At the beginning of the 17^{th} century, two different mathematicians were working to simplify tedious calculations in the field of astronomy (where numbers are often very large) by creating a function that would "translate" multiplication into addition. After all, it is much easier to add numbers like 177,320,045 and 9,317,032,566 by hand than it is to multiply them. This work by John Napier from Scotland and Joost Bürgi from Switzerland led to the creation of the function that we now call the logarithm. The precise nature of the logarithm has been refined over the years by other mathematicians, but its original use as a computational aid is exemplified by a familiar identity: for all positive real numbers x and y,

$$\log(x \cdot y) = \log(x) + \log(y)$$

It is useful to think of this property as "preserving an operation"; that is, the logarithm function preserves multiplication by translating it into addition.

1. Of course, most functions on the real numbers do not translate a multiplicative structure into an additive one.

   (a) Find a pair of positive real numbers x and y that demonstrates that $\sin(x \cdot y)$ need not equal $\sin(x) + \sin(y)$. Find a pair of non-zero values for which, by coincidence, this relationship does hold.

   (b) Find a pair of positive real numbers x and y that demonstrates that $\sqrt{x \cdot y}$ need not equal $\sqrt{x} + \sqrt{y}$. Find a pair of non-zero values for which, by coincidence, this relationship does hold.

   (c) Is there a pair of non-zero real numbers x and y for which $2(x \cdot y) = 2x + 2y$? If so, describe how to find all such pairs.

2. A classmate claims that if everyone collectively forgot how to multiply two numbers together, logarithms would be useful for overcoming the memory lapse. That is, logarithms would be useful for computing products. Describe how you might still be able to compute the product $2 \cdot 3$ using the table of values for the logarithm function, given below, and the identity $\log(x \cdot y) = \log(x) + \log(y)$.

x	1	2	3	4	5	6	7	8	9
$\log(x)$	0	.301	.477	.602	.699	.778	.845	.903	.954

3. Another classmate claims that since 72 is the product of 8 and 9, the table can be used to calculate $\log(72)$. They claim that this is possible even though 72 is not an entry in the first row of the table. Moreover, they claim that the table can be used to compute the logarithm of any number which is the product of numbers represented in the first row of the table. Use the table and the identity $\log(x \cdot y) = \log(x) + \log(y)$ to:

   (a) Calculate $\log(15)$.

   (b) Calculate $\log(24)$ in two different ways.

   (c) Estimate $\log(17)$. Why isn't it possible to calculate $\log(17)$ precisely?

The steps used by astronomers and other scientists to calculate the product $x \cdot y$ using a logarithm function looked like the following:

- First, calculate $\log(x)$ and $\log(y)$.
- Next, add these values together.
- Finally, find a positive real number whose logarithm is the sum $\log(x) + \log(y)$. This number is $x \cdot y$.

For the first and last step, charts called log tables were created which recorded, as efficiently as possible, $\log(x)$ for all real numbers x. A (simplified) log table is included below.

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	n/a	-1.000	-0.699	-0.523	-0.398	-0.301	-0.222	-0.155	-0.097	-0.046
1	0.000	0.041	0.079	0.114	0.146	0.176	0.204	0.230	0.255	0.279
2	0.301	0.322	0.342	0.362	0.380	0.398	0.415	0.431	0.447	0.462
3	0.477	0.491	0.505	0.519	0.531	0.544	0.556	0.568	0.580	0.591
4	0.602	0.613	0.623	0.633	0.643	0.653	0.663	0.672	0.681	0.690
5	0.699	0.708	0.716	0.724	0.732	0.740	0.748	0.756	0.763	0.771
6	0.778	0.785	0.792	0.799	0.806	0.813	0.820	0.826	0.833	0.839
7	0.845	0.851	0.857	0.863	0.869	0.875	0.881	0.886	0.892	0.898
8	0.903	0.908	0.914	0.919	0.924	0.929	0.934	0.940	0.944	0.949
9	0.954	0.959	0.964	0.968	0.973	0.978	0.982	0.987	0.991	0.996

For example, $\log(1.3)$ is found at the intersection of the row marked "1" and the column marked ".3", so $\log(1.3) = 0.114$. Log tables (and a related tool, the slide rule) were in use well into the 1900s, and were even used by NASA to make calculations for the Apollo 11 moon landing.

