To Be (a Circle) or Not to Be?

Hassan Boualem and Robert Brouzet

Hassan Boualem (hassan.boualem@univ-montp2.fr) received his Ph.D. in mathematics from the University of Montpellier in 1992, where he studied group actions and foliations theory and continues to teach. He is also interested in symplectic geometry and integrable systems. His main interests outside mathematics are cooking, drawing, and botany.

Robert Brouzet (robert.brouzet@univ-perp.fr) received his Ph.D. in mathematics from the University of Montpellier in 1991, where he studied the geometry of bi-Hamiltonian systems. He now teaches at the University of Perpignan, and his main interests are differential geometry and the history of mathematics. Besides mathematics, he is also interested in philosophy, music, and mountain climbing.

Is the curve Γ given by $\sqrt{x} + \sqrt{y} = 1$ for $(x, y) \in [0, 1]^2$ the arc of a circle? Plotting the curve, Figure 1, shows why this is a natural question. This exercise was given as an open question in a French high school. Despite the visual impression, this curve is not the arc of a circle. In a tour of properties and characterizations of the circle, we detail or outline 14 proofs!

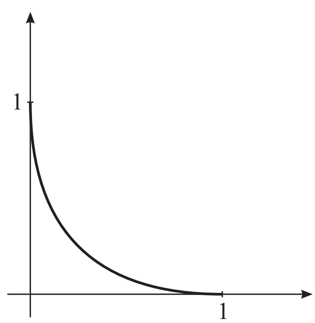

Figure 1. The curve Γ

After such a definitive resolution to the original question, we establish that the curve is indeed the arc of another conic section. Generalizing, our curve is part of a family Γ_r of curves whose Cartesian equations have the form $x^r + y^r = 1$. For which r is Γ_r an arc of a conic? Although general conic sections have fewer nice properties than circles, we answer the problem in two ways, using curvature and algebra.

http://dx.doi.org/10.4169/college.math.j.46.3.197
MSC: 53A04

The Γ_r are part of a family of curves studied by Lamé known as superellipses. We conclude by reviewing some of his results on these curves and looking at our questions in that context.

Many proofs for "not to be"

We begin with an abundance of proofs for the following claim. The 14 arguments, some of which are given as exercises, are divided among four methods: elementary geometry, analytic geometry, parametric curves, and calculus.

Proposition 1. *The curve Γ given by $\sqrt{x} + \sqrt{y} = 1$ for $(x, y) \in [0, 1]^2$ is not an arc of a circle.*

As an initial step, write $\sqrt{y} = 1 - \sqrt{x}$ and square to see that Γ is the graph of the function $f(x) = x - 2\sqrt{x} + 1$ for $x \in [0, 1]$.

Elementary geometry. Our first proof is based on the fact that three noncollinear points determine a circle. The subsequent exercise leads to a proof using the result that all perpendicular bisectors of chords on a circle must pass through the center.

Proof 1. It is easy to identify three noncollinear points of Γ, e.g., $(0, 1)$, $(1, 0)$, $(1/4, 1/4)$. The general equation of a circle is $x^2 + y^2 + \alpha x + \beta y + \gamma = 0$. Substituting the coordinates of our three points shows that α, β, γ satisfy the system

$$\beta + \gamma = -1,\ \alpha + \gamma = -1,\ \alpha + \beta + 4\gamma = -\tfrac{1}{2}.$$

An easy calculation gives $\alpha = \beta = -7/4$ and $\gamma = 3/4$. Thus, the only circle that can contain the curve Γ has equation

$$x^2 + y^2 - \tfrac{7}{4}x - \tfrac{7}{4}y + \tfrac{3}{4} = 0.$$

However, the point $(1/9, 4/9)$ on Γ does not satisfy this equation: Verify that its coordinates give $-1/81$ rather than 0. This gives a contradiction. ■

Exercise 2. Show that segments with endpoints in Γ do not have perpendicular bisectors that pass through the center of the circle.

Hint: Use the points in Proof 1.

Analytic geometry. Here, we give a very natural and elementary proof using the definition of a circle as the set of points in the plane whose distance to a fixed point (the center) is constant. Then we outline four related proofs whose details are left as exercises.

