1. Below are the first seven rows of the *arithmetic triangle*, also known as Yang Hui's triangle or Pascal's triangle, among other names. By custom, the rows and entries are numbered starting at 0.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

(a) Write down at least three patterns that you observe.

(b) Generate the next row of the triangle using some or all of these patterns.

In high school, Anton learned that binomial expansions can be computed using the arithmetic triangle, and he wants to investigate why the numerical pattern in the arithmetic triangle emerges from the algebraic process of computing powers of a binomial. As part of his investigation, Anton expanded each of the following expressions using the distributive property of multiplication over addition, without simplifying the expressions using the commutative or associative properties of multiplication. Below are the expansion he computed, and all of his computations are correct.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx+xy+yx+yy\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y)\\ &= xxxx+xxxy+xxyx+xxyy+xyxx+xyxy+xyyx+xyyy+yxxx+yxxy\\ &\quad +yxyx+yxyy+yyxx+yyxy+yyyx+yyyy\end{aligned}$$

Next, Anton looked for patterns in the types of each term, making a table listing the terms with the following attributes.

$(x+y)^2$

Two x's and Zero y's	One x and One y	Zero x's and Two y's
xx	xy yx	yy

$(x+y)^3$

Three x's and Zero y's	Two x's and One y	One x and Two y's	Zero x's and Three y's
xxx	xxy xyx yxx	xyy yxy yyx	yyy

$(x+y)^4$

Four x's and Zero y's	Three x's and One y	Two x's and Two y's	One x and Three y's	Zero x's and Four y's
$xxxx$	$xxxy$ $xxyx$ $xyxx$ $yxxx$	$xxyy$ $xyxy$ $xyyx$ $yxxy$ $yxyx$ $yyxx$	$xyyy$ $yxyy$ $yyxy$ $yyyx$	$yyyy$

2. Anton first looked for patterns in the total number of terms in each of his expansions. He noticed that his expansion of $(x+y)^2$ has 4 terms, his expansion of $(x+y)^3$ has 8 terms, and his expansion of $(x+y)^4$ has 16 terms. Without multiplying everything out and counting, how can you use the pattern Anton noticed to predict the number of terms that $(x+y)^5$ would have?

3. In his tables, Anton observed that the number of expressions of each type correspond to entries of the arithmetic triangle. For example, in his first table for $(x+y)^2$ he counted the following number of terms in each category which corresponded to Row 2 of the arithmetic triangle.

Two x's and Zero y's	One x and One y	Zero x's and Two y's
xx	xy yx	yy
(1)	(2)	(1)

   Use the pattern he found to conjecture why it is customary that the rows and entries of the arithmetic triangle are numbered beginning with 0.

1. Consider a string of four letters made up of only the letters x or y.

— — — —

   (a) How many such strings are there? Explain your reasoning.

   (b) List the strings that contain exactly three x's. How many are there? Explain using combinatorial reasoning.

   (c) List the strings that contain exactly one y. How many are there? Explain using combinatorial reasoning.

   (d) List the strings that contain exactly three y's. How many are there? Explain using combinatorial reasoning.

2. Reexamine the table Anton created in his quest to understand binomial patterns. Notice the terms in his expansion of $(x+y)^4$ with three x's and one y are: $xxxy, xxyx, xyxx$, and $yxxx$. Also notice that using the context of Problem 1, Anton has listed all of the strings with three x's and one y.

   (a) Explain to Anton why $\binom{4}{1}$ counts the number of such terms.

   (b) Now, use associative and commutative properties of multiplication to generate like terms. Explain to Anton why the coefficient of x^3y in the expansion of $(x+y)^4$ is exactly the same as the number of terms that have three x's and one y.

3. Look again at Anton's table for the expansion of $(x+y)^4$. Note that the $(x+y)^4$ entries in the column labeled "Two x's and Two y's" are of two types: those that start with x and those that start with y. Explain how to generate each of these entries by starting with appropriate entries in the table for $(x+y)^3$.

4. Suppose Anton were to expand

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

   From his earlier work, he knows that he will have 5 terms in the expansion, one each corresponding to x^5, x^4y, x^3y^2, x^2y^3, xy^4, and y^5.

   Determine the coefficients in the expansion, and explain to Anton how you determined the coefficients.

5. **Expanding Binomial Products.**
Use combinatorial notation to write the expansion of $(x + y)^n$. Explain how you determined this is the appropriate expression.

6. **Applying the Binomial Theorem.**

(a) What is the coefficient of the term that contains x^7 in the expansion of $(x + 4y)^{10}$? Explain how you determined this.

(b) Use the binomial theorem to expand $(3x - 2y)^5$. Show all of your work.

