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math HORIZONS

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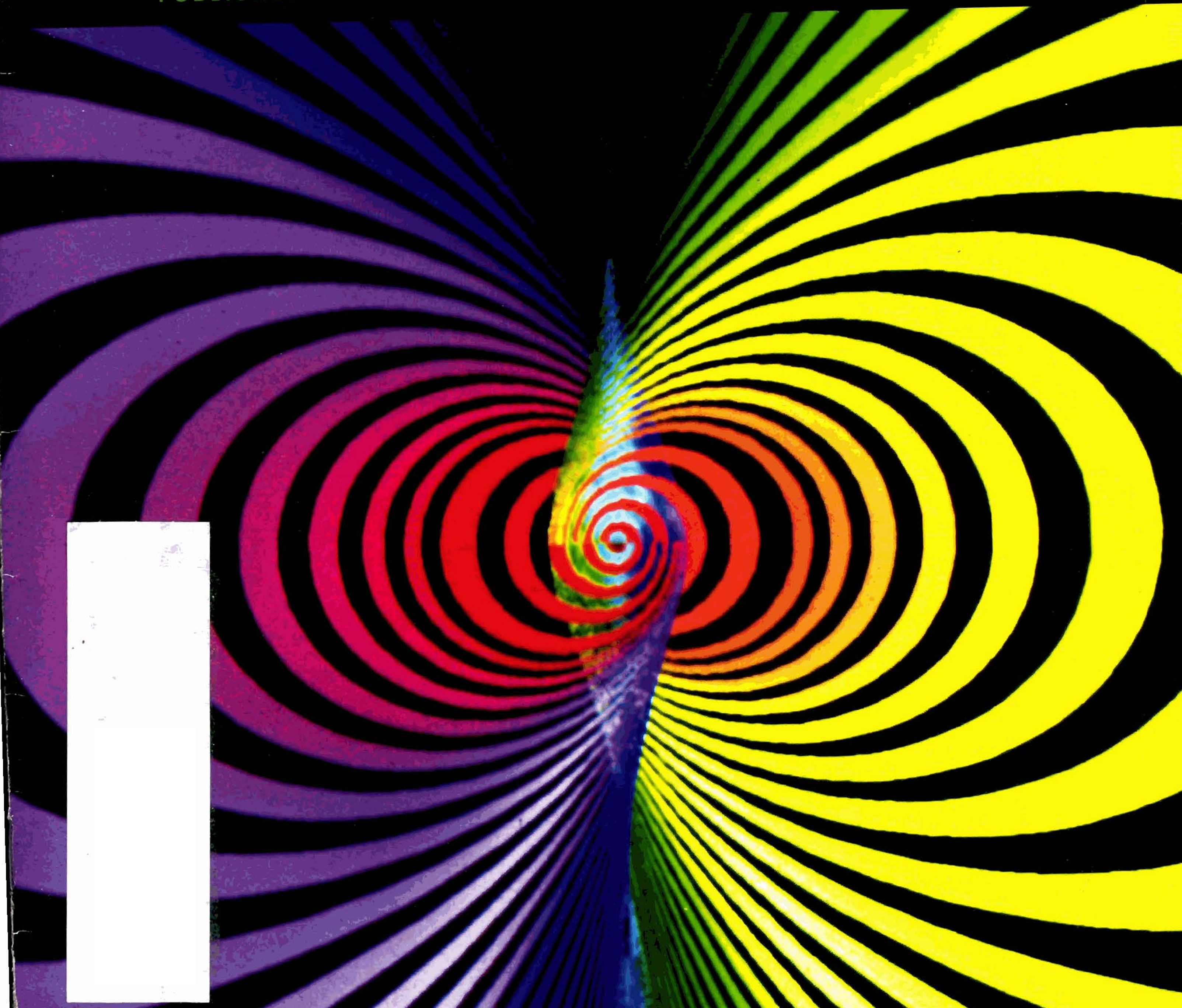
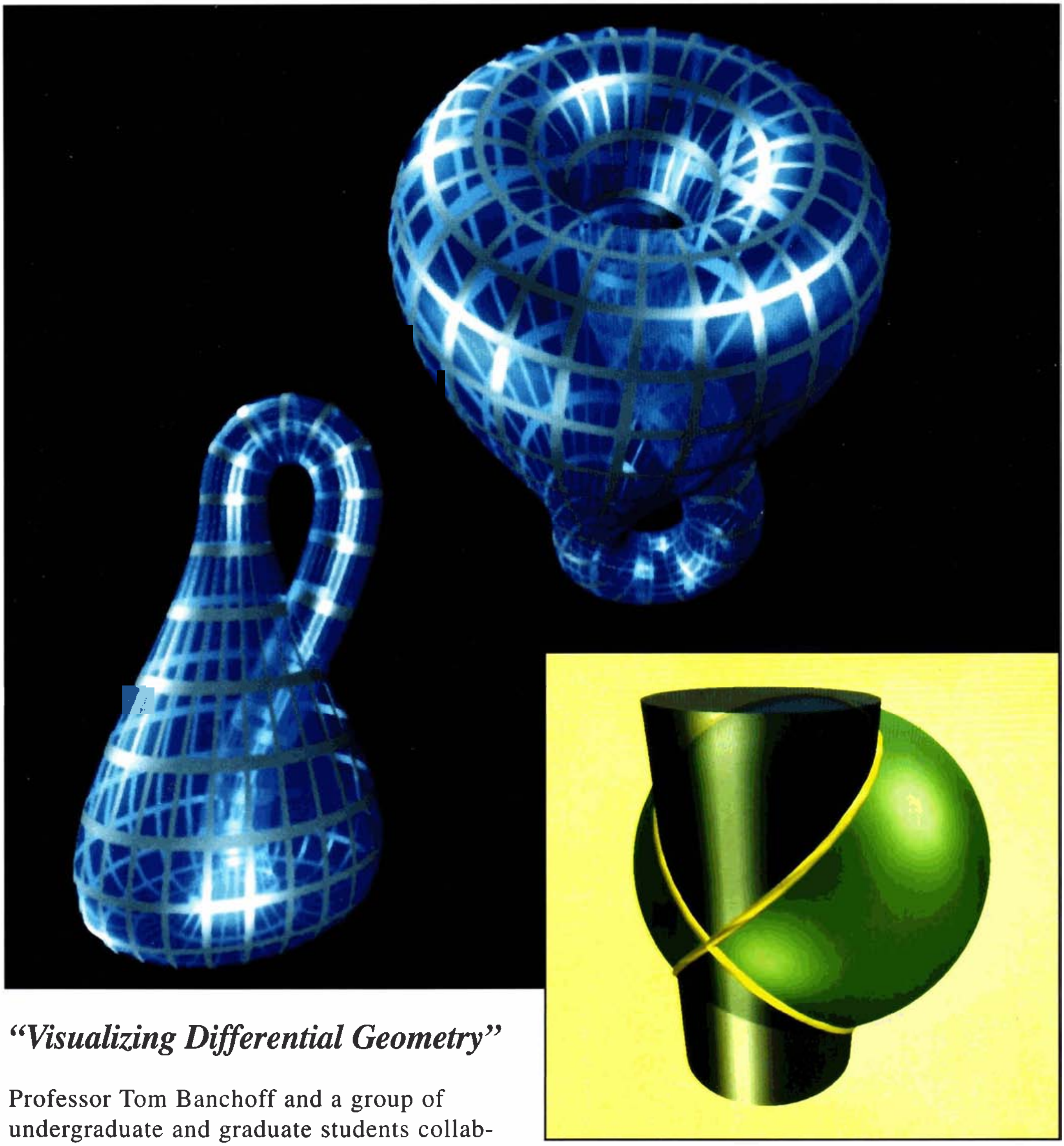


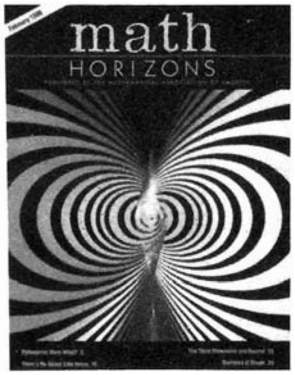


"Visualizing Differential Geometry"

Professor Tom Banchoff and a group of undergraduate and graduate students collaborated in the summer of 1994 on a computer animated videotape featuring a fly-through of a translucent Klein bottle (above) and the Temple of Viviani (top right).

Students in the photograph (right) from left to right are: Jeff Beall, Ezra Miller, Neel Madan, Julia Steinberger, Chenghui Luo, Laura Dorfman, Bin Wang and Cathy Stenson. Missing from the photo, Ying Wang.

Artwork courtsey of Tom Banchoff. An article on Banchoff begins on page 18 of this issue.



Math Horizons is for undergraduates and others who are interested in mathematics. Its purpose is to expand both the career and intellectual horizons of students.

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In this issue

Cover image courtesy of Tom Banchoff

The cover image is a projection of a twisted sphere in four-space that intersects itself in just one point (the way a figure eight represents a twisted circle in the plane.)

math
HORIZONS

Blizzards, Floods, and Paper Prices

This issue is reaching you a little bit late because the Blizzard of '96 prevented the staff of *Math Horizons* from getting to our offices for an entire week. And the warm weather and rapid melting resulted in floods, which further slowed our efforts to get this issue to you.

Our recent survey of readers of *Math Horizons* was very gratifying. You told us that you like what we're doing. A few of you asked for articles that are tougher, but most of you seem to think the level is about right. Please continue to tell us how to improve *Math Horizons*, since it is for you—the student.

During the past year, the cost of paper has risen dramatically. We have not increased the price of *Math Horizons* in two years, but the recent increases in paper costs force us to increase bulk subscription prices for the first time. Currently each issue is priced at $1.25. Beginning with the 1996–97 academic year, the price of each issue will be $1.50. Even at $1.50 per issue, we view *Math Horizons* as one of the best buys in our part of the universe.

We're having a great time bringing *Math Horizons* to all 27,000 of our subscribers. We could have an even better time if you would send us your articles, poems, problems, humor in the classroom, and suggestions.

Don Albers

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MARGARET WERTHEIM

Pythagoras' Trousers

Even in his own time, Pythagoras was a legend. Rumored to be the son of the god Apollo by a virgin birth to his mother, Pythais, he was said to have worked miracles, conversed with daemons, and heard the "music" of the stars. He was regarded by his followers as semidivine, and there was a saying that "among rational creatures there are gods and men and beings like Pythagoras." It is difficult to sort out fact from fiction about his life, for he lived in that brilliant but hazy zone where myth and history collide. None of his writing has survived, but ancient sources abound with references to him. Even in the works of that most logical of ancients, Aristotle, we find accounts of Pythagoras that mix tales of miracles with discussions of his mathematics and cosmology. Pythagoras' philosophy fully reflected his transitional age, for while it contained the seeds of mathematical science, it also maintained a role for the pantheon of gods. Both in thought and in life, this Samian sage was a bridge between two worlds.

In many respects the mythico–religious dimension of Pythagoras' life bears an uncanny resemblance to the life of Christ depicted in the New Testament. Both men are said to have been the offspring of a god and a virgin woman. In both cases their fathers received messages that a special child was to be born to their wives—Joseph was told by an angel in a dream; Pythagoras' father, Mnesarchus, received the glad tidings from the Delphic oracle. Both spent a period of contemplation in isolation on a holy mountain, and both were said to have ascended bodily into the heavens upon their deaths. Furthermore, both spread their teachings in the form of parables, called *akousmata* by the Pythagoreans, and a number of parables from the New Testament are known to be versions of earlier Pythagorean *akousmata*. One historian has suggested that early Christians may have taken elements of the Pythagorean myth and attributed them to their own prophet, for in the ancient world Pythagoras was known first and foremost as a religious figure. During the closing centuries of the Roman Empire, when Christianity was just another cult vying for religious supremacy, a great revival of Pythagoreanism occurred, and latter–day followers of the "Master" promoted him as a Hellenistic alternative to the "king of the Jews." As had Christ, the Samian sage had promised mystical union with the divine, and to his Roman followers his teachings offered a rational spiritual alternative to the rising tide of Christianity.

There was a saying that among rational creatures there are gods and men and beings like Pythagoras.

Pythagoras is known to have been born around 560 B.C. on Samos, a prosperous island in the Aegean Sea not far from the coast of Asia Minor. An important gateway to the commercial cities of the mainland, Samos was also religiously significant, being the site of a monumental temple to Hera, queen of the Olympian gods. On this island, Pythagoras was always something of an outsider, for while his mother is said to have been a native of Samos, his father was a foreigner, probably a Phoenician, who had been made an honorary citizen for giving grain to the Samians during a time of drought. As an ethnic half–caste, Pythagoras was not considered a true Greek, and furthermore, his mystical bent singled him out early as a misfit among the Samians. Later in life he turned away from Ionian culture and identified himself strongly with the East, an allegiance he symbolized by rejecting the long robes favored by the Greeks and adopting instead the Persian fashion of trousers.

As a wealthy merchant, Mnesarchus could afford to educate his son, and in this age of awakening, the young Samian was taught by some of the greatest of the new Ionian thinkers. His instructors included Anaximander, Pherecydes, and Thales, one of the legendary Seven Sages and the first true philosopher. But although Pythagoras was trained by the best philosophers of the time, he hankered for something more, and after absorbing the best of the West, he set out for the East—initially Egypt, and later Babylon. (Thales had recommended that if he wished to be the wisest man alive, Pythagoras should go to the land of the pharaohs, where

MARGARET WERTHEIM is an Australian science writer now living in New York City.

Illustration by Greg Nemec

geometry had been discovered.) There is controversy among historians about whether Pythagoras really made a trip to Egypt and Babylon or whether it was an invention of later disciples. But either way, historian David Lindberg has pointed out that the story encapsulates an essential historical truth: The Greeks inherited mathematics from the Egyptians and Babylonians, and Pythagoras is regarded as the person who introduced this treasure to the West. Because he was undoubtedly the first great Greek mathematician, we shall assume, along with the ancients, that the journey did in fact take place.

According to Iamblichus, his third-century Roman biographer, Pythagoras traveled to Egypt by way of the Levant, the lands bordering the eastern shores of the Mediterranean, His intention was to learn the sacred rites and secrets of the region's religious sects. Some people collect stamps, other coins; Pythagoras collected religions, and he made it his business to be initiated into as many as he could. In this some of his ancient detractors accused him of cynical motives, and even his supporters acknowledged the accusation was partly true. As a young man Pythagoras certainly aspired to a career as a public speaker, and he clearly understood the public relations value of exotic mystical experiences. Nonetheless, he was also a genuinely religious man.

When Pythagoras arrived in the land of the pharaohs, events didn't go entirely as he had hoped, for according to Porphyry, another Roman biographer, he was rejected by the priests at the temples of both Heliopolis and Memphis. Eventually, however, he was accepted at Diospolis, where he studied for many years. The ancients disagree about how long Pythagoras spent among the Egyptians, but it seems to have been at least a decade. Porphyry tells us the priests imposed harsh tests on their foreign aspirant, but what he learned from them will forever remain a mystery because Pythagoras always honored their fanatical secrecy, which he would later make a cornerstone of his own religious community.

Pythagoras' Egyptian sojourn came to an abrupt end in 525 B.C., when the Persians invaded Egypt and he was taken as a captive to Babylon. In that fabled city of the hanging gardens and the great ziggurat, he availed himself of the wisdom of the Babylonians. According to Porphyry, he studied under the sage Zaratas, from whom he learned astrology and the use of drugs for purifying the mind and body. He was also initiated into the mysteries of Zoroastrianism, with its opposing cosmic forces of good and evil. This dualism would profoundly influence his own thinking and would eventually be incorporated into his mathematico-mystical philosophy. The Babylonians were not only astrologers but also great astronomers and mathematicians. Lindberg notes their mathematics was of "an order of magnitude superior to that of the Egyptians." From them Pythagoras may well have learned the theorem for which he is still famous today: that for a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. Although we are taught in school that this is the Pythagorean Theorem, historians of mathematics believe it was almost certainly known to the Babylonians before him.

If Pythagoras had been an oddity on Samos before he left for the East, how much more a misfit he must have been upon his return after two decades spent with foreign priests and sages. Now he not only wore trousers, but also never cut his hair or beard—a habit that would later become a hallmark of Pythagorean followers. On Samos, he set himself up to teach philosophy and mathematics, offering lectures in the open air. Yet it soon became clear that his mystical leanings had little appeal to the Samians, so once again he left his homeland, this time forever. Pythagoras' aim now was to found his own community, where committed followers would dedicate themselves to a life of religious contemplation and study of the "divine." As the site of this utopian community he chose the town of Croton in southern Italy—a place at the very extremity of the Greek world.

Since none of the Pythagorean community's writings or records have been preserved, because of the group's fanatical secrecy, the details of its operations remain shrouded in mystery, but we do know that the lives of its participants combined elements of Greek religious practice with Egyptian-inspired rituals. In addition, the community also operated as a philosophical and mathematical school. Members were of two kinds: the *akousmatics* and the *mathematikoi.* The former lived outside the community and visited only for teaching and spiritual guidance. They did not study mathematics or philosophy but were taught through *akousmata,* which espoused a simple, nonviolent way of life. For them Pythagoreanism was essentially an ethical system with mystical undertones and Pythagoras was a purely spiritual leader.