Proof 3. We proceed by contradiction. Assume that Γ is an arc of a circle with center (a, b). In this case, the function defined on $[0, 1]$ by

$$\varphi(x) = (x - a)^2 + (x - 2\sqrt{x} + 1 - b)^2$$

is constant and so has a vanishing derivative on $(0, 1]$. Therefore, for all $x \in (0, 1]$,

$$\varphi'(x) = 2(x - a) + 2(x - 2\sqrt{x} + 1 - b)\left(1 - \frac{1}{\sqrt{x}}\right) = 0$$

and so (with $X = \sqrt{x}$) the polynomial $2X^3 - 3X^2 + (3 - a - b)X + b - 1$ has an infinite set of roots, a contradiction. ■

One can determine the only possible values for the center coordinates a and b in several ways. Here is a method using the fact that the tangent to a circle at a given point is perpendicular to the radius at that point.

The function f has a derivative on $(0, 1]$ given by $f'(x) = 1 - 1/\sqrt{x}$. Because $f'(1) = 0$, the tangent to Γ is horizontal at the point $(1, 0)$. Moreover, the curve is symmetric relative to the line $y = x$, so the tangent at the point $(0, 1)$ is vertical. Because the tangent at a point of a circle is perpendicular to the corresponding radius, the circle $\mathcal{C}$ potentially containing Γ has center $(1, 1)$ and radius 1. Figure 2 shows Γ and the relevant quarter of $\mathcal{C}$.

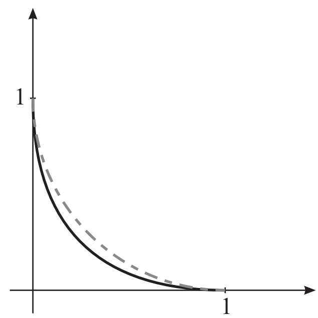

Figure 2. The curve Γ (full line) and the arc of circle $\mathcal{C}$ (dashed line)

The following exercises lead to other proofs of Claim 1 based on the necessary parameters of $\mathcal{C}$.

Exercise 4. Show that the only value of x satisfying $(x - 1)^2 + (f(x) - 1)^2 = 1$ is $x = 1$.

Exercise 5. Calculate the distance between the point $(1/4, 1/4)$ on Γ and $(1, 1)$.

Exercise 6. Show that each point of Γ must satisfy $x^2 + y^2 - 2xy - 2x - 2y + 1 = 0$. Compare this equation with that of the "necessary circle" $\mathcal{C}$.

Exercise 7. If Γ were an arc of a circle, then its tangent lines would be perpendicular to the radius at each point, and so its normal lines would pass through a fixed point (the center of the circle). Show that it is not the case.

Hint: Proceed by contradiction; give an equation of the normal line at $(a, f(a))$ and show that all normals must pass through $(1, 1)$. Use algebra to conclude that this will only be true for $a = 1$ and $a = 1/4$.

Parametric curves. Here are proofs based on two parametrizations of the circle.

Proof 8. We again assume that Γ is an arc of a circle with center (a, b) and radius R so that Γ can be parametrized as $x(\theta) = a + R\cos\theta,\ y(\theta) = b + R\sin\theta$ for θ in some interval $I \subset [0, 2\pi]$. We can specify I because we have already seen that, if Γ is an arc of a circle, then it is a quarter of the circle; thus, $I = [\pi, 3\pi/2]$. We also know from other proofs that $a = b = R = 1$. So, according to the expression of f, for all $\theta \in I$ we have $\sin\theta = \cos\theta - 2\sqrt{1 + \cos\theta} + 1$. Differentiating with respect to θ gives $\cos\theta = -\sin\theta + \sin\theta/\sqrt{1 + \cos\theta}$. Taking $\theta = 5\pi/4 \in I$, we obtain the equality

$$-\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} - \frac{\frac{\sqrt{2}}{2}}{\sqrt{1-\frac{\sqrt{2}}{2}}},$$

which simplifies to $2\sqrt{1-\sqrt{2}/2} = 1$. But this is impossible because, after squaring, we obtain $\sqrt{2} = 3/2$ (and, after squaring again, $8 = 9$). ■

Exercise 9. A circle is a unicursal curve, i.e., it admits a rational parametrization, namely the one obtained from the previous trigonometric parametrization by using the parameter $t = \tan(\theta/2)$. Show that if Γ was the quarter of a circle, then $1 - t = \sqrt{2(1+t^2)}$ for all $t \in (-\infty, -1]$ leading, for $t = -2$, to the impossibility $9 = 10$.

Calculus. We finish our proofs with ones using concepts from calculus: area, arc length, curvature, and smoothness.