7. Evelyn and Ivy were working on Problem 4 in the Class Activity where they determined the coefficients in the expansion of $(x+y)^5$. Evelyn says that the coefficient of x^3y^2 is $\binom{5}{3} = 10$ because she was counting the ways to place 3 x's in a five-character string. Ivy claims that the coefficient is $\binom{5}{2} = 10$ since she was counting ways to place 2 y's in a five-character string. Write two questions you could ask Evelyn and Ivy to help them find the common ground between their distinct approaches. Explain how your questions might help them.

8. Henry, a high school student, expanded $(2x-y)^4$ using the binomial theorem and made some errors. Below is his work.

$$(2x-y)^4 = 2x^4 - 8x^3y - 12x^2y^2 - 8xy^3 - y^4$$

   (a) What does Henry understand about the binomial theorem?

   (b) What does Henry not yet understand about binomial theorem?

(c) Consider the following questions that someone might ask Henry about his work.

i. Explain how the following question could help Henry to advance in his understanding of the binomial theorem:

How is $(2x - y)^4$ *similar to* $(x + y)^4$ *and how is it different?*

ii. Explain how the following question can help you assess what Henry understands about the binomial theorem:

Why doesn't $(-3y)^2 = -3y^2$*?*

iii. Explain why the following question would not help Henry:

Do the exponents and the coefficients look right?

1. Generalize the example from Problem 3 of the Class Activity to fill in the proof sketch for the following statement.

   Proposition. The counting numbers $\binom{n}{k}$ satisfy the arithmetic triangle pattern

   $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

   for all $n \geq k \geq 1$.

   Proof sketch. Each of the $\binom{n}{k}$ arrangements of k x's and $n-k$ y's has one of the following forms:

   (i) An x followed by an arrangement of $k-1$ x's and $n-k$ y's; or
   (ii) A y followed by an arrangement of k x's and $n-k-1$ y's

   The number of arrangements of type (i) is ______ and the number of arrangements of type (ii) is ______, so . . .

2. Students with a not-yet-complete understanding of high school algebra commonly argue that $(x+y)^2 = x^2 + y^2$. Below are three different approaches a teacher can use to help students see that this is not true.

   - One approach is to choose particular values for a and b, say $a = 1$ and $b = 2$, and ask students to plug those values into the left-hand side of the equation and simplify and then plug those same values into the right-hand side of the equation and simplify. Students can compare their answers to see that they are not equal. This approach can quickly show students that $(x+y)^2 \neq x^2 + y^2$ but doesn't give them insight into why the equality does not hold.
   - Another approach is to create an area model, such as the one below, which shows students that $(x+y)^2 \neq x^2 + y^2$. In this case the binomial $x+y$ is represented as a segment of length $x+y$, and the product $(x+y)^2$ is represented as the area of a rectangle whose side lengths are both $x+y$. This representation shows students which mathematical components they are missing in their answer, specifically the term $2xy$.

	x	y
x	x^2	xy
y	yx	y^2

   - A third approach is to apply the binomial theorem to $(x+y)^2$.

   (a) Use the binomial theorem to explain to a high school student why $(x+y)^2 \neq x^2 + y^2$.
   (b) Compare the "area model" approach with the "binomial theorem" approach. For what kinds of problems would each approach work? What insight does each approach highlight that will help students understand why $(x+y)^2 \neq x^2 + y^2$?

3. Charlie incorrectly expanded $(x+y)^2$ as follows.

$$\begin{aligned}(x+y)^2 &= 1x^2y^2 + 2x^1y^1 + 1x^0y^0\\ &= x^2y^2 + 2xy + 1\end{aligned}$$

   (a) What does Charlie understand about the binomial theorem?

   (b) What does Charlie not yet understand about the binomial theorem?

   (c) Write at least two questions that will help Charlie revise their work and develop a deeper understanding of the binomial theorem. Explain how those questions will help Charlie.

1. Consider the expansion of $(x+y)^{10}$.

   (a) Beyond "because the binomial theorem says so," explain why the coefficient of the x^4y^6 term is $\binom{10}{6}$.

   (b) One student states that the coefficient of the x^4y^6 term is $\binom{10}{4}$, and a second student states it is $\binom{10}{6}$. Beyond "because of symmetry," explain why each of them correctly computes the coefficient of x^4y^6.

2. Tencha, a high school student, expanded $(x-3y)^4$ using the binomial theorem and made some errors. Below is their work.

$$(x-3y)^4 = x^4 - 12x^3y - 18x^2y^2 - 12xy^3 - 3y^4$$

   (a) What does Tencha understand about the binomial theorem?

   (b) What does Tencha not yet understand about the binomial theorem?

   (c) Write two questions you can ask Tencha to help them revise their work. Explain how your questions could help guide their mathematical understanding.