The *mathematikoi,* however, lived inside the community and dedicated themselves to a Pythagorean life. That life was communistic in the sense that initiates had to give up their property to the community and renounce all personal possessions. Pythagoras believed this was necessary in order for the soul to be free from extraneous worries. Inspired by his life among the Egyptian priests, he was also greatly concerned with purification, and the *mathematikoi* were not allowed to eat meat or fish, or to wear wool or leather. It was said by ancient commentators that initiates had to undergo a probationary period of up to five years, during which they were to be silent to demonstrate their self–control. While it is unlikely that silence was complete, it is clear that only those who were truly dedicated made it into the inner circle, handpicked by Pythagoras, to hear the Master's most secret teachings and study mathematics. Following the model of the Egyptians, all knowledge was kept secret within the community, and one member was expelled when he revealed the mathematical properties of the dodecahedron, one of the five "perfect" solids. Pythagoras believed that, as divine knowledge, mathematics should be revealed *only* to those who had been properly purified in both mind and body, and the *mathematikoi* approached its study in the spirit of priesthood.

Pythagoras believed that... mathematics should be revealed only to those who had been properly purified in both mind and body.

The Pythagorean community at Croton is often referred to as the brotherhood, yet this is a misnomer because it also included women. Pythagoras himself was married with several children, and his wife, Theano, was an active member and teacher in the community. But the controversial question is not so much whether women could be Pythagoreans but whether they were allowed to become *mathematikoi,* philosopher–mathematicians, or only *akousmatics.* Because no record of the community survives, it is difficult to resolve this issue, but in the writings of a number of ancient commentators there is evidence that there were women *mathematikoi.* Theano, for example, is said to have written treatises on mathematics and cosmology. The idea that women could be members of Pythagoras' inner circle is also lent credence by the fact that Pythagorean communities in the fifth century B.C. also included women: Phintys, Melissa, and Tymicha are three whose names have come down to us. Finally there is the example of Plato, who was deeply influenced by Pythagoreanism and was the only one of the great Athenian philosophers who advocated the education of women. Unlike Aristotle, Plato allowed women into his famous Academy, where mathematics was taught. Thus it seems reasonable to conclude that among the original Pythagoreans women *did* participate in mathematical study. Given the nature of Greek society at the time, it is highly unlikely there were as many women as men among the *mathematikoi,* but given how misogynist the Greeks were soon to become, the community at Croton must be seen as one of the more gender–equitable havens of the Greek world. In the beginning, then, mathematical men acknowledged and accepted the presence of mathematical women.

The last years of Pythagoras' life are clouded in shadow. Between 510 and 500 B.C., a Croton nobleman, Kylon, led an uprising against the Pythagorean community, which resulted in its demise. This event has been variously described as religious persecution and as a democratic revolt against an aristocratic sect. Ancient champions of Pythagoras depict Kylon as a tyrannical man motivated by revenge after having been rejected by the Pythagoreans, yet some historians believe the backlash against the community was a response to its elitist and secretive nature. During the uprising Pythagoras fled and supposedly spent the rest of his long life wandering in Italy, spreading his teachings. It is said that on his death he ascended directly into the heavens from a temple of the muses. ■

The Ant on a 1 x 1 x 2

An ant is at corner A of a 1 x 1 x 2 box. It crawls along the surface along a geodesic, the shortest possible path, to a point B. Where is B located to make the path as long as possible?

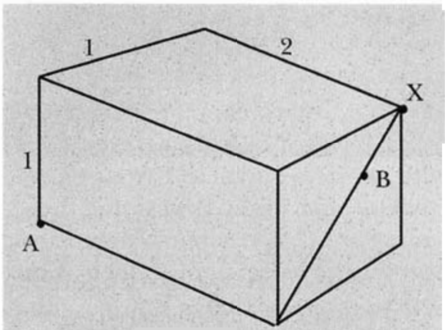


Figure 1 shows a 1 x 1 x 2 bicube, a solid formed by joining two cubes.

Intuitively one would guess B to be at the corner marked X because this is the point the farthest from corner A. Yoshiyuki Kotani, a professor of mathematics in Saitama, Japan, recently made a surprising discovery. Point B is not at X, but one-fourth of the way down the diagonal of the square face as shown!

The Geodesic from A to X is easily traced by unfolding the solid along a hinged edge as shown in Figure 2. The Pythagorean theorem gives the path as the square root of 8, or 2.828... If you trace the geodesic from A to B, the ant can take either of the two routes shown in Figures 3 and 4. By symmetry there are two similar routes along the hidden sides of the solid. Two of the four paths go over two sides, and the other two go over three sides. Applying the Pythagorean theorem to these four paths, they all have the same length of 2.850..., about .022 longer than the path from A to X!

I do not know whether Kotani generalized the problem to 1 x 1 x n solids. In any case, physicist Richard Hess, who first called my attention to the problem, and four mathematicians to whom I sent the problem (Ken Knowlton, Robert Wainwright, Dana Richards, and Brian Kennedy) each independently solved this more general case. I expected that calculus would be required, but it turns out that by unfolding the solid along hinged edges, and applying basic algebra, the formula for the location of B is not too difficult to find.

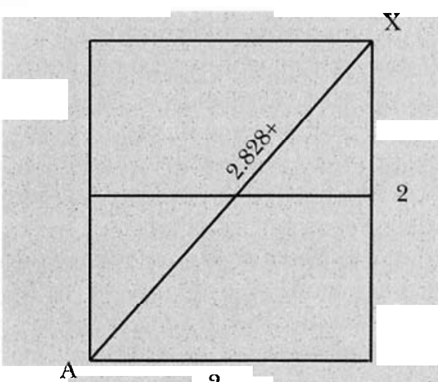


Figure 2

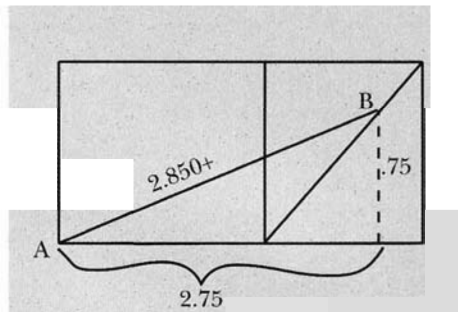


Figure 3

Label the diagonal of the square face with 0 at X and 1 at the diagonal's bottom corner. The distance of B from X, along this diagonal, as a function of n, is $(n-1)/2n$. If $n = 2$, the formula puts B a quarter of the way down the diagonal. If $n = 3$, B is 1/3 of the way down. As n approaches infinity, B approaches 1/2 at the limit, placing it at the center of the diagonal. Of course n can take any real value greater than 1 and not necessarily an integer.

Problems about a spider at spot A on the wall of a room that crawls along a geodesic to catch a fly at spot B are in many classic puzzle books. Henry Dudeney, the British puzzle maker, gives such a problem in **The Canterbury Puzzles**, and his American counterpart Sam Loyd has the same problem in his **Cyclopedia of Puzzles** (page 219). The French mathematician Maurice Kraitchik poses a similar puzzle in **Mathematical Recreations**, with an illustration showing all ways of unfolding the room.

MARTIN GARDNER is best known for his long running "Mathematical Games" column in *Scientific American*. He has published five books with the Mathematical Association of America.

Such problems can be further generalized to solids (or rooms) of dimensions $a \times b \times c$. A more difficult question, suggested by computer scientist Donald Knuth, is to find maximum-length geodesics on such solids. For example, the maximum geodesic on the 1 x 1 x 2 solid is not from the center of one square face to the center of the other, a distance of 3. Hess has made the surprising discovery that the maximum path has a length of slightly more than 3.01. But Hess's results on maximum-length geodesics, as yet unpublished, are a long story. ■

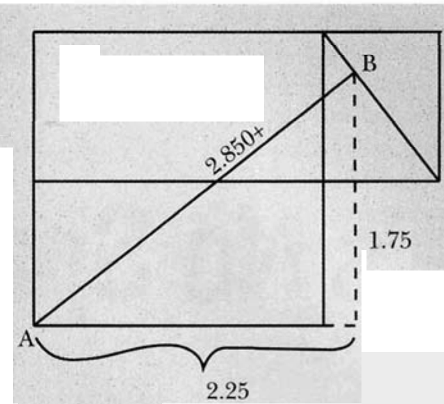


Figure 4

Gardner welcomes your comments, problems, and solutions. Write to him at the following address: Martin Gardner, 3001 Chestnut Road, Hendersonville, NC 28792.

DANIEL ASIMOV

There's No Space Like Home

To prepare for reading this article, try a few simple warm-up exercises. First, press your finger on the period at the end of this sentence, here: . Finished? Thank you. You have just touched numbers. Now raise your finger into the air and poke. You are prodding numbers. Now snap your fingers. You snapped numbers. More precisely, each of your gestures put you into contact with three numbers: the real numbers that, at least in this small region of the universe, describe the location of every point in space. There is no escaping those numbers. Wherever you go, whatever you do, you live and breathe and move amid a swarm of constantly changing coordinates. They are your destiny, the birthright of every denizen of three-dimensional space.

It is not clear who first conceived of a world saturated with numerical addresses. The idea of identifying points by longitude, latitude and altitude goes back at least to Archimedes, but it was not formalized until 2,000 years later, when the seventeenth-century French mathematicians Pierre de Fermat and Réne Descartes forged the link between geometry and algebra. Then, at some point in the nineteenth century, mathematicians took an important leap of logic. If ordered lists of numbers describe a space perfectly, they reasoned, why not say that those lists of numbers are the space? And in that case, why stop at three? They then boldly proceeded to define n-dimensional Euclidean space (n–space, for short), for any positive integer n, as the set of all n-tuples of real numbers $(x_1,\ldots, x_n)$. The symbol for such a space is an odd-looking R (for the real numbers) garnished with a superscript *n*: $\mathbf{R}^n$.

Having crossed that bridge, mathematicians found it fairly straightforward to extend geometric concepts such as line and sphere into higher dimensions. By applying the algebraic and trigonometric tools originally invented for analyzing the 2-tuples (ordered pairs) and 3-tuples (ordered triples) associated with ordinary geometric spaces, they were able to calculate analogues for lengths, angles, areas, volumes and a host of other quantities. What mathematicians could not do, and cannot do now, is visualize directly what higher-dimensional objects really look like. Instead, those of us who work with such concepts rely on calculation and intuition, supplemented, over the centuries, by a grab bag of techniques—including projections, cross sections and animations—to which computer graphics have added some valuable new twists. Nowadays the concept of n–dimensional Euclidean space permeates virtually all branches of mathematics: algebra, analysis, geometry, topology and even probability and statistics.

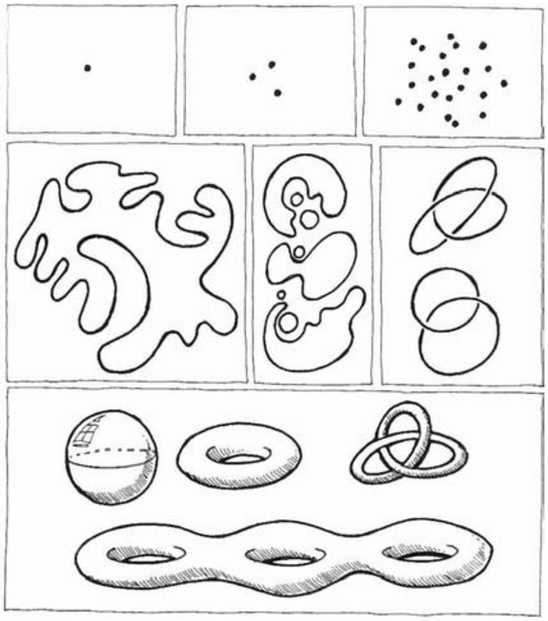
We are luckier than any hypothetical higher-dimensional beings: our space is the highest-numbered space with any hope of being sorted out.

You might expect that as the number of dimensions gets larger and larger, space gets stranger and more interesting. And so it does, in the trivial sense that any space has all the lower-dimensional spaces packed inside it. If planes (2-space) contain lines (1 -space) and three-dimensional space contains planes, then, in a way, anything that can take place on a line also takes place in 3-space, as well as in any higher-dimensional space. From a deeper point of view, however, every Euclidean space has its own character, and as far as the number of dimensions is concerned, more is often less.

Considered strictly on their own merits, higher-dimensional spaces tend to blur together into multidimensional sameness. It is often among the low-dimensional spaces that the most dramatic transitions take place: as the number of dimensions rises, fundamental properties suddenly flash into existence or vanish forever, never to change again. For many such properties, the birthplace or killing ground lies in the very space in which you are reading this article, "ordinary" three-dimensional space. Indeed, once you remove the blinders of familiarity, it turns out that 3-space is a fascinating place; in many important ways, there really is no space like home.

Are you skeptical? Then consider this statement: Three-dimensional space is the first Euclidean space in which it is possible to get hopelessly lost. To appreciate how that can be so,

Daniel Asimov is a mathematician and senior computer scientist at the NASA Ames Research Center in Mountain View, California, and a visiting scholar in the mathematics department at Stanford University.



Some 0-manifolds

1-manifolds

2-manifolds

it helps to see why it is impossible to get lost in one- or two-dimensional space.

Imagine that you are about to set out on a random walk through 1-space. You are perched like a tightrope walker on an infinitely long, infinitely narrow thread, the real-number line. To each side of you, stretching away endlessly as far as you can see, dangle tags representing the integers—positive to your right, negative to your left, each number exactly one step apart. The zero point, or origin, lies between your feet.

In one hand you hold a perfectly fair coin. Flip the coin and catch it. If it lands heads, step one number to the right; if tails, step one number to the left. Then flip the coin again. If you imagine (as mathematicians routinely do) that there is no death, fatigue or boredom to disturb your flipping and stepping, what is the probability that your random walk will eventually take you back to the origin? The answer turns out to be surprisingly simple: on a scale of zero to one, the probability is one—infinite likelihood.

Next, imagine repeating the random tour on a plane, taking your instructions not from a coin but from a four-sided die say, a regular tetrahedron, its faces labeled north, south, east, west. What is the probability that you will return to your starting point at least once in the course of your random walk? Again the answer is one, virtual certainty.

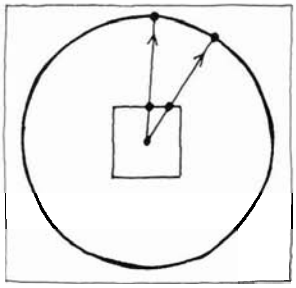
Now try the same thing in three dimensions, reading directions off an ordinary six-sided die with faces labeled north, south, east, west, up, down (in addition to being immortal and indefatigable, you have acquired the ability to levitate). If you can take infinitely many steps, what is the probability that you will return at least once to your starting point? If you sense a setup, you are right. This time the probability turns out to be only about 0.3405373, or roughly 34 percent. In higher dimensions the chances of returning to the origin are even slimmer. In spaces with a large number of dimensions, n, the probability of a return is approximately $1/(2n)$—the same as the probability that you will return to the origin on your second step. In other words, if you do not make it home at the first opportunity, you are probably lost in space forever. Wandering aimlessly is not likely to get you back; there are too many ways to go wrong. (See inside cover.)

In the physical world, random walks in space provide an ideal model for Brownian motion, the zigzag movement of molecules colliding in a gas. Because Brownian motion also describes the diffusion of heat, the properties of random walks have profound implications for heat conduction. Imagine, for instance, that mathematical space is made of a uniform, homogeneous substance that conducts heat. Imagine further that the space starts out at absolute zero, as cold as anything can possibly be. Now revisit each point to drop off a heat source—a perfectly symmetrical object that maintains a steady temperature, come what may, forever. (On the line such a symmetrical "space heater" would be a rod; on the plane, a disk; in 3-space, a ball.) What happens next?

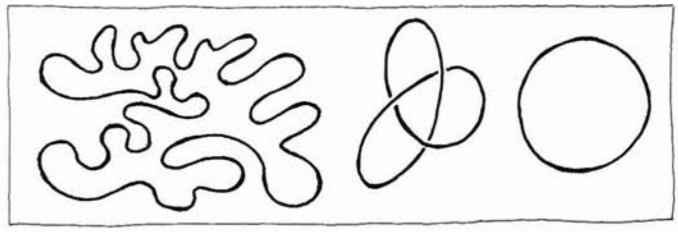
Eventually each space will warm up until it approaches an equilibrium distribution of temperatures. In 1-space and 2-space those temperatures are the same everywhere: they are the temperature of the heat source. But in 3-space the equilibrium temperature distribution is not constant; far from the ball, it will always approach absolute zero. No heater can ever generate enough heat to warm up all of three-dimensional Euclidean space or, for that matter, any other multi-dimensional space between 3-space and infinity. To rephrase Gertrude Stein, there is simply too much there there.