Exercise 10. If Γ were the quarter of circle we have discussed, then the area delimited by it would be $\pi/4$. But computing the area between $y = 1$ and f leads to the impossibility $\pi/4 = 1 - \int_0^1 f(x)\,dx = 5/6$.

Proof 11. Instead of the using the area of the quarter of circle, we can consider the arc length, which would be $\pi/2$. Since Γ is concave up over the region, we know that its arc length is greater than the combined length of the line segments from $(0, 1)$ to $(1/4, 1/4)$ and from $(1/4, 1/4)$ to $(1, 0)$. But this combined length is $\sqrt{10}/2$, which is greater than $\pi/2$ since $10 > \pi^2$. ■

Although the lower bound in Proof 11 suffices, we can compute the exact arc length of Γ as outlined in the next exercise.

Exercise 12. The length of Γ is given by the formula [**2, 3**]

$$\ell(\Gamma) = \int_0^1 \sqrt{1 + f'(x)^2}\,dx.$$

1. Prove that $\ell(\Gamma) = \sqrt{2}\int_0^1 \sqrt{u^2+1}\,du$.
2. Use the hyperbolic functions to show $\ell(\Gamma) = 1 + \left(\sqrt{2}\ln\left(1+\sqrt{2}\right)\right)/2$.
3. Using just the rough inequalities $\pi < 3.2$ and $e < 2.72$, prove that

$$\tfrac{\pi}{2} < 1 + \tfrac{\sqrt{2}}{2}\ln\left(1+\sqrt{2}\right).$$

Recall that curvature measures the intensity of the force needed to keep the particle on the track of the curve and also measures how quickly the curve changes direction. More precisely, when the curve is parametrized with respect to arc length, independent of the parametrization, the curvature is defined in terms of the acceleration vector. For example, the curvature of the straight line is identically zero so that the circle of radius R has constant curvature $1/R$.

Exercise 13. The curvature $c(x)$ of an arc with Cartesian equation $y = f(x)$ at the point $(x, f(x))$ is given by the formula [**1, 2, 3**]

$$c^2(x) = \frac{(f''(x))^2}{(1+(f'(x))^2)^3}. \tag{1}$$

Calculate the curvature of Γ at multiple values, say $1/2$ and 1, to show that it is not constant, as required for an arc of a circle.

Finally, recall that a smooth (C^∞) real function of one real variable has, at all defined points, derivatives of all orders.

Proof 14. Let s denote reflection across the vertical line $x = 1$. If Γ were an arc of a circle, because it would have a horizontal tangent at $(1, 0)$ (as already noted in others proofs), then $\Gamma \cup s(\Gamma)$ would be an arc of this same circle (actually a half circle, although we will not need that specificity). Let g represent the symmetric arc; for all $x \in [1, 2]$ we must have $g(x) = f(2 - x)$. Thus, a Cartesian equation of the set $\Gamma \cup s(\Gamma)$ is

$$y = \begin{cases} f(x) = x - 2\sqrt{x} + 1 & \text{if } x \in [0, 1], \\ g(x) = -x - 2\sqrt{2 - x} + 3 & \text{if } x \in [1, 2]. \end{cases}$$

The junction at $x = 1$ would be smooth if $\Gamma \cup s(\Gamma)$ were an arc of a circle. However,

$$f'(x) = 1 - \frac{1}{\sqrt{x}}, \qquad f''(x) = \frac{1}{2x^{3/2}}, \qquad f^{(3)}(x) = \frac{-3}{4x^{5/2}},$$
$$g'(x) = -1 + \frac{1}{\sqrt{2 - x}}, \qquad g''(x) = \frac{1}{2(2 - x)^{3/2}}, \qquad g^{(3)}(x) = \frac{3}{4(2 - x)^{5/2}},$$

so, at $x = 1$, we have $f(1) = g(1) = 0$, $f'(1) = g'(1) = 0$, $f''(1) = g''(1) = 1/2$, but $f^{(3)}(1) = -3/4$ and $g^{(3)}(1) = 3/4$. All is well for the two first derivatives but not for the third, so $\Gamma \cup s(\Gamma)$ is not smooth, and once again, Γ is not an arc of a circle. ■