1. Use the above process and the log table provided to find the product of the following numbers:

   (a) 1.5 and 2.0

   (b) 0.5 and 8.6

   (c) 1.2 and 1.2

2. Another setting in advanced mathematics where a function ”preserves” a difficult operation is found in linear algebra. Consider the set $GL_2(\mathbb{R}) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{R}, ad - bc \neq 0\}$, which consists of all invertible 2×2 matrices with real number entries, and the determinant function $det : GL_2(\mathbb{R}) \to \mathbb{R}$, which assigns to each matrix its determinant.

   Recall from linear algebra that for any two matrices M and N in $GL_2(\mathbb{R})$, we have that $\det(\mathrm{MN}) = \det(\mathrm{M}) \cdot \det(\mathrm{N})$. Choose two 2×2 matrices that you know are invertible and verify this identity holds for your two matrices.

3. Finn hopes that the determinant function (see previous question) can help them to avoid multiplying matrices, much like the logarithm helped astronomers avoid multiplication of large, real numbers. Finn models the following strategy based on the process given before Problem 1:

   - First, calculate $\det(\mathrm{M})$ and $\det(\mathrm{N})$
   - Next, multiply these values together.
   - Finally, find a matrix whose determinant is the product $\det(\mathrm{M}) \cdot \det(\mathrm{N})$. This matrix is MN.

   (a) Why won’t Finn’s strategy work?

   (b) What question might you ask Finn to help them see the flaw in this plan? Why do you think this question would be helpful?

4. What is a key property that the logarithm function applied to positive real numbers possesses that the determinant function applied to invertible matrices does NOT possess?

5. Between which two groups is the logarithm function an isomorphism?

6. Now that Finn understands that the function det is not a group isomorphism and will not help avoid matrix multiplication, they want to try and find another function from $(GL_2(\mathbb{R}), \cdot) \to (\mathbb{R}^*, \cdot)$ which is a group isomorphism, where $\mathbb{R}^*$ denotes the set of nonzero real numbers. Does such a function exist? Why or why not?

7. Finn shifts focus to finding a "better logarithm"; that is, a group isomorphism from $(\mathbb{R}^+, \cdot) \to (\mathbb{Z}, +)$ that allows them to multiply positive real numbers by adding integers instead of real numbers. Does such a function exist? Why or why not?

8. Maybe, Finn says, we could at least find a group isomorphism from $(\mathbb{Q}^+, \cdot) \to (\mathbb{Z}, +)$ that lets us multiply positive rational numbers by adding integers. Does such a function exist? Why or why not?

1. Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups and $f : G \to H$ be a homomorphism.

   (a) Show that the image of the group identity of G under f must be the group identity of H (that is, $f(e_G) = e_H$).

   (b) Use this result to show that, given an arbitrary element a in G, the image of the inverse of a under f must be the inverse of the image of a under f (that is, $\forall a \in G$, $f(a^{-1}) = f(a)^{-1}$).

   (c) Use the previous two results to show that the image of G under f is itself a group with the operation of H.

2. Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups and $f : G \to H$ be a homomorphism. Furthermore, let G be an abelian group. Anjali writes the following proof, which she claims shows that H is also an abelian group:

   Let g_1 and g_2 be two distinct elements of G. Then:

   $g_1 \cdot_G g_2 = g_2 \cdot_G g_1$ ← G is abelian

   $f(g_1 \cdot_G g_2) = f(g_2 \cdot_G g_1)$ f is a homomorphism

   $f(g_1) \cdot_H f(g_2) = f(g_2) \cdot_H f(g_1)$ $f(g_1)$ and $f(g_2)$ are in H

   $h_1 \cdot_H h_2 = h_2 \cdot_H h_1$

   So, H is also an abelian group.

