.]

   (a) Is this set closed under composition? Explain.

   (b) Do rotations commute with each other under composition? Explain.

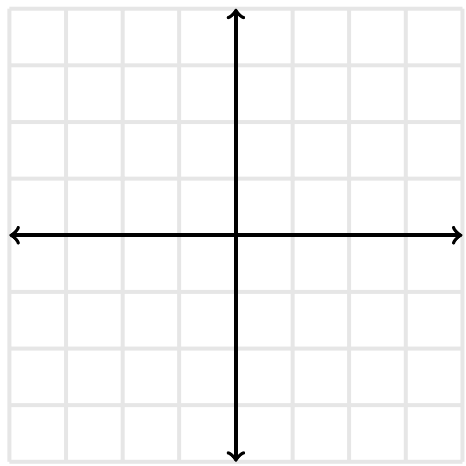
   (c) Do rotations about the origin satisfy the associative law under composition? Explain.

   (d) Does "order matter" when performing a series of rotations about the origin in the plane? Explain.

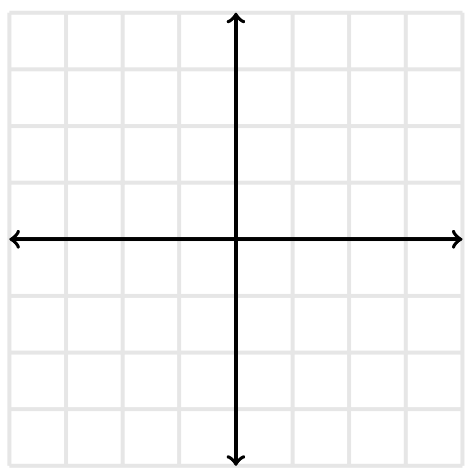
2. Consider the set Σ, given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

   (a) Calculate $A(\frac{\pi}{2})$. Choose three nonzero vectors v_1, v_2, and v_3 in $\mathbb{R}^2$ that are not all scalar multiples of one another. Compute $A(\frac{\pi}{2})v_1$, $A(\frac{\pi}{2})v_2$, and $A(\frac{\pi}{2})v_3$. Sketch all six vectors on the same coordinate plane.



   (b) Repeat the process of 2(a) with $A(\theta)$ for a different nonzero value of θ and the same vectors.

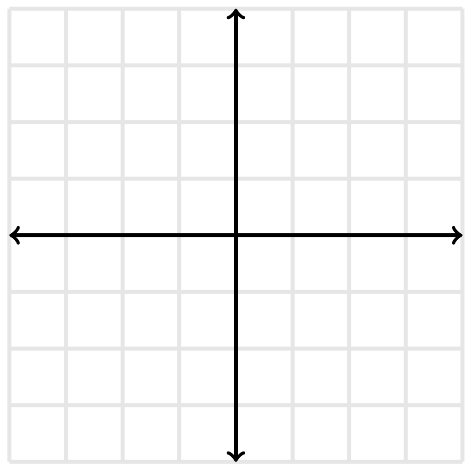


   (c) Write a geometric description of how an arbitrary matrix from Σ acts on vectors in $\mathbb{R}^2$ based on your sketches in Problems 2(a) and 2(b).

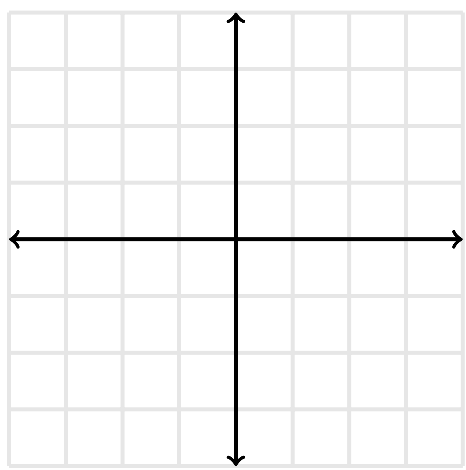
3. Consider the set Φ, given below.

$$\Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

(a) Repeat the process of Problem 2(a) with matrix $B(\frac{\pi}{2})$.



(b) Repeat the process of Problem 3(a) with $B(\theta)$ for a different value of θ and the same vectors.



(c) Write a geometric description of how an arbitrary matrix from Φ acts on vectors in $\mathbb{R}^2$ based on your sketches in Problems 3(a) and 3(b).

4. Recall that 2×2 matrix multiplication is associative. On the other hand, 2×2 matrices do not always commute under matrix multiplication. Give an example of a pair of 2×2 matrices that do commute and a pair of 2×2 matrices that do NOT commute under matrix multiplication.