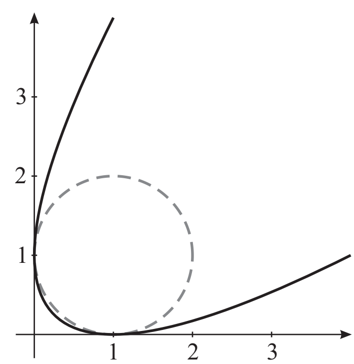

Figure 3. The circle and the parabola

Not a circle, but what? Now the reader should be *very* convinced that Γ is not an arc of a circle. So what is it? We have shown (in Exercise 6) that each point of Γ satisfies $x^2 + y^2 - 2xy - 2x - 2y + 1 = 0$. We recognize in this the equation of a conic [**1, 2**] whose nature is determined by the quadratic form $q = x^2 + y^2 - 2xy$. This one is degenerate because $q = (x - y)^2$, and so this conic is a parabola; see Figure 3. Precisely, in the new coordinates $X = x - y$ and $Y = x + y$ (i.e., use the two bisectors as new axes), we get the equation $X^2 - 2Y + 1 = 0$, and a translation of the origin brings us to $Y' = X^2$. So Γ is an arc of a parabola, not an arc of circle.

To explain that Γ is an arc of parabola, we can also give the following elementary argument (suggested by one of the anonymous referees). By rotating the axes 45° clockwise, the equation becomes

$$\sqrt{\frac{X}{\sqrt{2}}+\frac{Y}{\sqrt{2}}}+\sqrt{\frac{X}{-\sqrt{2}}+\frac{Y}{\sqrt{2}}}=1,$$

so $\sqrt{X+Y}=2^{1/4}-\sqrt{Y-X}$. Squaring leads to $\sqrt{Y-X}=2^{-3/4}-2^{-1/4}X$ and, squaring again, $Y=2^{-3/2}+2^{-1/2}X^2$, clearly the equation of a parabola.

The equal exponent family of curves

After the proofs that the curve $\sqrt{x}+\sqrt{y}=1$ for $(x, y)\in[0,1]^2$ is not an arc of a circle, we showed that it is an arc of a parabola. The curve Γ is the $r=1/2$ case of a family Γ_r with Cartesian equation $x^r+y^r=1$ for $r\in(0,+\infty)$ and $(x,y)\in[0,1]^2$. Of course, in the case $r=2$, we have an arc of a circle and, for $r=1$, an arc of a straight line, i.e., a segment. It is natural to ask for which r the family of curves contains arcs of conics [**1, 2**]. We show next that we have found all such.

Proposition 2. *The curve Γ_r given by $x^r+y^r=1$ for $r\in(0,+\infty)$ and $(x,y)\in[0,1]^2$ is an arc of a conic if and only if $r\in\{1/2,1,2\}$.*

Just the "only if" direction of the claim requires proof because we already know that in the cases $r=1/2,1,2$ the curves are conics. We want to prove this assertion using two methods, first using curvature, second by algebra.

Proof. We proceed in two steps:

(i) According to the value of $r\in(0,+\infty)\setminus\{1/2,1,2\}$, we show that the curvature approaches either 0 or infinity in a neighborhood of $x=1$.

(ii) We verify that conics do not have this property.

Choosing x for the parameter, the curvature formula (1) gives $c^2=y''^2/(1+y'^2)^3$. In our case, we have $y=(1-x^r)^{1/r}$ for $x\in(0,1)$. Easy calculations give

$$y'=-x^{r-1}\left(1-x^r\right)^{\frac{1}{r}-1},\quad y''=(1-r)x^{r-2}\left(1-x^r\right)^{\frac{1}{r}-2}$$

so that we can write c as

$$c(x)=\frac{|1-r|x^{r-2}(1-x^r)^{\frac{1-2r}{r}}}{\left(1+x^{2(r-1)}(1-x^r)^{\frac{2(1-r)}{r}}\right)^{\frac{3}{2}}}.$$

If $r<1$, then $c(x)\sim|1-r|(1-x^r)^{(1-2r)/r}$ as $x\to1$ with limit 0 if $r\in(0,1/2)$ and limit $+\infty$ if $r\in(1/2,1)$. If $r>1$, then $c(x)\sim|1-r|(1-x^r)^{(r-2)/r}$ as $x\to1$ with limit $+\infty$ if $r<2$ and limit 0 if $r>2$. These calculations show that for $r\in(0,+\infty)\setminus\{1/2,1,2\}$, the curvature can be as small or as large as we want, according to the values of r, in a neighborhood of the point $(1,0)$. We now proceed to our second step.

Again, curvature does not depend on the regular parametrization of the curve. So we can calculate curvature of the three kinds of conics using standard parametrizations, given below with their curvatures calculated from (1).

