1. Recall that $\mathbb{Z}_n$ is the set of *equivalence classes* on the integers, where two integers are in the same equivalence class if and only if they both have the same (smallest, non-negative) remainder when divided by n. The set $\mathbb{Z}_n$ contains n such equivalence classes which, canonically, are represented by the possible remainders when an integer is divided by n: $\{0, 1, \ldots, n-2, n-1\}$.

   (a) Fill in the following chart with the representative of each integer's equivalence class in $\mathbb{Z}_{10}$.

Integer	36	17	-4	-17
Representative in $\mathbb{Z}_{10}$				

   If we are careful, we can also (sometimes) represent non-integers as elements of $\mathbb{Z}_n$. For example, if we interpret the notation "$1/3$" as "the element you multiply by 3 to get 1," we would then consider 7 in $\mathbb{Z}_{10}$ a representative of "$1/3$", since $3 \cdot 7 = 21 = 1$ (where $21 = 1$ because 21 has remainder 1 when divided by 10). Furthermore, no other element of $\mathbb{Z}_{10}$ has this property.

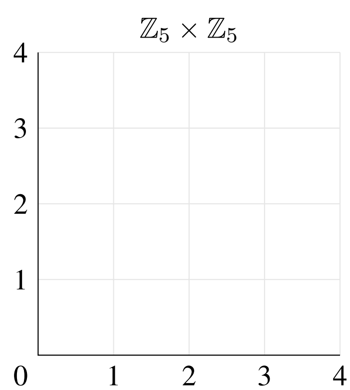
   (b) Fill in the following chart with the representative in $\mathbb{Z}_{10}$, if it exists.

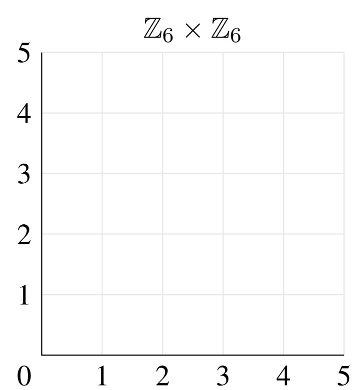
"Non-integer"	"$1/1$"	"$1/2$"	"$1/3$"	"$1/4$"	"$1/5$"	"$1/6$"	"$1/7$"	"$1/8$"	"$1/9$"
Representative in $\mathbb{Z}_{10}$	1		7						

2. Let A be a set of elements (numbers) with a well-defined notion of addition and multiplication. We define a *line over A* as the solution set to an equation of the form $ax + by = c$ for some fixed values of $a, b, c \in$ A. That is, a line is the set of all ordered pairs $(x, y) \in$ A $\times$ A that make $ax + by = c$ a true statement in A. The *graph of a line* is a scatter plot of the solution set on a coordinate plane, usually one with perpendicular axes marked by the elements of A.

   For example, in the system of real numbers, the set of all ordered pairs that make $y = 3x$ a true statement in $\mathbb{R}$ has a graph which is a continuous straight geometric line of slope 3 through the point $(0, 0)$ in our usual Cartesian coordinate system.

   Graph the line $y = 3x$ in the following spaces on the provided axes.





3. In solving the equation $x + 4 = 1 + 4x$ in $\mathbb{R}$, a student makes the following algebraic manipulations:

$$\begin{aligned} x + 4 &= 1 + 4x \\ 4 &= 1 + 3x \\ 3 &= 3x \\ 1 &= x \end{aligned}$$

The student then concludes that $x = 1$ is the solution set to $x + 4 = 1 + 4x$ in $\mathbb{R}$.

(a) Describe the mathematical justification for each step in the student's solution.

(b) To solve $x + 4 = 1 + 4x$ for x in $\mathbb{Z}_5$, are we allowed to repeat the process the student used (in $\mathbb{R}$) as presented above? Does this process yield the entire solution set to the equation? Explain.

(c) To solve $x + 4 = 1 + 4x$ for x in $\mathbb{Z}_6$, are we allowed to repeat the process the student used (in $\mathbb{R}$) as presented above? Does this process yield the entire solution set to the equation? Explain.

Consider the linear equation $y = 3x$ (and your corresponding graphs) from the Pre-Activity.

1. How many solutions to $0 = 3x$ exist in the following domains? What are they?

Domain	$\mathbb{R}$	$\mathbb{Z}_5$	$\mathbb{Z}_6$
# of Sol.			
Sol. Set			

2. How many solutions to $1 = 3x$ exist in the following domains? What are they?

Domain	$\mathbb{R}$	$\mathbb{Z}_5$	$\mathbb{Z}_6$
# of Sol.			
Sol. Set			

3. Based on his work in Problem 1, Omar guesses that, in $\mathbb{Z}_{10}$, the equation $0 = 3x$ will have multiple solutions.

   (a) Why do you think Omar might have made this hypothesis?

   (b) In $\mathbb{Z}_{10}$, for which non-zero value(s) of a does the equation $0 = ax$ have a unique solution? Was Omar's hypothesis correct?

