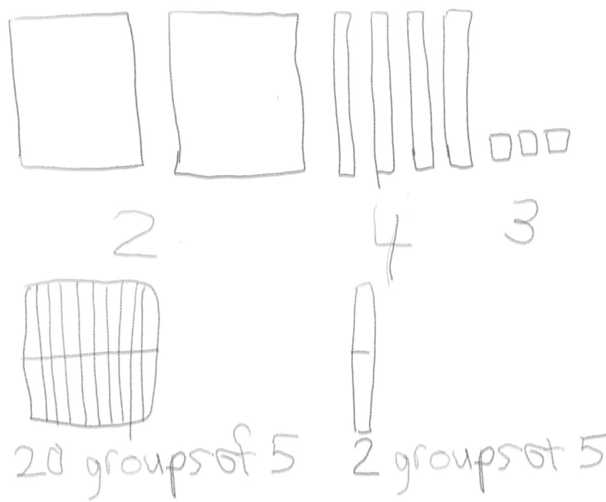
A fifth-grade class is discussing divisibility by 5. One student says, "My sister said you can tell if a number is divisible by 5 if it ends in a 0 or a 5." Another student adds, "Yeah, you can tell from skip counting by 5s: $5, 10, 15, 20, 25, \ldots$"

The teacher recognizes an opportunity to engage students in reasoning about properties of positive integers. The students have been using base-ten blocks to demonstrate whether various numbers are even or odd, so the teacher asks students to use their blocks to demonstrate whether the three numbers 243, 240, and 245 are divisible by 5.

Notice that there is a difference between **how to recognize** if an integer is divisible by 5 or not, which involves inspecting the digit in the ones place, and **proving** that an integer is divisible by 5, which involves determining whether there exists an integer m such that the integer can be written as $5m$.

Below is what one student, Rickie, wrote to explain that 243 is not divisible by 5, but 240 and 245 are.



I represented 243 as 2 groups of 100, 4 groups of 10, and then 3 ones. Each group of 10 forms 2 groups of 5, and each group of 100 forms 20 groups of 5. Then you have to look at the leftover blocks.
If the number ends in 0, then you haven't added any blocks and you can keep the groups you have. That's what happens with 240.
If the number ends in 1, 2, 3, or 4, then you have leftover blocks. That's what happens with 243.
If the number ends in 5, then you have 5 leftover blocks and you can put those into 1 group of 5. That's what happens with 245.
I guess you could also have 6, 7, 8 or 9 leftover blocks, but that's it.

State Rickie's test for divisibility by 5 as a biconditional statement. Explain how to generalize Rickie's argument about why the divisibility rule for 5 will always work.

1. **Test for Divisibility by 3:**
   An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

   (a) Use a sketch to demonstrate how base-ten blocks could show whether or not the integer 248 is divisible by 3.

   (b) Write a sequence of equivalent equations that numerically express the argument you made above with base-ten blocks.

   (c) Outline an algebraic argument that shows that the test for divisibility by 3 holds for any three-digit integer. Ensure you consider both directions of the biconditional statement.

2. **Test for Divisibility by 4:**
An integer is divisible by 4 if and only if the integer formed by the last two digits (that is, the integer between 0 and 99) is divisible by 4.

Colby provided the following argument for explaining why the divisibility test for 4 always works.

> *The* 100 *block is always divisible by* 4*, because* $4 \times 25 = 100$*. Any number of* 100 *blocks will be divisible by* 4*. So I just have to check whether the number formed by the tens and ones is divisible by* 4*.*

Quinn prefers to use divisibility by 2 to check for divisibility by 4.

> *First I check if the number is even, and if it is, then I know it is divisible by* 2*. So I divide the number by* 2*, and if that answer is even, then I know the number is divisible by* 4*.*

(a) Explain when Colby's method might have an advantage over Quinn's when trying to recognize if a four-digit integer is divisible by 4.