Euclidean space is a reassuring place—smooth, rectilinear, all-encompassing but it is also just a wee bit artificial. Real objects, after all, come in a multitudinous variety of shapes, some of them fiendishly complex. Just think of all the shapes you can make from one-dimensional "material": not only polygons but also every imaginable closed figure, every letter of every alphabet, every conceivable squiggle and doodle, including some so intricate that it would take eons to draw them. But however complicated such a figure may be, any small piece of it, viewed up close, resembles the one-dimensional Euclidean space of a straight line. Such shapes, created out of Euclidean space as modeling clay, are what mathematicians call manifolds, and they can come in any number of dimensions.

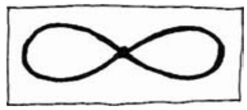
Faced with an unexplored manifold, you can find out quite a bit about it by showing that it is topologically equivalent to a simpler or better-known shape. To prove that two shapes are topologically equivalent, mathematicians set up a smooth one-to-one correspondence (known technically as a homeomorphism) between the points on the first shape and the points on the second one. The topological equivalence classes of the Geometric font of sans serif capi-



Radial projection provides a homeomorphism between a square and a circle. Note that this correspondence takes each point of the square to a point of the circle, and is both one-to-one and onto. Further, both the correspondence and its inverse (taking the circle onto the square) are continuous. Hence we know this is a topological equivalence.



Any simple closed curve is topologically equivalent to a circle.



A circle is NOT topologically equivalent to a figure-eight curve, since unlike the figure eight, there is no point of the circle whose removal will leave the circle disconnected in two pieces.

The sphere is NOT topologically equivalent to a torus since unlike the situation for the torus, every closed curve on the sphere can be continuously shrunk to a point.

tal letters are as follows: {C, G, 1, J, L, M, N, S, U, V, W, Z} (line segments), {E, F, T, Y} (lines with one "arm"), {K, X} (figures with four arms emerging from the same point), {D,O} (circles), {P, Q} (circles with one arm), {A, R} (circles with two arms). B and H each stand alone. (If you have trouble seeing how some of the correspondences work, remember that topology lets you rotate, bend, stretch and flip objects as necessary, to make, say, the bottom bar on the E match up with the lower part of the stem of the F.)

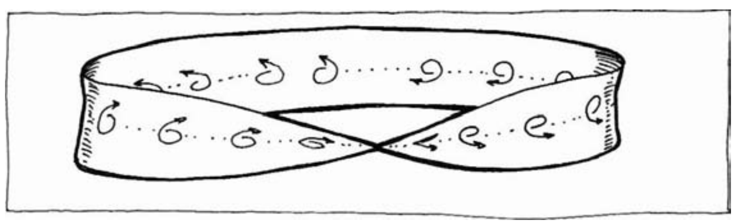
One simple but useful class of manifolds is made up of manifolds that are finite in extent, have no boundary and come in one piece. The one-dimensional manifolds, or 1-manifolds, with those properties include the letters D and 0 as well as ovals, ellipses, polygons and every other closed curve. All of them are homeomorphic to the circle, conventionary denoted S^1.

The 2-manifolds, or surfaces, are more interesting. They come in two varieties, orientable and nonorientable. A nonorientable surface is one that contains as a subset a Möbius band, the surface that results when you give a half-twist to a strip of paper and then tape the ends together. Orientable surfaces are easy to describe: each of them is homeomorphic to the surface of a button with some number of holes (possibly none). A button with no holes is topologically equivalent to the ordinary sphere S^2. The surface of a button with one hole is homeomorphic to a doughnut, which topologists call a torus.

Three-dimensional space can also be bent and stretched into any number of shapes. Because such distortions may veer into higher dimensions, people cannot visualize them "from the outside," but one can imagine what they would look like from the inside. To see how that works, imagine a two-dimensional creature—a paper-doll cutout, perhaps—living on the surface of a torus. The creature cannot stand outside the surface and observe the torus as a doughnut in three dimensions, but it can observe the peculiar properties of the surface itself. For example, light rays that shoot off to its right suddenly reemerge on its left, just as objects do in many video games. As a result, by looking in either direction, the creature can see the back of its head, and beyond that, as if in a hall of mirrors, infinitely many copies of itself and its surroundings. Gazing upward, the creature can see the light from its feet—and beyond that, an infinite number of receding copies of its environment as seen from below. From those observations, a paper-doll topologist could model its two-dimensional universe as a simple flat square, the top edge identified with the bottom edge, the left edge identified with the right.

Similarly, one of the simplest 3-manifolds is the 3-torus, which we three-dimensional creatures can visualize as a "glued" cube. The top of the cube is identified with ("glued" to) the bottom, the north face with the south face, and the east face with the west face. Denizens of such a manifold would see infinite repetition in all directions.

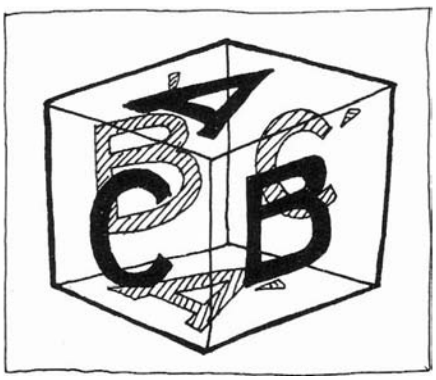
By gluing, twisting and otherwise identifying segments of complex surfaces, one can build up a bewildering array of possible 3-manifolds. But by establishing classes of topologically equivalent manifolds, topologists have made remarkable progress in narrowing down the possibilities. In that respect we are luckier than any hypothetical higher-dimensional beings. It has been shown mathematically that, in dimensions four and up, manifolds get so complicated that no algorithm can identify all of them. Nobody knows whether that is true in three dimensions. Here, then, is something else special about 3-space: In an important sense, it is the



The Möbius band can be modeled with a strip of paper whose ends are taped together after a half-twist. It is not orientable—which means that only one kind of spin, not two, is possible. By contrast, the cylinder (or plane, or sphere, or torus) is orientable: clockwise and counterclockwise spins can be distinguished from each other.

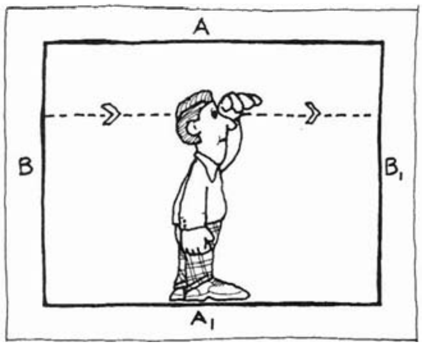
highest-numbered space that has any hope of being sensibly sorted out.

Given the variety of shapes a blob of space can assume, it always comes as a relief when an unfamiliar manifold turns out to be topologically equivalent to a simpler one. In any space, the pinnacle of simplicity is the manifold known as the unit sphere, defined as the set of points in $\mathbf{R}^n$ that lie precisely one unit from the zero point, or origin, of the space. In the plane $\mathbf{R}^2$, the unit sphere is just the familiar unit circle, called the 1-sphere, or S^1; it can be thought of as a one-dimensional line segment bent around and glued to itself. In 3-space the unit sphere is the ordinary spherical surface S^2 ; it does not include the points inside the surface. (The superscript of S^n refers not to the dimension the sphere resides in but to the stuff it is made of.)



The 3-torus is a 3-manifold and can be conceptually modeled from a solid cube by gluing the top-face to the bottom face, the left face to the right face, and the front face to the back face. These gluings cannot all be carried out in 3-space, but they can be done in 4-space.

In 1904 the French mathematician Henri Poincaré suggested a possible simple test for classifying three-dimensional manifolds. It might be true, he hazarded, that a 3-manifold is topologically equivalent to the 3-sphere if any loop in the manifold can be continuously shrunk down to a point. The loop test shows whether a manifold has any holes in it; spaces that pass the test are said to be simply connected. A 2-sphere is simply connected (as are its spherical cousins in every higher dimension); a 2-torus is not. That is why you can tie a firm slipknot through an iron ring but not around a basketball.



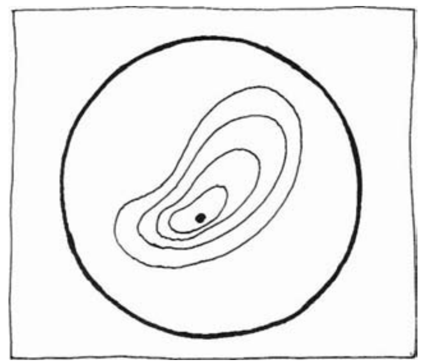
A torus may be modeled by a square whose top edge A is identified or "glued" to its bottom edge A_1 and whose left edge B is "glued" to its right edge to B_1. A two-dimensional being who lives in such a torus could see the back of its own head. A torus is homeomorphic to an inner tube or the surface of a bagel.

Poincaré's Conjecture is clear, elegant and potentially quite useful. Unfortunately, neither Poincaré nor any other mathematician to this day has proved that it works for all 3-manifolds. Proving what has come to be known as the Poincaré conjecture remains one of the great unsolved problems of mathematics.

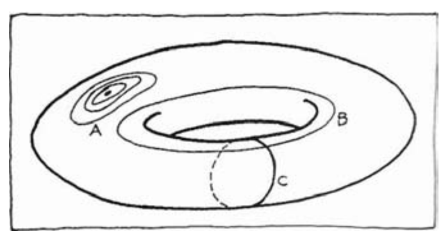
Strangely enough, an expanded version of the conjecture has been proved for all dimensions higher than three. The so-called Generalized Poincaré Conjecture is slightly more complicated than Poincaré's original conjecture. The mathematical objects that must be shrunk to a point include not just loops (1-spheres) but also ordinary spheres (2-spheres), 3-spheres, 4-spheres and so on, up to half the number of dimensions of the manifold in question. For example, to prove that a 4- or 5-manifold is homeomorphic to a four-or five-dimensional sphere, you must be able to shrink both loops and ordinary 2-spheres; for a 6- or 7-manifold the tests involve loops, 2-spheres and 3-spheres; and so on.

In 1960 Stephen Smale of the University of California, Berkeley, astonished the mathematical world by proving the Generalized Poincaré Conjecture for dimensions five and higher. Smale's ingenious proof involved decomposing a manifold into pieces called handles and then rearranging the handles until almost all of them canceled one another. What remained could be easily identified as homeomorphic to the sphere. In 1966 Smale was awarded the Fields Medal—the mathematical equivalent of the Nobel Prize—for that work.

Only the third and fourth dimensions remained. There Smale's techniques failed, because, in a sense, low-dimensional space is just too cramped. In dimensions five and higher there is plenty of maneuvering room to rearrange the handles to cancel one another. In dimensions three and four,



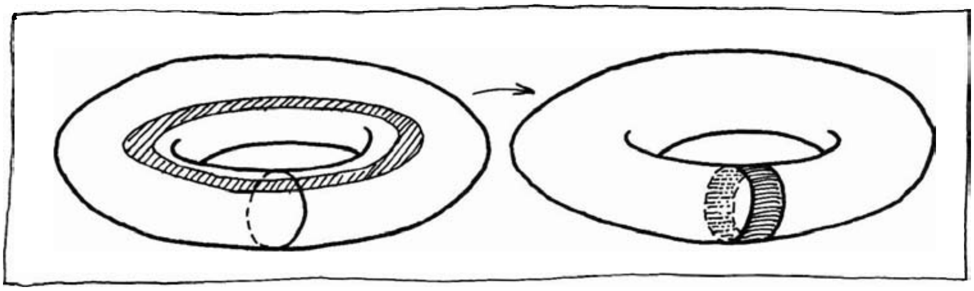
On a 2-sphere, any closed curve can be continuously shrunk to a point. This means the sphere is "simply-connected."



On the torus, however, some curves (like B or C) cannot be shrunk to a point. So the torus is not simply-connected.

however, the handles wind up hopelessly tangled. In 1982 the mathematical world was again astonished when Michael H. Freedman of the University of California, San Diego, proved the four-dimensional Poincaré Conjecture.

The ABCs of Topological Surgery



The torus has many interesting homeomorphisms to itself. For example, its two kinds of "holes" can be interchanged: Surgery on a 3-manifold involves first removing a solid torus, then gluing it back in differently by using an interesting homeomorphism of its boundary (an ordinary torus) to itself.

Even though Freedman's proof covered only a single dimension, it was much more complex than even Smale's had been. In 1986 Freedman, too, was awarded the Fields Medal for his achievement.

And so the matter stands: the Poincaré Conjecture has been proved in every dimension except three—the dimension in which it was originally stated.

Why is the three-dimensional case so hard to prove? Nobody knows, but one can speculate. For any n-manifold, the conjecture requires tests involving a certain number of spherical objects. For even-numbered manifolds the number is n/2; for odd-numbered ones, (n–1)/2. For dimensions four and above, the number of tests ranges between 40 and 50 percent of the number of dimensions. For three dimensions, however, it is a mere 33 1/3 percent. It may be no coincidence that the toughest nut to crack is the space with the fewest constraints relative to the number of degrees of freedom.

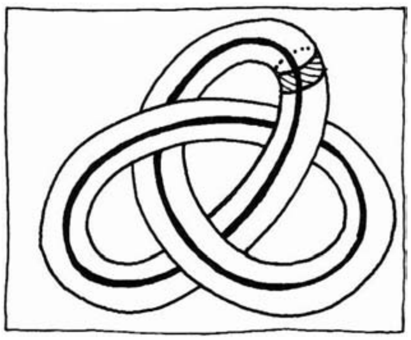
The Poincaré Conjecture aside, in recent years the study of 3-manifolds has surged forward. Some of the most important advances have come from work on the large class of manifolds that are finite in extent, orientable and made of a single piece. In one particularly amazing and useful breakthrough, topologists have discovered that they can start with the 3-sphere and, via a process of cutting and pasting called surgery, refashion it to produce any other finite, orientable, one-piece 3-manifold. Though more roundabout than a homeomorphism, the route from the sphere to a manifold can give much important information about the manifold itself.

How does topological surgery work? In the space defined by the 3-sphere, draw a simple closed curve. (Any collection of separate closed curves would work as well, but it is simpler to consider one.) Now let the curve expand to annex all the points within a small distance of it, as if they were frost forming on the cooling coil of a freezer. The result is a solid torus, the shape of a doughnut. Now transport the points inside the torus somewhere else, leaving only the surface wrapped around the doughnut-shaped hole in space, and stuff the points back into the hole in a different way. You have changed the 3-sphere into a new, but still finite, orientable, single-piece 3-manifold. By repeating the operation enough times, you can transform the original sphere into any finite, orientable, single-piece manifold you like.

Return to the first step in the surgery, a simple closed curve in 3-space. Topologists have a word for such a curve: knot. Confusingly, topological knots include loops that any nonmathematician would say are not knotted at all. Such rubber-band-like loops and their topological equivalents are considered unknotted knots (in much the same way that, in arithmetic, you might think of zero as a numberless number). Real knots—knotted knots—reveal another unique aspect of the third dimension. Amazingly enough, 3-space is the only Euclidean space in which they are possible. In all spaces with higher dimensions, any such curves can be transformed, without being cut, back into unknotted loops. (There are consolations, however. In 4-space you gain the ability to tie knots in a 2-sphere.)

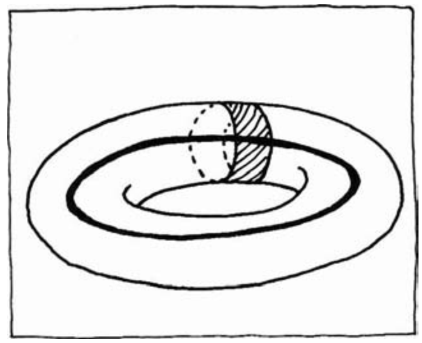
The discovery of surgery has made possible some of the most dramatic recent advances in topology. They resulted from the work of the mathematician William P. Thurston, then at Princeton University and now the director of the Mathematical Sciences Research Institute in Berkeley. In the early 1980s Thurston discovered that each manifold in a large class of 3-manifolds can be carved up into pieces with relatively simple geometric properties. Eight kinds of pieces, he determined, are enough to do the job: those with constant positive curvature; those with constant negative curvature; those with zero curvature; and five hybrids.

Step 1: Remove solid torus from 3-manifold, leaving solid torus shaped hole.



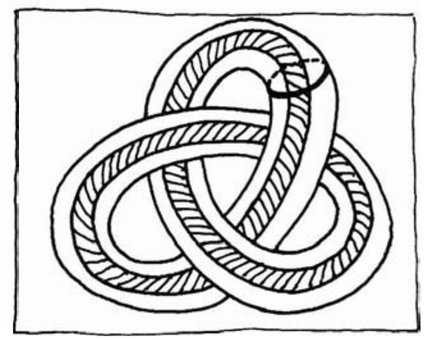
Solid torus-shaped hole

Step 2: Replace solid torus back in 3-manifold by using an interesting homeomorphism of its boundary (a torus to the boundary of the hole, which also is a torus).



Solid torus that was removed

Result: a (usually) topologically distinct 3-manifold from the original one.



From that standard kit of eight basic components, a wide variety of manifolds can be assembled.