   (a) Explain why Anjali's proof does not show that H is an abelian group. What has she proven instead?

   (b) Given the same groups $(G, \cdot_G)$ and $(H, \cdot_H)$ let $f : G \to H$ now be an **isomorphism**. Add to Anjali's proof to show that, under these conditions, H is now an abelian group.

3. Imagine that you are a high school mathematics teacher who has noticed that one of your students, Hai, has assumed that $\log(x + y) = \log(x) + \log(y)$ when simplifying logarithmic expressions on a homework assignment.

   (a) Find a pair of real numbers x and y demonstrating that Hai's assumption is not always true. For which pairs of positive real numbers (x, y) is Hai's assumption true?

   (b) Using what you have learned about logarithms in this lesson, how might you help Hai understand that $\log(x \cdot y) = \log(x) + \log(y)$ is the correct identity? Why is your explanation helpful? Make sure your explanation is appropriate for a high school student.

4. Recall that $\mathbb{Z}_n$ is the set of *equivalence classes* on the integers, where two integers are in the same equivalence class if and only if they both have the same (smallest, non-negative) remainder when divided by n. The set $\mathbb{Z}_n$ contains n such equivalence classes which, canonically, are represented by the possible remainders when an integer is divided by n: $\{0, 1, \ldots, n-2, n-1\}$.

(a) Let θ be the map from $(\mathbb{Z},+)$ to $(\mathbb{Z}_n,+)$ given by $\theta(z)=r$, where z is an integer and r is its remainder when divided by n. Show that θ is a group homomorphism. You may assume without proof that $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+)$ are in fact groups.

(b) Explain why θ cannot be an isomorphism in two ways: by showing that θ is not a bijection and by finding a difference in the algebraic structures of $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+)$ (other than that their orders differ).

5. Let F be the set of continuous, real-valued functions on the interval $[0,1]$. Let σ be the map from $(F,+)$ to $(\mathbb{R},+)$ given by $\sigma(f)=\int_0^1 f(x)dx$ for all f in F. The operation "+" on F is defined by $(f+g)(x)=f(x)+g(x)$ for all f, g in F and $x\in[0,1]$. The operation "+" on $\mathbb{R}$ is the usual real number addition.

   (a) Show that σ is a group homomorphism. You may assume without proof that $(F,+)$ and $(\mathbb{R},+)$ are in fact groups.

   (b) Describe the function that represents the identity element in $(F,+)$. Now, choose a different element of $(F,+)$ which maps to the identity element of $(\mathbb{R},+)$ under the map σ. Draw a graph of your chosen function and explain how you know it meets this criteria.

   (c) Let $ker(\sigma)$ be the set of **all** elements of $(F,+)$ that map to the identity element of $(\mathbb{R},+)$ under the map σ. Show that $ker(\sigma)$ is a subgroup of $(F,+)$.

1. Let $M_2(\mathbb{R}) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{R}\}$ and define the function $tr : (M_2(\mathbb{R}), +) \to (\mathbb{R}, +)$ to be the map $tr(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = a + d$. That is, tr assigns to each matrix its trace.

   (a) Show that tr is a group homomorphism. You may assume without proof that $(M_2(\mathbb{R}), +)$ and $(\mathbb{R}, +)$ are in fact groups.

   (b) Is tr an isomorphism? Why or why not?

2. Abina claims that there must be an isomorphism between $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ since the orders of the groups are the same.

   (a) Based on her claim, what do you think Abina understands about isomorphisms?

   (b) Explain why the two groups in question cannot be isomorphic.

   (c) What question would you ask Abina to help her understand her mistake? Why do you think your question would be helpful?