   (c) In $\mathbb{Z}_{10}$, for which non-zero value(s) of a does the equation $1 = ax$ have a solution?

   (d) Look back at your answers to Problems 3(b) and 3(c). What relationship do these integers have with 10, the modulus of $\mathbb{Z}_{10}$?

For problems 4–6, consider the quadratic equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{10}$.

4. What are some ways that you might attempt to solve this equation for x?

5. Notice that we can factor the left-hand side of this equation to obtain $(x-2)(x-3) = 0$, from which we find that $x - 2 = 0$ or $x - 3 = 0$. This yields the solutions $x = 2$ and $x = 3$.

   (a) Verify that $x = 7$ is also a solution. Are there any more? Why do you think factoring did not yield *all* the solutions?

   (b) What important property of $\mathbb{R}$ are we using when we find the roots of a factored expression and claim those roots constitute the entire solution set?

6. Attempt to apply the quadratic formula to the above equation. What difficulties do you encounter?

1. In $\mathbb{R} \times \mathbb{R}$, for any two distinct points A and B, there exists a unique line containing them. Show this statement is not true in $\mathbb{Z}_6 \times \mathbb{Z}_6$ by finding the equations of two distinct lines that both contain the points $(1, 2)$ and $(3, 4)$. [Recall that for a set A of elements (numbers) with a well-defined notion of addition and multiplication, we define a *line over A* as the solution set to an equation of the form $ax + by = c$ for some fixed values of $a, b, c \in$ A. That is, a line is the set of all ordered pairs $(x, y) \in$ A $\times$ A that make $ax + by = c$ a true statement in A. The *graph of a line* is a scatter plot of the solution set on a coordinate plane, usually one with perpendicular axes marked by the elements of A.]

2. Artyom says that since $\mathbb{R}$ is an integral domain, then the set of ordered pairs $\mathbb{R} \times \mathbb{R}$ must also be an integral domain under the operations given below:

$$(a, b) \oplus (c, d) = (a + c, b + d)$$
$$(a, b) \otimes (c, d) = (a \cdot c, b \cdot d)$$

   (a) Why is Artyom incorrect?

   (b) What question would you ask Artyom to help him understand his error? Why would your question be helpful?

3. Let R be a commutative ring in which the multiplicative identity and additive identity are distinct elements.

   (a) Prove that if R is an integral domain, then for $a, b, c \in$ R and $a \neq 0$, $a \cdot b = a \cdot c \Rightarrow b = c$.

   (b) Prove that if $\forall\, a, b, c \in$ R with $a \neq 0$ we have that $a \cdot b = a \cdot c \Rightarrow b = c$, then R is an integral domain.

4. When looking for solutions to the equation $x^3 = 1$ for $x \in \mathbb{Z}_{13}$, we see that $x = 1$ clearly works. To find other solutions, it might be useful to observe that every element in $\mathbb{Z}_{13}$ corresponds to a value 2^n for some n by completing the following table of values in $\mathbb{Z}_{13}$. [Hint: Double the values in the table from left to right, remembering to convert to modulo 13 when appropriate]

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
				3								1

   Now, to find other solutions, we might use the table above to help; for example, $x^3 = 1 = 2^{12} = (2^4)^3 = 3^3$. Thus, 3 is also a solution. It turns out there is only one more solution to this equation. Find it and justify your answer by using powers of 2.

5. How many solutions does the equation $ax = 0$ have in $\mathbb{Z}_{12}$ for each nonzero a? Use your answer to make a hypothesis about the number of solutions to the equation $ax = 0$ in $\mathbb{Z}_n$ when a is nonzero.

6. Solve the system of linear equations given below in the following rings, if possible.

$$2x + y = 4$$
$$x + 2y = 0$$

(a) $\mathbb{Z}_5 \times \mathbb{Z}_5$

(b) $\mathbb{Z}_6 \times \mathbb{Z}_6$

(c) Was your process for solving parts (a) and (b) the same? Why or why not? What difficulties arose in parts (a) and (b)?

NAME:

1. List all the nonzero values of a which give the equation $ax = 0$ a unique solution in the following rings:

   (a) $\mathbb{Z}_{13}$

   (b) $\mathbb{Z}_{14}$

2. Thuy's work for finding solutions to $x^2 - x = 0$ in $\mathbb{Z}_4$ is shown below.

   $x^2 - x = 0$
   $x(x-1) = 0$
   Therefore, either
   $x = 0$ or $x-1=0$
   The solution set is $\{0, 1\}$

   (a) From her work, what assumption does Thuy seem to be making about $\mathbb{Z}_4$? Is this assumption correct?

   (b) Thuy checks each element of $\mathbb{Z}_4$ and verifies that her solution set is correct. Her teacher asks her to attempt to solve the same equation, this time in $\mathbb{Z}_6$. What is the teacher hoping Thuy will understand about her approach by working in $\mathbb{Z}_6$?