One of the eight components is vastly more useful than the others. In an argument that relied heavily on the techniques of surgery on knots and links, Thurston showed that most of the 3-manifolds he had in mind can be broken down solely into pieces with constant negative curvature, or what is known as a hyperbolic structure. That makes such negatively curved manifolds extremely useful for studying 3-manifolds in general.

Thurston's work earned him the Fields Medal in 1983. By that time he had published a conjecture that, if true, would be even more sweeping: that not just "a large class" but all 3-manifolds can be decomposed into the eight geometric components. A proof of Thurston's geometrization conjecture would be a stunning advance for topology. Among other things, the three-dimensional Poincaré conjecture would follow as an instant corollary. But do not expect to see it in tomorrow's headlines: some mathematicians working in the field suspect the Poincaré conjecture is false.

In grappling with curvature, Thurston strayed beyond the bounds of topology. Topology deals only with shapes, not quantities; clearly, any discipline that considers a bagel equivalent to a beer stein is filtering out a lot of information. To determine curvature, however, you must be able to measure the distance between any two points on a manifold, and measurement comes under the purview of geometry.

In dealing with manifolds, another useful quantity to measure is volume. The 3-manifolds Thurston considered (technically known as complete hyperbolic manifolds) may be either finite or infinite in extent, but each of them encloses a finite volume. Volume marks yet another way in which 3-manifolds are unique. In all other dimensions the possible volumes of complete hyperbolic manifolds crop up along the positive side of the real-number line as tidily as telephone poles. For 2-manifolds (surfaces) with constant negative curvature, the volumes fall at points exactly 2π apart; for 4-manifolds, the distance is $4\pi^2/3$.

Plot the possible volumes for complete hyperbolic 3-manifolds, on the other hand, and the pattern you get is incomparably more interesting and beautiful. Instead of spreading out at prim intervals, the volume-measures cluster alongside accumulation points-numbers with an infinitude of possible volume-measures jammed next to them, creeping up from below. (For an idea of how that looks, try plotting the sequence 1/2, 2/3, 3/4, 4/5 and so on. The accumulation point is 1.)

Now, hold on to your hats! For if you ignore the original volume points and look only at the accumulation points bracketing them, you will see that those points have accumulation points themselves; and those accumulation points have accumulation points; and so on ad infinitum, each new accumulation point marking the volume of at least one complete hyperbolic 3-manifold. And with that endlessly unfolding display of numerical pyrotechnics, unparalleled in any other dimension, I wish you a pleasant return to your native manifold. ■

Illustrations by John Johnson

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ANDY LIU

Counting Sides and Angles

A partition of a polygon into triangles is called an anti-triangulation if no two triangles share a complete common side. Figure 1 shows an anti-triangulation of a triangle.

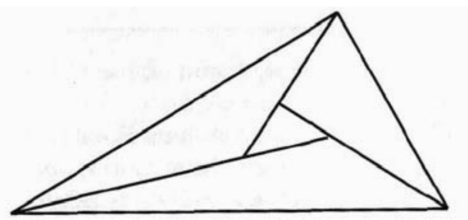

Figure 1

A very attractive and challenging problem was proposed by Nicolas Vasiliev of Moscow for the international Mathematics Tournament of the Towns conference at Beloretsk, Russia, in August of 1993. Part of it is to determine all positive integers n for which there exists an anti-triangulation of a convex n-gon.

We urge the reader to experiment, formulate a conjecture and try to prove it.

Nicolas was probably the proposer of the following problem in the Fall 1993 tournament. A convex 1993-gon is partitioned into convex 7-gons. Two adjacent 7-gons share a complete common side, and each side of the 1993-gon is a side of a 7-gon. Prove that there exist three consecutive sides of the 1993-gon which are sides of the same 7-gon.

This problem can be solved by the technique of counting sides and angles.

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Let f denote the number of 7-gons, v the number of their vertices inside the 1993-gon, and e the number of their sides inside the 1993-gon.

Each 7-gon has 7 sides. Hence the total number of sides, counting multiplicities, is $7f$. Each side of the 1993-gon contributes 1 to this total, while each interior side contributes 2. It follows that $7f = 1993 + 2e$.

The sum of the angles of each 7-gon is 5π. Hence their total measure is $5f\pi$. The angles of the 1993-gon contribute 1991π to this total, while the angles around each interior vertex contribute 2π. It follows that $5f=1991+2v$. Eliminating f, we have $3972=10e-14v$.

Let b denote the number of vertices of the 1993-gon which lie on at least one interior side. Since the 7-gons are convex, each interior vertex lies on at least 3 interior sides. Hence $2e \geq 3v+b$. It follows that $3972 \geq 15v + 5b - 14v > 5b$ or $795>b$.

This means that more than half of the vertices of the 1993-gon do not lie on any interior sides. Hence there exist two such vertices which are adjacent. The three consecutive sides of the 1993-gon around them must be the sides of the same 7-gon.

We now return to the anti-triangulation problem, which can be solved by the same technique.

Consider an arbitrary partition of a convex n-gon into triangles, as illustrated in Figure 2 with $n=4$. Denote by f the number of triangles. Here, $f=9$. There are two types of vertices on the boundary of the n-gon. Type-C vertices are those of the n-gon. Type-B vertices are those which lie on the sides of the n-gon. There are n of the former, and denote by b the number of the latter. Together, they divide the boundary of the n-gon into $n+b$ boundary segments.

There are also two types of vertices inside the n-gon. Type-D vertices are those which are vertices of every triangle to which they belong. Type-A vertices are those which lie on a side of at least one triangle. Denote their numbers by d and a respectively.

An interior segment is defined to be any side of a triangle inside the n-gon which is not a proper subset of a side of another triangle. Denote their number by e. In figure 2, $e = 10$. C_3A_2 and A_2A_1 are not segments because both are proper subsets of C_3A_1. Similarly, C_4D_1 is a segment but C_4A_1 and A_1D_1 are not.

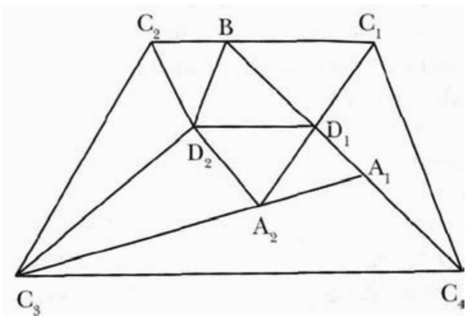

Figure 2

If such a partition is to be an anti-triangulation, we must have at least one Type-A vertex on each interior segment. Thus a necessary condition is $e \leq a$.

Each triangle has 3 sides. Hence the total number of sides, counting multiplicities, is $3f$. Each of the $n+b$ boundary segments contributes 1 to this total. Each interior segment contributes 2 plus the number of Type-A vertices on it. It follows that $3f = n + b + 2e + a$, since each Type-A vertex lies on exactly one interior segment.

The sum of the angles of each triangle is π. Hence their total measure is $f\pi$. The angles of the n-gon contribute $(n-2)\pi$ to this total. The angles around each Type-A and Type-B vertex contribute π, and those around each Type-D vertex contribute 2π. It follows that $f = n - 2 + b + a + 2d$.

Eliminating f, we have

$$e - a = n - 3 + b + 3d \geq n - 3.$$

If $n > 3$, then $e > a$ and we cannot possibly have an anti-triangulation. Hence the only convex polygon which can be anti-triangulated is the triangle!

For an account of the Beloretsk Conference, see [1]. For another example of the technique of counting sides and angles, see [2].

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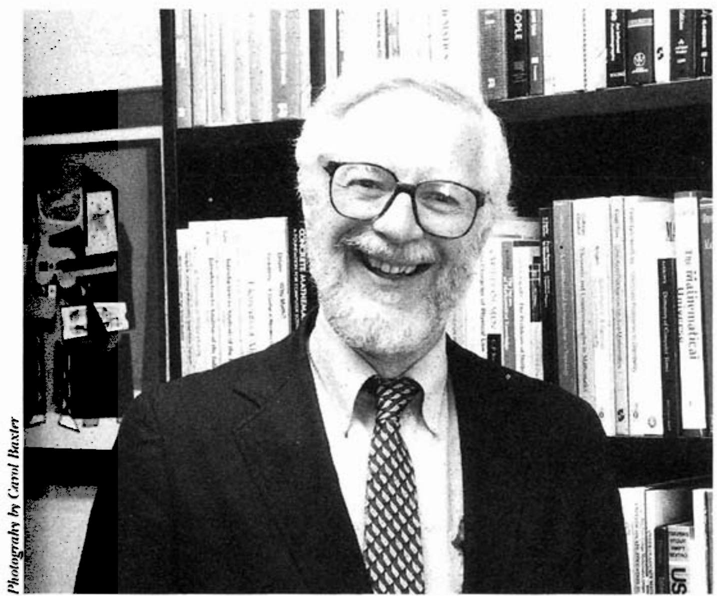
1 Liu, A., "A Mathematical Journey," *Crux Mathematicorum* 20 (1994) 1–5.

2 Niven, I., "Convex polygons that cannot tile the plane," *American Mathematical Monthly* 85 (1978) 785–792.

DONALD J. ALBERS

Tom Banchoff: Multidimensional Mathematician

Professor Thomas Banchoff is one of the first people on planet Earth to interact with the fourth dimension. Thanks to a blend of his powerful imagination and computer technology, he has made it possible for all of us to see beyond the three-dimensional space in which we live. The images that he displays on his computer screen are beautiful, exciting, and often surprising. To be more precise, what Banchoff enables us to "see" are three-dimensional shadows of four-dimensional objects. "We're trained from very early childhood," Banchoff explains, "to interpret the two-dimensional shadows of three-dimensional objects. We all learn to infer the shapes of three-dimensional objects from their shadows. So if we want to visualize a four-dimensional object, the best thing to do is work with three-dimensional shadows."



Professor Tom Banchoff, just back from another encounter with the fourth dimension.

Captain Marvel

Banchoff has been fascinated with the fourth dimension since he was ten years old. He first read about it in a comic book, "Captain Marvel Visits the World of Your Tomorrow." One of the panels shows a boy reporter going into a futuristic laboratory where his guide says, "This is where our scientists are working on the seventh, eighth, and ninth dimensions." A thought balloon rises above the hero's head: "I wonder what ever happened to the fourth, fifth, and sixth dimensions." Banchoff, too, has been wondering ever since. He decided to keep trying until he understood the fourth dimension completely or until it became boring. Very soon he realized that he would never be able to figure it all out, and that it would never get boring.

He really got into thinking about the fourth dimension as a student at Trenton Catholic Boys' High School. By the time he was a sophomore, he had developed a full–fledged theory of the Trinity. He remembers one afternoon cornering Father Jeffrey, a sympathetic biology teacher, and trying as hard as he could to explain his theory—if God came from the fourth dimension into our three-dimensional world, all we would see is a 'slice' person who would look like us, but there would still be two other parts of God that we couldn't see, and that's where the Trinity comes in. Father Jeffrey was amused by his earnestness and asked why it was so important for the theory to be validated that day. He answered, "Because tomorrow I'm going to be sixteen years old."

Banchoff grew up in Trenton, New Jersey. His father, who was a payroll accountant, impressed upon him the fact that English and arithmetic were the most important subjects in the curriculum. He was also very concerned that Tom should be a "regular guy." His father was actually rather suspicious of intellectuals. He knew people who read books and discussed them, but he himself wasn't a reader. His favorite example was a middle-aged, somewhat eccentric neighbor who spent most of his time in the library and carried his laundry in a paper sack. "My father warned me," recalls Banchoff, "You don't want to grow up like him." On the other hand, his mother, a kindergarten teacher, was a great reader and was very encouraging.

So Tom proceeded to become a regular guy. He played on the tennis

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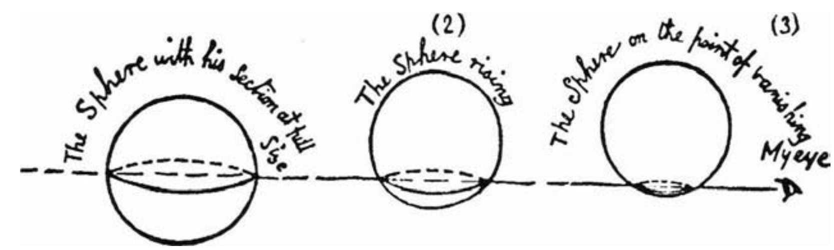
Flatland

In 1884, a British schoolmaster, Edwin Abbott Abbott, wrote the classic introduction to the dimensional analogy. His small book *Flatland* is narrated by A Square, living on a two-dimensional flat universe and as incapable of comprehending our geometry as we are when we try to conceptualize a fourth spatial dimension. We are invited to empathize with the experiences of A Square, first of all in his own two-dimensional world, a social satire on Victorian England, and then as he is confronted with a visitation by a being from a higher dimension, A Sphere, who passes through his universe, giving to the two-dimensional onlooker the impression of a growing and changing circular figure. This experience challenges A Square to rethink all that he had previously taken for granted about the nature of reality. Analogously, we are challenged to imagine the experience of being visited by beings from a fourth spatial dimension.



As A. Sphere passes through Flatland, the two-dimensional section changes, starting as a point, reaching maximal extent as a circular section, and then reducing to a point as it leaves flatland.

and soccer teams. In addition, he was in the school play, orchestra, band, and debating team as well as editor of the school paper and the yearbook. At his high school, it was considered all right to get high grades as long as it was clear that you weren't spending all the time studying, so he maintained a 99+ average over four years. He won the regional science fair, represented his district in the National Student Congress, and was in the first class of National Merit Scholars. And he was an altar boy, too!

In January 1996, Banchoff won one of the MAA's Distinguished Teaching Awards. He cares deeply about teaching and is proud of his award. He always knew that he would be a teacher, but he wasn't sure at first what it was he wanted to teach. For several years it was a toss-up between mathematics and English. At Notre Dame, he remembers, "In mathematics when I made a mistake and the teacher pointed out a counterexample or a flaw in the argument, I could accept that. I didn't have the tolerance for ambiguity that was necessary to become a scholar of literature. But in mathematics, I knew by that time that I could come up with original ideas, and the courses definitely were challenging. So I decided to become a mathematics teacher."

Many teachers influenced Banchoff. In his freshman year of high school, Father Ronald Schultz stood out. "Although I never took a class from him, he was the first one who really listened to my mathematical ideas and encouraged me, especially in geometry."

Shadows

Banchoff discovered his first geometry theorem as a freshman. "Every Friday morning, the whole school would file into church for Mass, and our home room was the first to enter. While waiting for the rest to come in, there was plenty of time to contemplate the shadows advancing across the tiles at the base of the altar rail. When we first arrived, the narrow of the altar rail covered only a small portion of the triangular tiles, and by the end of Mass, almost the entire triangle was in shadow. When, I asked myself, did the shadow cover half the area? I hadn't studied any formal geometry yet, but I figured that if you cut an isosceles right triangle in half by a line perpendicular to the hypotenuse, then one of those halves could be rotated to give the triangle that remains when the shadow was covering half of the original triangle. It surprised me that the line did not pass through the centroid of the triangle! To this day, I still use that example when I teach calculus students about centroids."

Another early influence was Herbie Lavine, who was three years older than Banchoff and "real smart." Herbie worked in his father's grocery store, and when Tom was in grammar school, he would teach him the mathematics he was learning. His father would remind him that he was supposed to be unloading packing crates and not doing algebra on them! When Tom was in seventh grade, Herbie told him about a classic problem involving twelve billiard balls, one of which was either heavier or lighter than all the rest. How could you find the 'odd ball' in three weighings using just a balance scale? Tom couldn't solve it right away and Herbie was about to show him the answer. Tom said "no," he wanted to work it out by himself. "We both forgot about it," Banchoff recalls, "and soon afterward Herbie went off to college, while I started high school. Once again it was at one of the Friday morning masses that I received an inspiration: a pattern on one of the stained glass windows reminded me of the billiard ball problem from three years earlier, and the pattern gave me the idea for solving it. I sent my solution off to Herbie at the University of Michigan, and got a letter back saying it was right. It made me feel good to know that I could solve a problem that took a long time, and not just the usual problems that you can either do immediately or not at all." (Herbie went on to become an actuary and a professional bridge player. He and his wife and son visited Brown a couple of years ago, and Tom took them out to lunch. He astounded them by telling them that Herbie had been his mathematical hero when he was young.)