(b) Explain why Quinn might have suggested this method of checking for divisibility by 4 while studying prime factorization.

(c) Explain why students might have an easier time determining whether an integer is divisible by 5 than determining whether an integer is divisible by 4.

3. **Student Reasoning About Multiples:**
A student, Malik, tells his teacher that he has noticed that "when you skip count by 3s, you get the pattern odd-even-odd-even-odd-even, but when you skip count by 2s or 4s, you only get evens." Malik asks why that happens.

(a) Explain why skip counting by 2s (as in, "$2, 4, 6, 8, 10, \ldots$") or 4s (as in, "$4, 8, 12, 16, 20, \ldots$") yields only even integers. Use the definition of even integers in your explanation.

(b) Explain why skip counting by 3s (as in, "$3, 6, 9, 12, 15, 18, \ldots$") yields both odd and even integers.

(c) Explain how the two sketches below, representing 12 and 15, might help Malik to understand why the pattern holds.

(d) Explain how the following question might help Malik to understand why the pattern holds.
If I skip count by 3s and the last number I said was an even number, will the next multiple of 3 be even or odd? Draw a picture to show why.

1. Adam and Charlotte learn that a number is divisible by 6 if it is divisible by both 2 and 3. Each attempts to apply similar reasoning to state a divisibility rule for 20. Adam says that "because $20 = 2 \times 10$, if a number is divisible by both 2 and 10, then the number is divisible by 20." Charlotte states that "because 20 is divisible by 4 and 5, if a number is divisible by both 4 and 5, then the number is divisible by 20."

   (a) Why doesn't Adam's rule work, but Charlotte's rule does? What is the key difference between their two rules?

   (b) Adam's proof to their conjecture is shown below. Identify the error in their proof and explain why it is an error.

   > Let n be any integer that is divisible by 2 and 10. By the definition of divisibility, since n is divisible by 2, there is an integer k where $n = 2k$. Since n is also divisible by 10, that means that k must be divisible by 10. By the definition of divisibility, there is an integer l where $k = 10l$. Using substitution, $n = 2k = 2(10l) = 20l$. Since $n = 20l$ for some integer l, n is divisible by 20.

   (c) Their teacher recognizes that Adam was probably testing the number 20 or 40 when they conjectured their rule. What are some other integers the teacher could encourage Adam and Charlotte to experiment with that would help them to understand divisibility by 20?

2. A student, Isla, tells you that she has created a "test" for divisibility by 7. She claims that an integer n is divisible by 7 if and only if the rightmost two digits of n form an integer that is a multiple of 7. Provide a counterexample showing that Isla's test for divisibility by 7 doesn't work, and explain why Isla may believe her test works.

3. Proving Tests for Divisibility

   (a) Prove that an integer is divisible by 5 if and only if its last digit is 0 or 5.

   (b) Prove that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

   (c) Prove that an integer is divisible by 4 if and only if the integer formed by its last two digits is divisible by 4.

   (d) Prove that an integer is divisible by 6 if and only if the integer is divisible by both 2 and 3.

1. An integer n is divisible by 10 if and only if the final digit of the integer is 0.

   (a) Prove that this is true.

   (b) Describe how you relied on properties of integers in your proof.

2. In class, Olivia learned that a number is divisible by 6 if it is divisible by both 2 and 3 and used that to conjecture a divisibility rule for 60. Olivia says that because 60 is divisible by 6 and 10, you can tell which numbers are divisible by 60 by checking if the number is divisible by both 6 and 10.

   (a) Rewrite Olivia's conjecture as a biconditional statement.

   (b) One direction of the biconditional statement is true and the other is false. State which direction is false and find a counterexample showing that it is false.

   (c) Explain how the following question might help Olivia to understand that her rule doesn't always work.

   What is the least common multiple of 6 *and* 10*?*

   (d) Provide two reasons why you think Olivia made this conjecture.