1. Consider the set Σ given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

Restate the following equations in terms of the geometric effect matrices in Σ have on vectors in $\mathbb{R}^2$. Do not make any calculations—your answers should be written sentences or labeled sketches. A sample solution to part (a) and a partial solution to part (b) are provided.

(a) $\mathrm{A}(\theta_1)[\mathrm{A}(\theta_2)\mathrm{A}(\theta_3)] = [\mathrm{A}(\theta_1)\mathrm{A}(\theta_2)]\mathrm{A}(\theta_3)$

When rotating a vector by three different angles, the way that I group the rotations before doing them does not affect the overall rotation.

(b) $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_1 + \theta_2)$

Rotating a vector by _____ radians and then by _____ radians is equivalent to rotating it by ____________ radians.

(c) $\mathrm{A}(\theta)^{-1} = \mathrm{A}(-\theta)$

(d) $\mathrm{A}(2\pi) = \mathrm{I}$

(e) $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_2)\mathrm{A}(\theta_1)$

2. Explain whether Σ is an abelian group under matrix multiplication by analyzing your responses to Problem 1. Does "order matter" when multiplying together a string of matrices from Σ?

3. Consider the set Φ given below.

$$\Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

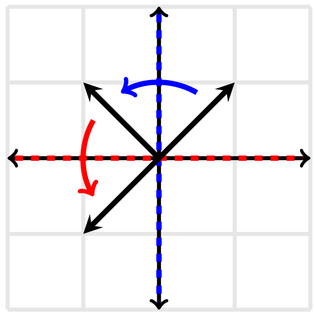
Answer the following questions by considering the geometric effect matrices in Φ have on vectors in $\mathbb{R}^2$. Do not make any calculations—your answers should be written sentences or labeled sketches.

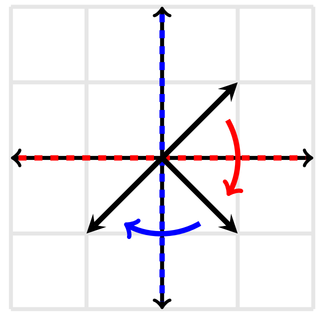
(a) Let $\mathrm{B}(\theta_1)$ be a particular matrix in Φ. Do you think $\mathrm{B}(\theta_1)$ has a multiplicative inverse in Φ? That is, is there a value θ_2 for which $\mathrm{B}(\theta_2) = \mathrm{B}(\theta_1)^{-1}$? Explain why or why not.

(b) Does Φ contain the identity matrix? That is, is there a value θ for which $\mathrm{B}(\theta) = I$? Explain why or why not.

(c) Explain how your answers to Problems 3(a) and 3(b) can be used to determine that Φ is NOT closed under matrix multiplication.

(d) Luisa sketches two diagrams, pictured below, to illustrate two different orders in which an arbitrary vector might be reflected over both the x and y axes.





Luisa then verifies arithmetically that $\mathrm{B}(\frac{\pi}{2})\mathrm{B}(\pi) = \mathrm{B}(\pi)\mathrm{B}(\frac{\pi}{2})$. Based on the diagrams and her supporting calculations, she concludes that elements of Φ commute under matrix multiplication.

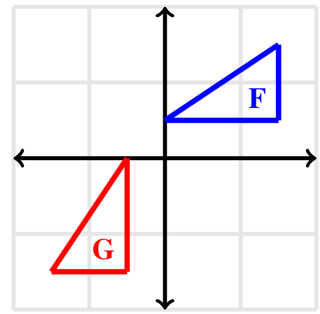
i. Luisa's conclusion is incorrect. What understanding has Luisa demonstrated in her work? What has she proven?

ii. Explain the error in Luisa's reasoning.

iii. Give two questions you could ask Luisa to help her understand her error. Explain why your questions are helpful.

4. Explain whether Φ is a group under matrix multiplication by analyzing your responses to Problem 3. Does "order matter" when multiplying together a string of matrices from Φ?

5. Todd, a high school geometry student, is attempting to show that the two triangles pictured to the right are congruent. To do so, he must use some combination of reflections and rotations to move triangle F on top of triangle G. Todd concludes that he should:



   - Reflect F over the y-axis.
   - Rotate F counterclockwise $90°$ about the origin.

   To move F back to its original position, Todd says he can make these same two transformations in reverse order. That is, once F has been moved to the same position as G, he would:

   - Rotate F counterclockwise $90°$ about the origin.
   - Reflect F over the y-axis.

   (a) Why might Todd expect this procedure to work?

   (b) Explain the error in Todd's reasoning.