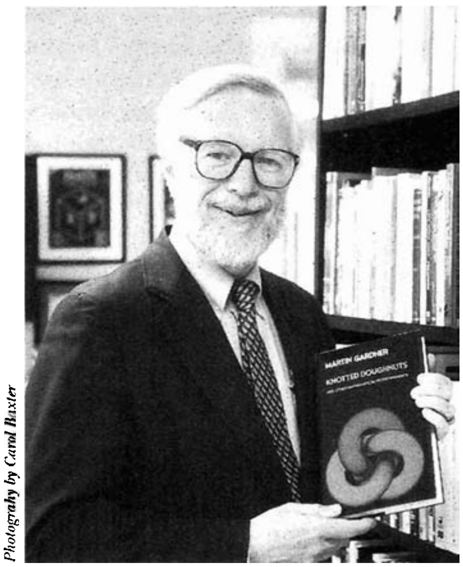
In high school, one of the things Tom liked most about mathematics was that he was asking questions that were different from the ones that his classmates and teachers were asking. "When I got to college, I realized that was still true. I knew that most of the things I observed had been seen before, but I thought even then that maybe I might have some insights that nobody else would have, that I would prove something that nobody ever would have thought of if I hadn't done it. And that was very appealing to me. I loved the creative aspect of mathematics. I was lucky enough to realize something about the creative aspect of mathematics when I was young Individually, the theorems I proved are almost trivial things, but I remember them very clearly. Curiously enough some of them keep showing up—I'm still watching shadows and cutting things in half!"

During his senior year of high school, Banchoff's math project on three-dimensional graphs of complex valued functions of a real variable won the regional science fair and earned him a trip to the National Science Fair in Oklahoma. On his return trip, he rerouted himself so he could make his first visit to the University of Notre Dame, where he had just been admitted. He met the Dean of the Arts and Letters College who introduced him to the well-known research mathematician, Professor Ky Fan. When he started explaining his science project, Fan interrupted him to say that he should spend his time learning mathematics, not trying to do original projects.

The next person he met was the mathematics department chairman, Dr. Arnold Ross. "I hear you are interested in becoming a mathematician," said Ross. "I was until five minutes ago," responded Banchoff. "Oh? Tell me about it," he said in his very fatherly way. He listened as Banchoff explained his project, and then said, "There is a mistake in this expression for the fourth root of -1. Go up to the blackboard right now and find the correct form." I said, "You mean, go to the blackboard right now and just do it?" He nodded, so I went up and figured it out. As I turned around, as surprised as I was proud, he smiled. Just like that I wanted to be a mathematician again."

You will never be a mathematician

As a senior at Notre Dame, Banchoff received from Ky Fan the only C of his life in a second year graduate course in general topology. He didn't realize at the time that he didn't really have enough background in advanced analysis to appreciate the very formal generalizations and proofs. "It was in that class that I began to appreciate how it feels to be lost most of the time. I was also taking courses in literature and philosophy at the time, and Dr. Fan had formed the impression I was not a serious mathematics student. I was so intimidated by the lectures that I would study the notes from the previous year before each class just in case he would ask a question. One day my fears were realized. He turned and said, 'Banchoff, what is a set of the second category?' I started my answer, "Well, a set of the..." and that was as far as he let me get. He shouted back, "'Well, you say 'Well'? This is not an English class, this is a mathematics class. When I ask a mathematical question I want a mathematical answer, not 'well'." I then responded: "A set of the second category is a set that cannot be expressed as a countable union of nowhere dense sets." "Why didn't you say that the first time? 'Well'!" It was pretty clear that the chemistry between us wasn't very good."



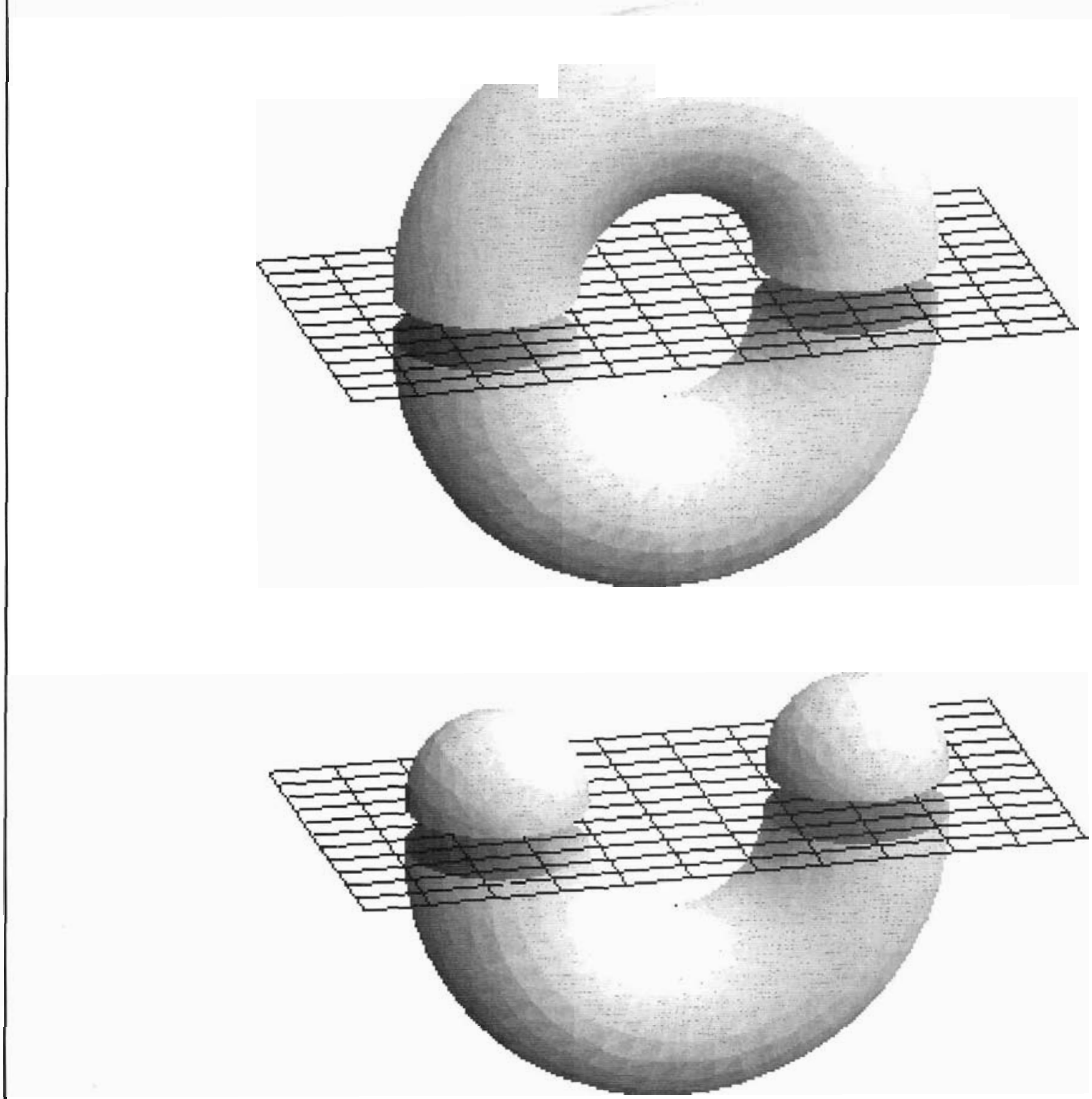
Photography by Carol Baxter

Banchoff is particularly fond of shadows, especially those of four-dimensional objects.

He stayed in that course for the second semester and gradually began to catch on so that he was able to raise his grade to a B. After missing class one day, he managed to infuriate Dr. Fan by a question he asked him in the hallway after class. At the top of his lungs he delivered his estimation of Banchoff, loud enough for the whole department to hear: "Mr. Banchoff, you will never be a mathematician, never! never!" At that moment he seriously wondered if he was in the right field.

After receiving his bachelor's degree in 1960, Banchoff began graduate studies at the University of California, Berkeley. Algebraic and combinatorial topology with topologist Edwin Spanier and differential geometry with visiting professor Marcel Berger introduced him to the areas that became his specialty. He became a research assistant to the differential geometer Shiing-Shen Chern who suggested a thesis project in total absolute curvature, the study of surfaces like a torus of revolution, that are "as convex as they can be." After making a great many drawings and models, he found an elementary way of interpreting that condition called the Two-Piece Property (TPP), and he found himself in the unusual position of being able to explain what his thesis topic was about, even to non-mathematicians. Using the TPP, he could consider polyhedra as well as smooth surfaces, and he came up with some models to show the difference between the smooth and the polyhedral cases. He was making some progress, but he still didn't have a big breakthrough.

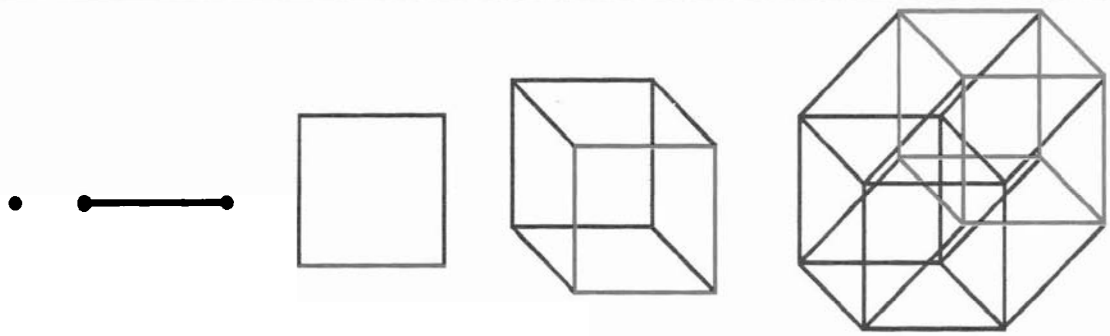
One day Professor Chern told Banchoff that he wanted to introduce him to Nicolaas Kuiper, a special visitor from Holland. "You two think alike" he said. Kuiper showed him some of his



Two-Piece Property

Certain items on a breakfast table have the property that they fall into at most two pieces when they are sliced with a long straight knife, for example an orange, or a hard-boiled egg. Other objects, like a fork or a sufficiently curved banana will fall into three or more pieces when cut in a certain direction, so they do not have the two-piece property (TPP). Any convex object has the TPP, but there are also non-convex objects with this property, for example a doughnut or bagel, or a half-cantaloupe, or a stemless apple (but not a pear or a peach). The study of smooth surfaces that have the TPP involves "total absolute curvature," a notion from differential geometry, the subject that applies calculus techniques to problems in the geometry of curves and surfaces in the plane, in three-space, and in higher-dimensional spaces. A closed curve with the TPP must lie in a plane, whether it is smooth or polygonal. By a theorem of Nicolaas Kuiper, a smooth surface with the TPP has to lie in a five-dimensional space, but, surprisingly, there are polyhedral surfaces with the TPP in six-space that do not lie in any five-dimensional subspace. More generally, in n-space there are TPP polyhedral surfaces not lying in any space of lower dimension, although for higher and higher dimensions, the surfaces must become more and more complicated. This is the primary contribution of Thomas Banchoff's Ph.D. thesis.

Images courtesy of Tom Banchoff. These images may be viewed on Banchoff's World Wide Web Page. WWW address: http://www.geom.umn.edu/~banchoff/



Hypercube

What does a shadow of a four-dimensional cube look like? To draw a two-dimensional shadow of a three-dimensional cube, we start with a parallelogram that is the shadow of a face of the cube, then move the parallelogram along a third direction in the plane and connect the corresponding vertices. Similarly we can obtain a three-dimensional shadow of a four-dimensional cube, or "hypercube," by moving a parallelepiped along a fourth direction in three-space and connecting the corresponding vertices. If we collect the shadows of an ordinary cube as it rotates, then we create an animated film that we learn to interpret as the images of a cube. In a similar way, if the hypercube rotates in four-space, then its shadows in three-space will produce an animation that we can interpret first as shadows of an object moving in three-space, but then some unfamiliar movements occur. With a good deal of experience it is possible to predict the changes that occur in the shadows of a rotating hypercube.

Illustration by Tom Banchoff.

recent papers on smooth surfaces with minimal total absolute curvature, and told him about his key result that says there are no smooth examples of this phenomenon in dimensions higher than five. He suggested that Banchoff might find the polyhedral analogues of these theorems, and so he went to work.

Benefits of Doing Laundry

A week later, while Kuiper was still visiting, he was folding his wash in the laundromat, and at the same time trying to come up with an argument to show why there were no TPP polyhedral surfaces in six-dimensional space, when all of a sudden, he saw how to construct one! He made a paper model that could be folded together in six-space and showed it to Kuiper the next day. He was astonished. He said to Banchoff, "What you have here is a gold mine. I'll give you six months to write a thesis about it. If you haven't done so, I'll give the problem to one of my students. It's too good a problem not to be done by somebody." He finished the thesis project in three months, and he also started studying Dutch. He knew where he wanted to do his postdoctoral work!

After receiving his Ph.D. in 1964, he was a Benjamin Peirce Instructor at Harvard for two years. He then spent a year with Kuiper in Amsterdam, and continued to work with him as a colleague up until his death in 1994.

Shortly after taking up his faculty position at Brown in 1967, Banchoff met Charles Strauss, an applied mathematician with special talents in computer graphics, which was then a brand new field. Strauss was looking for new problems for his interactive "three-dimensional blackboard," and it was clear that these new programs could not only show two-dimensional images of complicated three-dimensional objects, but also they could produce rotating shadows of objects from four-dimensional space. Together, Banchoff and Strauss produced a series of computer-animated films, the most notable being "The Hypercube: Projections and Slices," first shown at the International Congress of Mathematicians in Helsinki in 1978. It is a grand tour of a basic four-dimensional object that has never been built and never can be built in our three-dimensional space.

Almost a quarter century has passed since Banchoff began using computers to enhance his interaction with the fourth dimension. Tremendous advances with hardware and software have enabled Banchoff to help all of us see beyond the third dimension

"Right up until the time I got my Ph.D., I had this recurring nightmare that my advisor Professor Chern would run into Dr. Fan and that my name would come up in their conversation. Then Professor Chern would come back and tell me, regretfully, that he had learned that I would never be a mathematician, never, never. As it happens, several years later I saw Dr. Fan at a mathematics meeting in New Orleans. I went over and introduced myself as one of his former students. He tried to place me. "You were in my freshman course?" "No, you're thinking of Jim Livingston." "You were interested in the four-color problem?" "No, that was Jim Wirth." "Ah, yes, 'Banchoff'" He paused, then he said, "Mr. Banchoff, I am happy to see that you have developed into a mature mathematician." ■

Available Films

"The Hypercube: Projections and Slicing," International Film Bureau, 332 South Michigan Ave., Chicago, IL. 60604, (312) 427-4545

"The Hypersphere: Foliation and Projections, and Fronts & Centers," The Great Media Company, PO Box 98, Nicasio, CA 94946 (415) 662-2426

JOEL CHAN

Prime Time!

The ancient Greeks were among the first to look at prime numbers, but mathematicians learned most of the interesting stuff about primes during the 19th century. As we all know, prime numbers are natural numbers which are not multiples of any smaller positive integer except 1.

Here are the numbers 1 through 100. Take the number 2 and circle it, then cross out all multiples of 2. Then circle the smallest uncrossed number and cross out all its multiples. Repeat the process.

1	②	③	~~4~~	⑤	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

This is known as the Sieve of Eratosthenes, which was invented around the 3rd century BC. Of course, this raises a few questions. One can imagine that the distribution of primes becomes smaller as we look at bigger numbers. So how many prime numbers are there?

Theorem 1. *There are an infinite number of primes.*

Poof†. Assume that there are n primes. Call them $p_1, p_2, p_3, \ldots, p_n$. Now consider

$$N = p_1 p_2 p_3 \cdots p_n + 1.$$

Is N prime? If it is, we've got a problem, since there are now more than n primes! So N must be composite. But if it is composite, then it can be factored into a product of primes. But N divided by any of the primes $p_1, \ldots, p_n$ gives a remainder of 1. This means that N has to be factored by primes other than those n primes. Contradiction. Fini.

A mathie would be quick to state that I've assumed a famous statement, known as the Fundamental Theorem of Arithmetic:

Theorem 2. *Every positive integer greater than 1 can be expressed as a product of primes in only one way.*

Poof. Take a course in number theory.

One of the most intriguing problems in mathematics is determining whether a large number is prime. So far, the algorithms that mathematicians have found are not very efficient (this is actually a good thing—see the article "Three Guys and a Large Number" in the February 1995 *Math Horizons*), but at least checking primes provides constructive work for supercomputers.

The largest known prime is:

$$2^{859433} - 1.$$

It would take too many pages to type the number. It has 258,716 digits! This number was discovered on January 4, 1994 using a CRAY supercomputer.

As for the distribution of primes, Carl Friedrich Gauss examined a table of primes in 1792—he was 15!—and conjectured (that's mathiespeak for "guessed") that the number of primes less than or equal to x is asymptotically equal to

$$\int_2^x \frac{dt}{\log t}$$

where $\log t$ is the natural logarithm of t. Since the integral is approximately equal to $x/\log x$ as x gets very large, this led to the famous prime number theorem, which was independently proven in 1896 by J. Hadamard and C.J. de la Valleé Poussin.

Theorem 3. *The number of primes not exceeding x is asymptotically equal to $x/\log x$.*

Poof. Take an advanced course in number theory.

Surprisingly enough, Gauss' integral is much more accurate than $x/\log x$ in estimating the number of primes less than x. Denoting $\pi(x)$ as the actual number of primes less than or equal to x,

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† A poof is one of the following:
- A proof that sneaks up on you and hits you like an uncountable number of bricks and then gets erased from the chalkboard before you absorb it.
- The main point of such a proof.
- A highly improbable construction (especially nonconstructive) which gives rise to such a proof. (The rabbit that gets pulled out of the hat.)
- Something which some students supply when asked to give a proof, particularly on tests. Said students do not necessarily continue in mathematics.
- Proof by intimidation: "You all see this, don't you!?!"

$$\pi(10^{16}) = 279.238.341.033.925$$

$$\int_2^{10^{16}} \frac{dt}{\log t} \approx 279.238.337.819.293$$

$$\frac{10^{16}}{\log 10^{16}} \approx 287.042.630.878.318.$$

Of course, why bother with estimation?

Theorem 4. $\pi(x) = -1 + \sum_{j=1}^{\lfloor x \rfloor} f(j)$, *where*

$$f(j) = \left\lfloor \cos^2 \frac{(j-1)!+1}{j/\pi} \right\rfloor.$$

Those weird brackets denote the greatest integer of that mess.

Before I forget, here is a cute observation that was proven by Euler in 1737.

Theorem 5. $\sum_p \frac{1}{p}$ *diverges to infinity.*

That is to say, the sum of the reciprocals of the prime numbers diverges towards infinity.

Now that Andrew Wiles has spoiled all the fun and proven Fermat's Last Theorem, a nice question to ask ourselves is: what is the most intriguing mathematical mystery that hasn't been solved? A simple criteria should be whether or not the mathematical problem can be understood by the general public, regardless of whether there are any applications or results that come as a consequence. Mathematicians (and computer scientists) might vote for the problem of whether or not P = NP, a problem involving computational complexity, but it cannot be explained in layman's terms. There are certainly many questions involving prime numbers that have not been solved, so let's look at a few of those problems which baffle mathematicians today.

Goldbach's Conjecture. *Every even integer greater than 2 can be written as the sum of two primes.*

Not only is Goldbach's Conjecture a prime candidate (excuse the pun!) as one of the most well-known unsolved mysteries in mathematics, G. H. Hardy also described this problem as one of the most difficult. The conjecture came about in a letter from Christian Goldbach (1690–1764) to Leonhard Euler in 1742, when Goldbach speculated that this was true.

The consensus seems to be that Goldbach's Conjecture is indeed true. The conjecture has been proven correct for all even integers up to 20,000,000,000. There have also been many proofs which have shown that if the integer is sufficiently large, then it can be expressed as a sum of two primes.

Goldbach also conjectured that any odd integer greater than 7 can be expressed as the sum of three odd primes. This is also an open problem, but it was shown in 1937 that this conjecture is true for any integer bigger than $3^{3^{15}}$! You might say, "OK, well, let's use a supercomputer and check for all odd integers less than $3^{3^{15}}$," but this number is over seven million digits long!

As an exercise, show that Goldbach's Conjecture is equivalent to the statement: Every integer greater than 5 can be expressed as the sum of three primes.

Twin Primes Conjecture. The odd integers p and $p+2$ are twin primes if both p and $p+2$ are prime. *There are infinitely many twin primes.*

Like Goldbach's conjecture, numerical evidence suggests that the Twin Primes conjecture is true. But here is a reasonable argument that the conjecture is true.

Poof. By the prime number theorem, the number of primes that are less than n (when n is sufficiently large) is about $n/\log n$, or the probability that a certain sufficiently large odd integer is prime is about $1/\log n$. So the probability that two consecutive sufficiently large odd integers are prime is about

$$\frac{1}{\log n} \cdot \frac{1}{\log(n+2)} \approx \frac{1}{(\log n)^2}.$$

That is, given a sufficiently large odd integer n, there are about

$$\frac{n}{(\log n)^2}$$

pairs of twin primes. Observe that

$$\lim_{n\to\infty} \frac{n}{(\log n)^2} = +\infty,$$

so this strongly suggests that there are infinitely many pairs of twin primes.

A similar unsolved problem is whether or not there are infinitely many prime triplets of the form $(p, p+2, p+6)$. It is a trivial exercise to show that there are finitely many sets of primes of the form $(p, p+2, p+4)$.

Mersenne Primes Conjecture. A *Mersenne prime* is a prime number that can be written in the form $2^n - 1$. The largest known prime which was stated earlier is a Mersenne prime. *There are infinitely many Mersenne primes.*

Even though only 33 Mersenne primes are known, Paulo Ribenboim has conjectured that there are infinitely many! Like the twin primes conjecture, Ribenboim uses a probabilistic argument to convince us that this is probably true.

As an aside, if a Mersenne number is prime, then the number $2^{n-1}(2^n - 1)$ is perfect, in other words, it is exactly the sum of its proper divisors.

Sierpinski's Postulate. *For any positive integer n, there exists a prime number between n^2 and $(n+1)^2$.*

This is based on Bertrand's Postulate—for any integer n there exists a prime number between n and $2n$—which was proven true in the late 19th century. Another similar open-ended question is the following:

Brocard's Conjecture. *Between the squares of two successive primes greater than* 2 *there are at least four primes, i.e.,* $\pi(p_{n+1}^2) - \pi(p_n^2) \geq 4$ *for* $n \geq 2$.

There is another question that remains unsolved: *Is there a simple (nonconstant) formula that generates every prime, or only primes?* So far, no "simple" formula is known. For instance, it can be easily shown that there is no nonconstant polynomial f with integer coefficients such that $f(n)$ is prime for every positive integer n.

Finally, we look at useless primes. We take them from Paulo Ribenboim's book *The Little Book of Big Primes,* which I highly recommend.

First we note that 11 is prime and all of its digits are 1s. The next such prime is

$$1,111,111,111,111,111,111.$$

$\frac{10^{1031}-1}{9}$ is the largest known prime with all its' digits equal to 1.

$$10^{5004} + 12323210 \times 10^{4998} + 1 = 1\underline{0\ldots0}12323210\underline{0\ldots0}1$$

is the largest known palindromic prime. (The underlined stuff indicates 4997 zeroes!)

And finally the champion of useless primes:

$$7532 \times \frac{10^{1104}-1}{10^4-1}$$

is the largest known prime whose digits are all prime!

References

Tom M. Apostol, *Introduction to Analytic Number Theory,* Springer-Verlag, New York, 1976.

Paulo Ribenboim, *The Book of Prime Number Records,* Springer-Verlag, New York, 1988.

——, *The Little Book of Big Primes,* Springer-Verlag, New York, 1991.

The largest known primes home page: http://www.utm.edu/research/primes/largest.html.

A Bunch of Nonsense[1]

Deanna B. Haunsperger
Stephen F. Kennedy

The game of creating specialized collective names for groups of like individuals has enriched our language with such linguistic gems as *a parliament of owls* and *a murder of crows.* In *A Gaggle of Geeks* [*Math Horizons,* September, 1995] the authors asked *Math Horizons'* readers to join us in playing the mathematical version of this game. We were delighted with the response and present here *a constellation of highlights.*

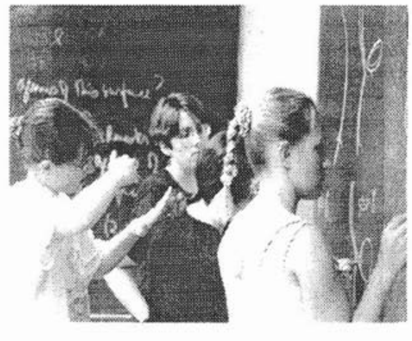
Many people poetically particularized mathematical specialities: *a residue of complex analysts* [Karl David], *a clique of graph theorists* [Roger Kirchner], *a bundle of algebraic topologists* [Jon Barwise], *an uncertainty of physicists* [Zine Smith]. Andy Tubesing proposed *a polygon of geometers* for an indeterminate number of them and suggests the appropriate specific polygon be used when we know how many there are, i.e., five geometers are *a pentagon of geometers.* Mark Krusemeyer took Andy's idea to its natural limit with his *a circle of geometers.*

An alluring allotment of alliterative amalgams also alighted in our mailbox: *a tangle of topologists* [Anonymous], *a problem set of pocket-protected people* [Alisa Walz-Flannigan], and *a survey of statisticians* [Shasta Willson]. And, of course, piles of people proposed passels of puns: *a figment of Newton* and *a bit on the Heaviside* [Sandy Keith], *a vice-presidency of algorithms* [Kathryn Jones], and *a following of corollaries,* our favorite entry, which earns Keith Durham a year's subscription to *Math Horizons.*

Readers suggested the following for a group of mathematicians: *a pattern of mathematicians* [Kathryn Jones], *a tautology of mathematicians* [Kathy Treash], *a mess of mathematicians* [Jon Barwise], and *a pit of adders* [Sandy Keith]. Matthew Shaffer, a junior math major at Cal State Hayward, must have been having a tough semester: he proposed *an obscurity of mathematicians* and *an absurdity of mathematicians.* Finally, we were charmed by the elegant pithiness of David Jones's *sum nerds.*

We would like to thank all those who joined us in this game. All the entries are accessible on the World-Wide Web [http://www.mathcs.carleton.edu/faculty/skennedy/venery]. Please continue to send us your mathematical terms of venery; we will post them on the Web. In particular, our search for the perfect term of collection for a set of mathematicians continues.

[1]The authors are indebted to Zalman Usiskin whose reaction to our original article provided us with a title for this follow-up.



DAVID L. WHEELER

The Statistics of Shape

A mathematician uses morphometrics to analyze the brains of schizophrenics

In the scientific world, mathematics often goes unnoticed until it is applied to a famous problem. So it was with Fred L. Bookstein's brand of math, known as morphometrics.

Using a "statistics of shape," he has studied bee wings, mole skulls, and the effects of jaw surgery on the human face. But his method, a mathematically rigorous analysis of biological forms, did not find a broad audience until he applied it to the human brain.

Mr. Bookstein, a mathematician at the University of Michigan, began his brain research after a psychiatrist suggested that he use morphometrics to find out what was different about schizophrenic brains. Beginning in 1994, Mr. Bookstein examined the cross-sectional images of schizophrenic brains produced by brain scans in a search for clues to causes of the disease. A morphometric analysis, aided by powerful computer graphics, revealed that the region of schizophrenics' brains that connects the two hemispheres is narrower than in "normal" brains.

Other researchers are checking the finding and are trying to understand its meaning. But Mr. Bookstein believes the discovery is a milestone for morphometrics.

"The schizophrenia example is the single most compelling argument I can imagine for the relevance of this kind of methodology," he says. "If we are right, we have made a discovery that could not have been made without having this method."

DAVID WHEELER is an assistant editor for the Chronicle of Higher Education.

Photography by Paul Thacker, Biomedical Communications, University of Michigan

Fred Bookstein

His brain research, done in collaboration with John DeQuardo, an assistant professor of psychiatry, and William D. K. Green, a mathematician, both also at Michigan, has lured other neuroscientists to his method.

Brain research, says Michael F. Huerta, associate director of the division of neuroscience and behavioral science at the National Institute of Mental Health, needs "sophisticated ways of approaching data, because we are drowning in it."

Variation in Size and Shape

Mr. Huerta says scientists are trying to find out how much of the enormous variation in the sizes and shapes of human brains and their components is normal and how much is related to disease. "The tools Fred is developing will let us answer that kind of question."

Other neuroscientists are beginning to use morphometrics to analyze the brains of those who are paralyzed, who have Alzheimer's disease, or who were born with fetal-alcohol syndrome. Scientists in other fields, looking for mathematical help in analyzing biological shapes, may join the neuroscientists.

"All my previous work has culminated in data sets that, while charming and scientifically interesting, were of absolutely no political importance," says Mr. Bookstein, who is quick with words as well as with computer keyboards.

He is proud of a book that he and several colleagues wrote about morphometrics and the evolution of fish that was published in 1985.

"It was brilliant," says Mr. Bookstein. "But it was about fish, and not from the fisherman's point of view, either."

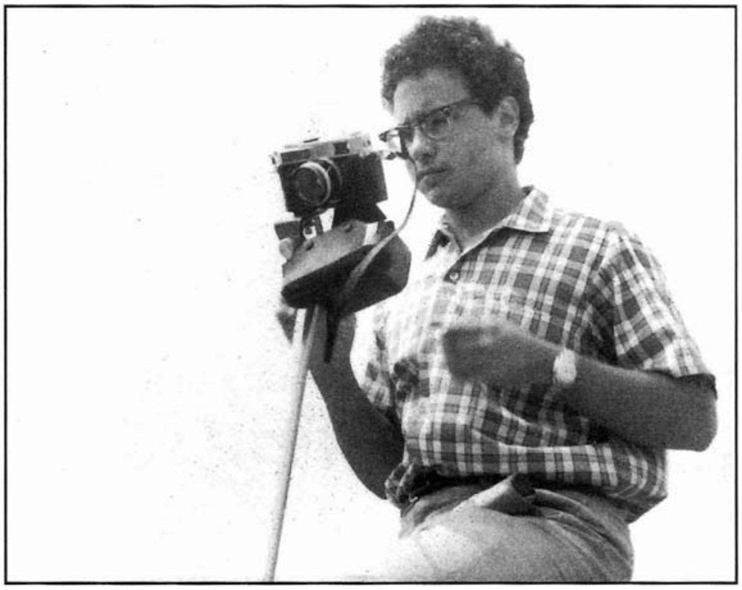
Now he has come upon schizophrenic brains and the Human Brain Project, the federally coordinated effort to develop tools for understanding the human brain. The project supports his research and has helped put morphometrics software into the hands of neuroscientists.

In previous work examining schizophrenics' brains, psychiatrists and neurologists have studied the volume of various portions and found them different from those of control brains. For instance, the fluid–filled ventricles in the brain are often larger in schizophrenics, and their brains are often smaller. But the differences, says Mr. Bookstein, were too small to lead to theories of causation and not useful enough to classify psychiatric patients.

The difference that he and his colleagues found, however, was three times more statistically significant than any found before. "If it's true, it's changed the discussion by an order of magnitude," he says.

Mr. Bookstein came to morphometrics after some sharp twists and turns in his early career. He grew up in Detroit and was a child prodigy in mathematics, winning a state high-school competition at the age of 14. He came to the University of Michigan as a freshman when he was 15. Although he graduated in three years, with a major in mathematical physics, the work became more and more difficult for him as time went on. "As is often the case with child prodigies," he says, "I lost it."

He started graduate work in mathematics at Harvard University at the age of 19 but dropped out after four weeks. "If the problem didn't have an answer in the back of the book," he says, "I couldn't solve it."



The protomorphometrician, aged 19, earnestly studying the statistics of natural form. The object of concern is most likely a rock or a waterfall.

Failure to Progress in Sociology

He transferred into graduate work in sociology, became a tutor in political philosophy, and taught a course on the concept of freedom. He earned a master's degree in sociology but was eventually discharged from the department for "failure to progress."

"Whatever I was doing, I wasn't turning into a sociologist," he says.

Embarrassed by his status as an outcast from sociology, Mr. Bookstein took a job as a computer programmer at the Harvard School of Public Health. In 1972, someone asked his department to describe the effects of orthodontic treatments, such as braces, on the shapes of children's skulls.

Inspired by the problem, Mr. Bookstein began his quest to devise a statistics of shape. His mathematical powers seemed to return—but this time they were in geometry instead of algebra or calculus. He went back to Michigan as a graduate student, feeling that now he had a chance to create a new kind of statistics, one that would make it possible to compare biological forms scientifically.

He took his first stab at creating morphometrics and received a Ph.D. in statistics and zoology. He had never taken a course in either field and remains self-taught in the mathematics he uses. "For a person pursuing an elementary problem that has been overlooked," he says, "it turned out to be exactly the right background."

It Began with Dürer

The story of morphometrics, Mr. Bookstein says, does not begin with him, or with the small cadre of scientists who developed it in the late 1970s and early '80s. Morphometrics, he says, began with the Renaissance painter and engraver Albrecht Dürer. In a book published in 1528, Dürer put grids on faces and then distorted the grids and the lines drawn within them. He used this method to explore what happened to faces as the proportions of various features changed—where an ear belongs on a long face, for instance.

The British scientist, Francis Galton, who invented weather maps and explored patterns in fingerprints in the late 19th century, apparently did not know about Durer's work, but continued to develop the foundation for morphometrics. Galton, who was part of the burgeoning eugenics movement, attempted to use a science of shape to describe the distinctive faces of criminals and to determine a "Jewish type." Galton, says Mr. Bookstein, was on the right track with his math, even if his attempts to apply it were wrong. Galton used key points, such as the tip of the chin or the nose, as landmarks for numerical coordinates to describe the face.

In the early part of this century, other scientists laid down the foundation of modern statistical methods. They learned how to relate many measurements of a population, for example, and how to examine differences among groups. "By 1950, everything the modern applied statistician uses for the core of inferential statistics was in place," says Mr. Bookstein.

But it wasn't until about 10 years ago that statistics was successfully fused with geometry into morphometrics. Mr. Bookstein says that he and other mathematically oriented scientists created the synthesis that made a powerful numerical analysis of shape possible.