   (c) Find a sequence of transformations that will move F back to its original position. Explain, using vocabulary or notation from this course, how you know your steps are correct.

6. It turns out that the set of all rotations and reflections, $\Sigma \cup \Phi$, is itself a group under multiplication (you do not need to prove this). Does "order matter" when multiplying together a string of matrices from $\Sigma \cup \Phi$?

1. Recall that Σ represents the set of rotations about the origin and that Φ represents the set of reflections across lines through the origin. These sets are given below:

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\} \quad \Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

Reasoning geometrically, Jordan finds that $\mathrm{B}(\frac{\pi}{4})v = v$ for any vector v whose vertical and horizontal components are equal.

   (a) What geometric understanding might Jordan have of the set Φ that enabled them to draw this conclusion without making any computations?

   (b) Jordan claims that the work above shows that $\mathrm{B}(\frac{\pi}{4})$ is the identity matrix. Explain the error in Jordan's reasoning.

   (c) Give two questions you could ask Jordan to help them understand their error. Why would your questions be helpful?

2. Consider the operation $\diamond$ given by $a \diamond b = a^{\log(b)}$ on the set of positive real numbers, $\mathbb{R}^+$.

   (a) Is $\diamond$ closed on this set? If so, justify your conclusion. If not, provide a specific example of $a, b \in \mathbb{R}^+$ for which $a \diamond b \notin \mathbb{R}^+$.

   (b) Is $\diamond$ an associative operation on this set? If so, justify your conclusion. If not, provide a specific example of $a, b, c \in \mathbb{R}^+$ for which $a \diamond (b \diamond c) \neq (a \diamond b) \diamond c$.

   (c) Is $\diamond$ a commutative operation on this set? If so, justify your conclusion. If not, provide a specific example of $a, b \in \mathbb{R}^+$ for which $a \diamond b \neq b \diamond a$.

   (d) Does "order matter" under this operation? Explain why or why not.

3. Aisling, a high school student, has made an 84 and a 72 on her first two precalculus assignments. She calculates her average in the course to be a 78. The following day, she receives a 90 on her next assignment. She makes the following calculation to compute her new average:

$$\tfrac{1}{2}(78 + 90) = 84$$

   (a) What error has Aisling made?

   (b) Show that the operation $*$, given by $a * b = \frac{1}{2}(a + b)$ where $a, b \in \mathbb{R}^+$, is commutative. Does "order matter" under this operation? Explain why or why not.

   (c) Consider the following questions that you might ask Aisling:

      i. Explain why the question below might not help Aisling:

         Should your average be lower than 84?

      ii. Explain how the question below might help you advance Aisling's understanding:

         What would your average be if you had made a 90, then a 72, then an 84?

4. Let G be a set with associative operation $*$ and with identity element e. Assume that every element of G has a left inverse: that is, $\forall\, a \in G, \exists\, b \in G$ such that $b * a = e$.

   (a) Show that b must also be a right inverse of a: that is, we also have $a * b = e$.

   (b) Explain how the associativity of $*$ plays a key role in your proof for 4(a).

   (c) Examine a list of axioms that you've seen presented in the definition of a group. How does your work in this problem affect your understanding of these axioms?

5. We have encountered matrices which represent rotations and reflections of vectors in $\mathbb{R}^2$. Does there exist a 2×2 matrix which represents the translation of vectors? If so, write it down and justify how you know it represents translations. If not, explain.

6. Show that the product of any two reflection matrices is a rotation matrix. [Hint: You will need the angle subtraction formulas for sine and cosine]. Using this result, give a geometric description of when two reflection matrices will commute.

1. Consider the set Δ, given below.

$$\Delta = \left\{ \mathrm{C}(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R}^+ \right\}$$

   (a) Describe the geometric effect that the matrix $\mathrm{C}(a)$ has on a vector $v \in \mathbb{R}^2$.

   (b) Is Δ an abelian group under matrix multiplication? Demonstrate why or why not.

2. Serena is working with the set Σ of rotations about the origin, given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

She knows Σ is a group under multiplication. Reasoning geometrically, Serena argues that the matrices $A(-\frac{\pi}{4})$ and $A(\frac{7\pi}{4})$ both act as inverses of matrix $A(\frac{\pi}{4})$.

(a) What geometric understanding might Serena have of rotations about the origin that enabled her to draw this conclusion without making any computations?

(b) Serena claims that her work above shows that the group of rotations is a counterexample to the claim that all group elements have a unique inverse. Explain the error in Serena's reasoning.

(c) What question would you ask Serena to help her understand her error? Why is your question helpful?