Filled with False Starts

The history of morphometrics, he says, is filled with false starts. He confesses to having taken a wrong turn himself in his doctoral dissertation, which he later retracted. "I understood the problem," he says, "but I had the wrong solution entirely."

Modern morphometrics, he says, has the qualities that statisticians like. When used correctly, it takes advantage of the information available in a shape and gives meaningful averages of groups of shapes.

The use of morphometrics has made it possible, for example, to come up with a picture of an average brain, which is helpful to neuroscientists. They previously had difficulty summing up the wide variation in brain shapes and sizes.

The software for morphometrics that Mr. Bookstein and his collaborators have developed, called "Edgewarp," generates a grid on the figure being analyzed. Distortions in the grid lines, which look like a spider's web being pulled out of shape, point the way to variations from the norm. The software is available free on the Internet. (It is available by anonymous FTP from brainmap.med.umich.edu in the directory pub/edgewarp.)

Mr. Bookstein and his collaborators are extending the program so that it can analyze three-dimensional data, something that will be necessary for a more powerful analysis of brains.

Both Mr. Bookstein and Dr. DeQuardo, the psychiatrist, would like to apply morphometrics to the brain scans of those who are at risk of schizophrenia but do not yet have the disease, which frequently strikes in late adolescence. Families with a history of schizophrenia often ask if physicians can check the brains of their children to see if they will get it, too.

Some evidence indicates that early treatment greatly alleviates the symptoms of the disease, and Dr. DeQuardo wonders if antipsychotic drugs administered before its symptoms even appear might head it off altogether.

Right now, says Dr. DeQuardo, he has nothing to offer those who don't yet have schizophrenia but fear getting it. Someday, he hopes, he will have something that can help. ■

Morphometrics

FRED BOOKSTEIN

Morphometrics combines geometry and statistics in tools for the scientific description of biological or medical shape variation and shape change. Its tactics differ over the different kinds of information from pictures of organisms or organs locations of named points, locations of boundaries (such as curving outlines), or quantitative pictorial content, such as darkness or color, over much of the whole picture. (If the "picture" is a CAT scan or similar three-dimensional medical image, the "boundaries'" might be whole two-dimensional surfaces instead of curves.) Our techniques are best developed for the first type of data, locations of named points, and that will be my main topic here.

Where do shape data come from?

Figure 1 is typical of the way shape data arise from pictures. This particular image is a "slice through the head"—a magnetic resonance image just off the midline of a normal brain. The ten labelled *landmark points* represent some reliable anatomical structures that intersect this plane. My exemplary data set includes this image and 27 others like it.

In ordinary language, the *shape* of an object is described by measurements that do not vary when the object is moved, rotated, enlarged, or reduced. The translations, rotations, and changes of scale we are ignoring constitute the *similarity group* of transformations of the plane. When the "objects" are point sets like that in Figure 1, it turns out to be useful to say that their shape simply is the set of all point sets that "have the same shape." This kind of definition may already be familiar to you–we just defined the shape of a set of points as the equivalence class of that point set, within the collection of all point sets of the same cardinality, under the operation of the similarity group.

We need a distance measure for shapes defined this way. If we were talking just about sets of labelled points, a reasonable formula for squared distance would be the usual Pythagorean sum of (squared) distances between corresponding points over the list. Since shapes are equivalence classes of these point sets, it is reasonable to define shape distance as the minimum of these sums of squares over the equivalence classes—over the operation of the group of similarities that shape is supposed to ignore. The squared shape distance between one point set A and another point set B might then be taken as the minimum summed squared Euclidean distances between the points of A and the corresponding points in point sets C as C ranges over the whole set of shapes equivalent to B. For this definition to make sense, we have to fix the scale of A. The mathematics of all this is most elegant if the sum of squares of the points of A around their center-of-gravity is constrained to be exactly 1. (A small adjustment of the definition is

Figure 1. Landmark data ordinarily arise from the identification of particular named points in routine biomedical images. Here we identify ten landmarks in a parasagittal magnetic resonance image of a normal brain. Bottom to top: bottom of cerebellum, bottom of pons at medulla, tentorium at dura, obex of fourth ventricle, top of pons, optic chiasm, top of cerebellum, superior colliculus, splenium of corpus callosum, genu of corpus callosum. The landmarks in this data set were located by Dr. John DeQuardo of the University of Michigan Department of Psychiatry.

required to make it symmetric in A and B.)

The series of steps that is involved in this computation can be followed down the rows of Figure 2. The top row shows two quadrilaterals of four landmarks (the dots and the X's) that might have come from a data source like Figure 1. We connect each landmark to the centroid of its own form and rescale so that the sum of squares of the distances shown are exactly 1 in each form (second row). We then simply pick up one of the forms (here, the one to the left, with the X's) and put it down with its centroid directly over the centroid of the other form, the one made out of dots. This gives us the distances between corresponding points shown at the left in the third row. The final step in computing Procrustes distance consists in computing the rotation that minimizes the sum of squares of those residual distances. This can be computed analytically using complex algebra, or one can simply experiment with a few tentative reorientations, like the one third row right, and minimize the sum of squares numerically. Eventually one arrives at the rotation that minimizes this residual sum-of-squares, the one shown at the left in the bottom row. The Procrustes distance between the forms, which is the sum of squares of those residuals at its minimum, can be seen graphically as the total area of the circles shown at the lower right, divided by pi.

How does one average a set of shapes?

To average ordinary numbers, you add them up, then divide by their count. Because we can't add up shapes or divide, instead we use a different characterization of the ordinary average of a list of numbers: it is also the "least-squares fit" to those numbers, the quantity about which they have the least sum of squared distances. Since we already have a distance between shapes, we inherit a notion of average in this way as soon as we have an algorithm for minimizing that sum of squares. That turns out not to be too difficult. The average of the 28 shapes like that of the red dots in Figure 1 is shown at the left in Figure 3.

After we've computed the average, we can put each individual shape down over the average using the similarity transformation that made the sum-of-squares from the average a minimum for that particular case. There results the picture in the center of Figure 3. These points, the *Procrustes shape coordinates* of our sample, describe the variation of the whole set of shapes around the average in terms of variations "at" the component points separately. (The description is not complete until we have linked up the changes across the different points of the same case, using techniques I will not review here.) The shape coordinates help us to carry out many familiar operations of ordinary scientific statistics. At the right in Figure 3, for example, is the computation of two averages for subsets of this one data set. Now it is fair to tell you that the 28 cases here consisted of points from brains of 14 normal people and 14 patients with schizophrenia. The averages in Figure 3c are for these two subgroups.

How can we read the ways in which two shapes differ?

To render shape differences legible, we turn to an idea as old as the invention of artistic perspective in the Renaissance. We can show the displacements of points in Figure 3c as one coherent graphical display by imagining one of the averages, say, the normals (the dots), to have been put down on ordinary square graph paper. Call it the *starting shape.* We deform the paper so that the dots now fall directly over the *other* set of points, the triangles constituting the *target shape.* Figure 4 shows what happens to the grid.

Naturally it matters what deformation one uses. Morphometricians prefer one particular choice, the *thin-plate spline,* that minimizes yet another sum-of-squares. In this context, we are minimizing the summed squared second derivatives (integrated over the whole plane) of the map in the figure—something like the summed squared deviations of the shapes of the little squares from the shapes of their neighbors, and thus a measure of local information in the mapping. That minimum actually turns out to be a quadratic form in the coordinates of the points of the target shape, with coefficients that depend on the starting shape. This quadratic form, the *bending energy*, joins the one we've already exploited (Procrustes distance) and the one that the statistician will automatically contribute (the covariance matrix). Morphometrics seems unusual among branches of modern applied statistics in the centrality of this set of *three* quadratic forms rather than the usual two.

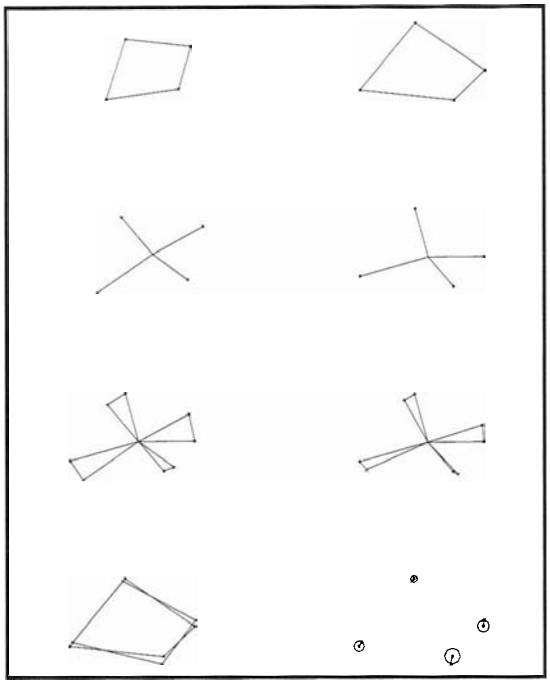


Figure 2. Steps in computing the Procrustes shape distance between a pair of forms. (top row) The data: two quadrilaterals of landmarks. (second row) Each form is scaled separately to unit sum of squares around its own centroid. (third row) The centroids are superposed and one form rotated with respect to the other. (bottom row) Procrustes distance is the minimum sum of squared distances between corresponding points over these rotations; it is proportional to the sum of the areas of the circles drawn here

Any scientist worth her salary would check the "finding" in Figure 4 to see if it is "statistically significant"—if it is larger than would plausibly arise by chance from groups that vary within themselves as in Figure 3b. The grid in the figure is, indeed, comfortably statistically significant. We are thereby encouraged to argue that those ten-landmark sets do systematically differ in shape between the schizophrenic and normal groups. As the figure indicates, the difference is highly localized to the relation between the landmarks at upper center in Figure 1, the segment from colliculus to splenium.

What about the rest of the picture?

Here's another old idea (Francis Galton, the inventor of weather maps and the regression coefficient, was already doing something like this in the 1880's). We can "unwarp" each case of each group to its own group average image by running the thin-plate spline "in reverse"—warping the average onto the landmarks of each case and then copying back the picture contents from the case onto a fixed grid of little pixel cells around the average landmarks. Once unwarped, images can be averaged the old-fashioned way--add them up, pixel by pixel and group by group, then divide. We arrive at the pair of averaged images in Figure 5 (where the normals are on the

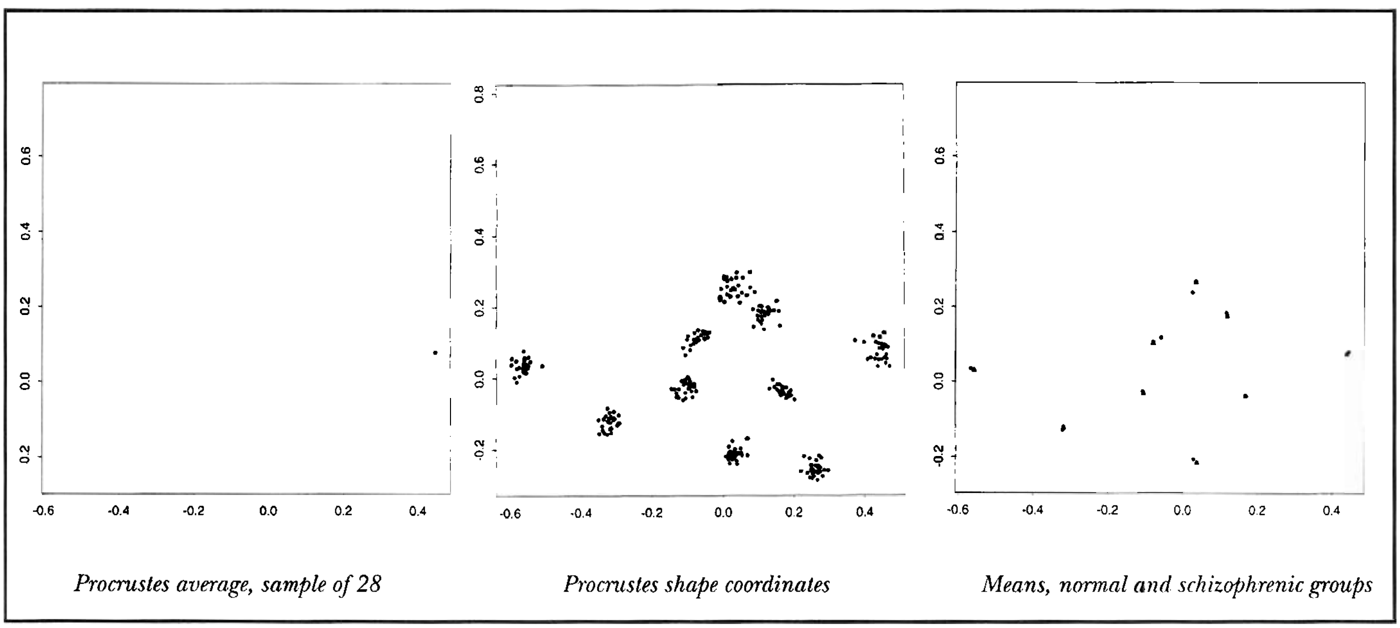


Figure 3. What Procrustes shape coordinates are for. (left) The Procrustes average of 28 shapes like that of the set of ten landmark points in Figure 1. (center) Procrustes shape coordinates come from fitting every individual case back over the sample average (left) by the procedure of Figure 2. (right) One can carry out ordinary statistical manipulations of these coordinates. Here we show the average shapes of the two subgroups making up our original sample. Triangles, schizophrenics; dots, normals.

left, the schizophrenics on the right). Near the landmarks, these images are gratifyingly clear, so the registration is working well for lining up information about curves, not just about the landmarks we used. At the same time, the cortex and the facial bones are badly blurred, mainly because they are well outside the hull of this landmark set. (Locating cortical points usually requires more data than is contained in any single two-dimensional picture.)

Now an additional group difference has become clear: the shape of the *splenium*, which is the blobby structure at the right end of the thick white arch at center. The scientist (in this case, a psychiatrist) can go back to the original images and measure more carefully in this region, and thus even more sharply localize the characterization of this particular image plane for this particular sample of schizophrenics and normals.

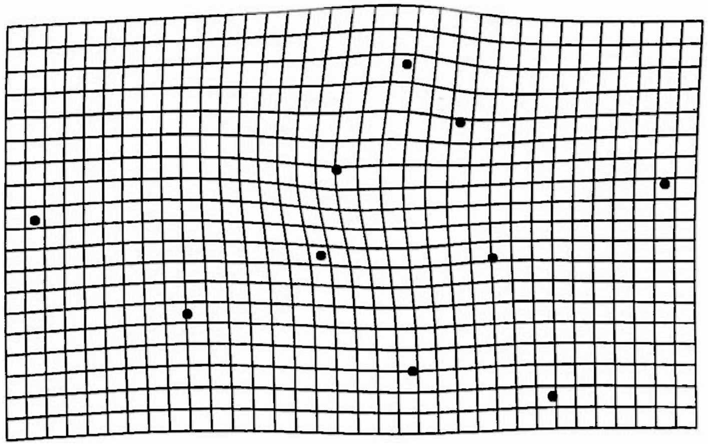


Figure 4. The thin-plate spline is an interpolation function between pairs of points sets that minimizes the variation of its affine derivative. The grid here, the spline from the dots to the triangles in Figure 3 (right), leads the eye immediately to the strongly local nature of the shape difference shown there.

All the techniques I have just reviewed for you are elementary, and yet all are remarkably new by the standards of our profession. The logical flow through which I just led you was not familiar even to specialists until 1990 or so. As an efflorescence of new tools, morphometrics is thus of remarkably sudden onset. It remains a new discipline, then, so that not only the canonical set of maneuvers I have just shown you but also many of its open questions are still quite elementary. (For instance: can an algorithm be trained to find the difference in splenium shape I pointed out across the panels of Figure 5?) To learn more, you might enjoy browsing my monograph *Morphometric Tools for landmark Data* (Cambridge University Press, 1991). If you have access to a Unix workstation, my colleague William D. K. Green distributes free software for experimenting with the thin-plate spline and producing average shapes and averaged unwarped pictures like these. Point your Internet browser to the file README.EDGEWARP at ftp://brainmap.med.umich.edu/pub/edgewarp and follow the FTP instructions there. ■

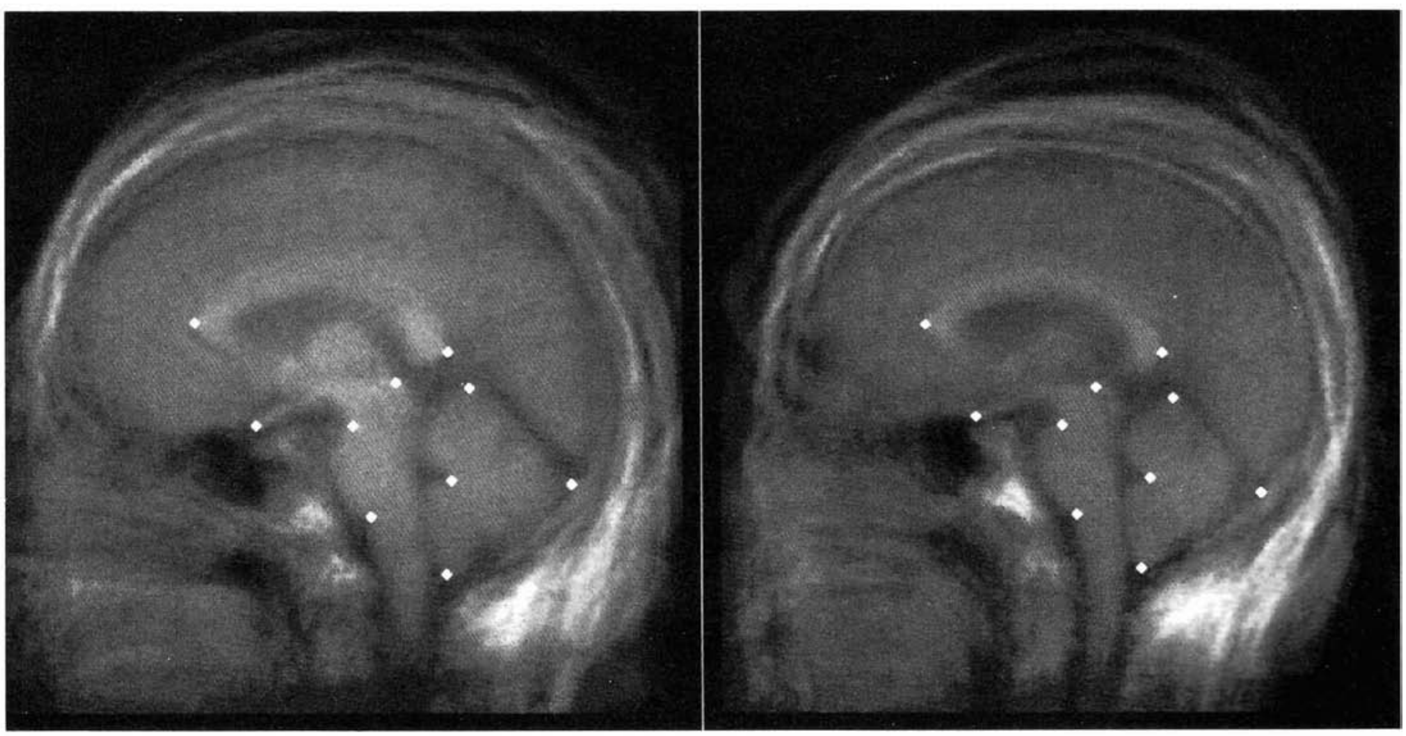


Figure 5. When the original images are averaged after being unwarped back to their own group mean landmark shapes, by a version of the same thin-plate spline, there emerge further systematic group differences in neuroanatomy. Left, average of 14 normals; right, of 14 schizophrenics. The shapes of the dots are the same as those in Figure 3 (right).

Problem Section

Editor

Murray Klamkin
University of Alberta

This section features problems for students at the undergraduate and (challenging) high school levels. As new editor, the problems will of course reflect my tastes as well as the submissions of its readers. My preferences are for problems that are not highly technical so they can be easily understood by the general reader. There should be a certain elegance about the problems; the best problems are elegant in statement ("short and sweet"), elegant in result, and elegant in solution. Such problems are not easy to come by. Nevertheless, any problem submitted should include a solution and be elegant in at least one of the three categories. Original problems are preferred but this does not rule out elegant problems which are not well known (these will be indicated by a dagger (†)). For the latter, any known information about them should be included. Also to be included are "Mathematical Quickies." These are problems which can be solved laboriously but with proper insight and knowledge can be solved easily. These problems will not be identified as such except for their solutions appearing at the end of the section (so no solutions should be submitted for these problems).

All problems and/or solutions should be submitted in duplicate in easily legible form (preferably printed) on separate sheets containing the contributor's name, mailing address, school affiliation, and academic status (i.e., high school student, undergraduate, teacher, etc.) and sent to the editor, Math. Dept., University of Alberta, Edmonton, Alberta T6G 2G1, Canada. If an acknowledgement is desired an e-mail address or a stamped self-addressed postcard should be included (no stamp necessary for outside Canada and the US).

Proposals

To be considered for publication, solutions to the following problems should be received by June 15, 1996.

Problem 41. Proposed by J. O. Chilaka, Long Island University.

Consider the locus of a point P on a segment of length ℓ whose endpoints lie on the nonnegative parts of the x and y axes of a rectangular coordinate system as the angle the segment makes with the x-axis varies from 0 to $\pi/2$. Determine (i) the maximum area bounded by the locus and the x, y axes; (ii) the minimum length of the locus.

Problem 42. Proposed by the Problem editor.

Given any continuous curve joining the points $P_1(0,0)$ and $P_2(h,k)$ in a rectangular coordinate system where h and k are integers which are not relatively prime. Prove that there exist two distinct points $P(x_1, y_1)$, $Q(x_2, y_2)$ on the curve (other than the endpoints) such that both $x_2 - x_1$ and $y_2 - y_1$ are integers.

Problem 43. Proposed by E. M. Kaye, Vancouver, B.C.

Solve the differential equation $[xD^3 + 9D^2 + 9]y = 0$ if it is given that $F(x)$, $G(x)$, and $H(x)$ are three linearly independent solutions of $[xD^3 + D^2 + 1]y = 0$ ($D \equiv d/dx$).

Problem 44. Proposed by Paul Wagner, Chicago, Illinois.

Determine $P(n+1)$ if $P(x)$ is a polynomial of degree n such that $P(k) = 3^k$ for $k = 0, 1, \ldots, n-1$ and $P(n) = 1$.

Problem 45. Proposed by K. M. Seymour, Toronto, Ontario.

Determine the extreme values of

$$\sum \frac{1}{1 + x_1 + x_1x_2 + \cdots + x_1x_2\cdots x_{n-1}}$$

where $x_1x_2\cdots x_n = 1$, $x_i > 0$, and the sum is cyclic over the indices $1, 2, \ldots, n$.

Solutions

Problem 31: An Inequality

Prove that

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \geq \sqrt{a_1 + 3a_2 + \cdots + (2n-1)a_n}$$

where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$.

Equivalent solutions by Christopher Ackler (undergraduate) Fitchburg State College, R. J. Covill, Atkinson, NH, and R. T. Sharp, McGill University.

$$\left\{\sum \sqrt{a_i}\right\}^2 = \sum a_j + 2\sum \sqrt{a_i a_j} \geq \sum a_i + 2\sum a_j$$
$$= \sum\{(2j-1)a_j\}.$$

Also solved (mostly) inductively by James Camacho, Jr., John Christopher, J. M. Coker, Q. H. Darwish, David Doster, Fresno Problem Solving Group, Geoff Goodson, L. C. Helenius, D. K. Johnson, Andrew Miller, C. A. Minh (undergraduate), J. S. Rombough (undergraduate), and E. T. H. Wang.

Editorial note. By letting $a_{n-k} = x_n + x_{n-1} + \cdots + x_{n-k}$, $k = 0, 1, \ldots, n-1$, the given inequality becomes

$$\sum_{i=1}^{n} \sqrt{x_i + x_{i+1} + \cdots + x_n}$$
$$\geq \sqrt{x_1 + 4x_2 + 9x_3 + \cdots + n^2 x_n}$$

which is an inequality due to D. J. Newman which he establishes by an immediate application of Minkowski's inequality. More generally, one can show in a similar fashion that

$$\sqrt[r]{a_1} + \sqrt[r]{a_2} + \cdots + \sqrt[r]{a_n} \geq \sqrt[r]{b_1 a_1 + b_2 a_2 + \cdots + b_n a_n}$$

where $r > 1$ and $b_k = k^r - (k-1)^r$.

Problem 32†. Derivative Evaluation

Evaluate $\{a_n[D+n/x]^m - a^m[D+m/x]^n\}e^{ax}$ where $D = d/dx$.

Solution by Fresno Problem Solving Group, California State University. It follows that

$$[D+k/x][D+r/x] = [D+(r+1)/x][D+(k-1)/x].$$

Using this identity successively, we find that $[D+n/x]^m D^n = [D+m/x]^n D^m$. Then

$$\begin{aligned} a^n[D+n/x]^m e^{ax} &= [D+n/x]^m D^n e^{ax} \\ &= [D+m/x]^n D^m e^{ax} \\ &= a^m[D+m/x]^n e^{ax}. \end{aligned}$$

Hence the given expression is zero. Also the Group submitted a second long expansion solution.

Editorial comment. For proving operator identities, the exponential shift theorem is often very useful, i.e., $e^{\int p\,dx}F(D) = F(D-p)e^{\int p\,dx}$. So that

$$\begin{aligned} x^{m+n}[D+n/x]^m D^n &= x^m D^m x^n D^n \\ &= x^n D^n x^m D^m \\ &= x^{m-n}[D+m/x]^n D^m \end{aligned}$$

since $x^m D^m$ and $x^n D^n$ commute. The latter follows since $x^r D^r$ expands into a sum of powers of xD with constant coefficients.

Problem 33. Extremal Cones

A cone with a plane base is inscribed in a sphere of radius R. Determine (i) the maximum volume of the cone (no calculus please), (ii) the maximum lateral area of the cone.

Solution of (i) by D. K. Johnson, Valley Catholic High School. It is clear that for a maximum volume, the cone will have to be a regular one. Letting its height be $R+x$, its base radius squared $r^2 = R^2 - x^2$. Hence $6V/\pi = (2R-2x)(R+x)(R+x)$. Since the sum of the latter three factors is $4R$, it follows by the A.M.-G.M. inequality that the maximum of the latter product equals $(4R/3)^3$ and is taken on when $2R - 2x = R + x$ or $x = R/3$ and then $\max V = \pi(4R/3)^3/6$.

Solution of (ii) by the proposer. It is intuitive that the cone must be a right circular one. To prove this, let the equation of the sphere be $x^2 + y^2 + z^2 = R^2$ and let the base be parallel to the xy plane with its center at $(0, 0, -p)$. If the vertex be at $(q, 0, h)$ with $q^2 + h^2 = R^2$, then the slant height ℓ from the vertex to the point $(a\cos\theta, a\sin\theta, -p)$ on the base (here $a^2 = R^2 - p^2$) is given by $\ell^2 = (q - a\cos\theta)^2 + a^2\sin^2\theta + (h+p)^2 = 2R^2 + 2hp - 2qa\cos\theta$. The lateral area is now given by

$$A = \frac{1}{2}\int_0^{2\pi} a\ell\, d\theta$$

or

$$\begin{aligned} 2A = &\int_0^{\pi} \left\{(R^2-p^2)(2R^2+2hp-2qa\cos\theta)\right\}^{1/2} d\theta \\ &+ \int_0^{\pi} \left\{(R^2-p^2)(2R^2+2hp+2qa\cos\theta)\right\}^{1/2} d\theta. \end{aligned}$$

Then by the power mean inequality,

$$A \leq \int_0^{\pi} \left\{(R^2-p^2)(2R^2+2hp)\right\}^{1/2} d\theta$$

with equality if $q = 0$ or $h = R$. It now remains to maximize $(R^2 - p^2)(2R^2 + 2Rp)$ and as in (i), this occurs when $p = R/3$. Finally $\max A = 8\pi R^2\sqrt{3}/9$.

Problem 34. Prime Congruence

Prove that for any prime $p > 5$,

$$(p^2 - 1)/24 \equiv 0 \quad \text{or} \quad 2 \bmod 5.$$

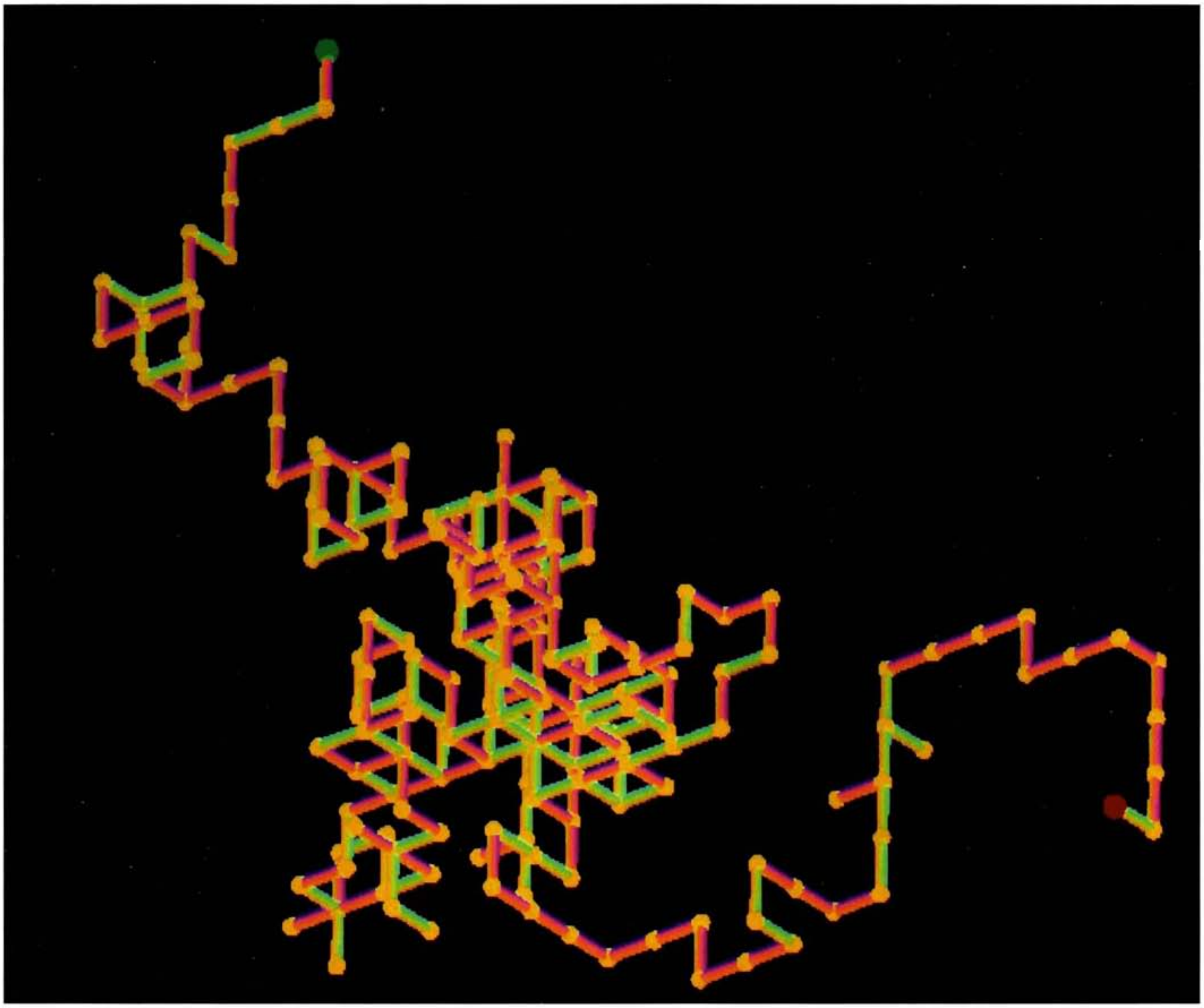
Solution by John Christopher, California State University. It follows easily that $p^2 - 1 \equiv 0$ mod 3 and mod 8. Hence, $(p^2 - 1) = 24q$ where q is an integer. The only possibilities for p are $p = \pm 1$ or ± 2 mod 5. If $p = \pm 1$ mod 5, then $p^2 - 1 = 24q = 0$ mod 5, hence $q = 0$ mod 5. If $p = \pm 2$, then $p^2 - 1 = 24q = 3$ mod 5 or $24(q - 2) \equiv 0$ mod 5, hence $q = 2$ mod 5.

Editorial comment. More generally it can be shown that if n is not a multiple of 5 and $(n^2 - k^2)/(25 - k^2)$ is an integer m for $k = 1, 2, 3$, or 4, then $m = 0$ or 2 mod 5.

Also solved by Christopher Ackler, Ryan Buschert (undergraduate), J. M. Coker, R. J. Covill, Fresno Problem Solving Group, Don Hancock, D. K. Johnson, Jahangeer Kholdi, Andrew Miller, C. A. Minh, R. T. Sharp, and the proposer.

Problem 45. (Quickie) Constant Sum

In the sum multiply the numerator and denominator of the 2nd term by x_1, the 3rd term by $x_1x_2, \ldots$, the nth term by $x_1x_2\cdots x_{n-1}$. After using $x_1x_2\cdots x_n = 1$, all the terms now have the same denominator and thus the given sum $= 1$.



Take a random hike...

This computer graphic shows a random walk of 289 steps in 3-dimensional space, starting from the green dot and ending up at the red dot. Each step was equally likely to go forward, backward, right, left, up or down. Extended indefinitely, the walk has only about a 34% chance of ever returning to its starting point. This is in sharp contrast to the 1- and 2-dimensional versions of such a random walk, in which returning to the starting point is indefinitely likely.

The artwork is courtesy of Daniel Asimov. His article appears on page 10 of this issue.