- The ellipse $x(t)=a\cos t,\ y(t)=b\sin t$ (we assume $a\ge b>0$) has curvature

$$c_E^2(t)=\frac{a^2b^2}{(a^2\sin^2 t+b^2\cos^2 t)^3}.$$

- The hyperbola $x(t) = a\cosh t,\ y(t) = b\sinh t$ (again $a \geq b > 0$) has curvature

$$c_H^2(t) = \frac{a^2 b^2}{(a^2 \sinh^2 t + b^2 \cosh^2 t)^3}.$$

- The parabola $y = x^2/2p$ has curvature $c_P^2(x) = p^4/(p^2 + x^2)^3$.

For an ellipse, because $a^2 \sin^2 t + b^2 \cos^2 t = b^2 + (a^2 - b^2)\sin^2 t \geq b^2$, since $a \geq b > 0$, we see that, for all real t,

$$\frac{a^2 b^2}{(a^2 + b^2)^3} \leq c_E^2(t) \leq \frac{a^2}{b^4}.$$

Since that curvature cannot approach zero or infinity, our arc cannot be an arc of an ellipse.

The two others curvatures have upper bounds, namely

$$c_H^2 \leq \frac{a^2}{b^4}, \quad c_P^2 \leq \frac{1}{p^2}.$$

These can approach zero at infinity, but that is not a problem since our arc is compact. Thus, our arc cannot be an arc of a hyperbola or a parabola either. ■

Proof. The general equation of a conic is $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ with not all of a, b, c, d, e being zero [**1**].

If our arc is an arc of a conic, then there exists some real numbers a, b, c, d, e, f such that we have, for all $x \in [0, 1]$,

$$ax^2 + 2bx(1 - x^r)^{\frac{1}{r}} + c(1 - x^r)^{\frac{2}{r}} + dx + e(1 - x^r)^{\frac{1}{r}} + f = 0.$$

The idea now is to look at the limited expansion near 0 of the function appearing in the left-hand side. In this expansion, we can find constant terms and x, x^2, x^r, x^{2r}, x^{3r}, $\ldots$; x^{r+1}, x^{2r+1}, x^{3r+1}, $\ldots$. Of course, these terms can be the same for certain values of r. We have two cases.

- $r > 1$ and $r \neq 2$. In this case, the only x^2 term is ax^2, so $a = 0$. The only x^{r+1} term is $-(2b/r)x^{r+1}$, so $b = 0$. Then only dx remains as the only x term, so $d = 0$. Now $x = 1$ gives $f = 0$. Simplifying the remaining express gives

$$c\,(1 - x^r)^{\frac{1}{r}} + e = 0.$$

  Again $x = 1$ implies $e = 0$ and finally $c = 0$. So we obtain $a = b = c = d = e = 0$, a contradiction.
- $0 < r < 1$ and $r \neq 1/2$: The powers of x in the six terms are 2, $kr + 1$, lr, 1, lr, 0, successively, for positive integers k, l. The terms x^r and x^{2r} appear only in the c and e factors (because r is not $1/2$), thus

$$-\frac{2}{r}c - \frac{1}{r}e = 0, \quad \frac{2}{r}\left(\frac{2}{r} - 1\right)c + \frac{2}{r}\left(\frac{1}{r} - 1\right)e = 0,$$

  so $2c + e = 0$ and $(2 - r)c + (1 - r)e = 0$. It follows that $c = e = 0$ and then also the constant term $f = 0$. Now there is only one x^{r+1} term, so $b = 0$. We conclude $a = d = 0$ and so get the same contradiction as in the previous case. ■

Among negative values for the common exponent, $r = -1$ gives an arc of hyperbola: $1/x + 1/y = 1$ for $(x, y) \in (0, 1)^2$ is equivalent to $(x-1)(y-1) = 1$, which we recognize a hyperbola with center $(1, 1)$. We remark without proof that the arcs are not arcs of conics for other negative values of r.

Lamé's curves

The Γ_r curves belong to the family of curves studied by G. Lamé [**4**] in the 19th century and now called Lamé's curves, or superellipses. Namely, such a Lamé's curve has Cartesian equation

$$\frac{x^\alpha}{a^\alpha} + \frac{y^\alpha}{b^\alpha} = 1,$$

where a and b are positive real numbers and α is a real number. For some values of α (e.g., $\alpha = 2$), the Cartesian coordinates x and y can be negative, but for some other values (e.g., our previous case $\alpha = 1/2$), they must be positive. In order to avoid supplementary difficulties and to stay in the previous framework, we only work with positive coordinates and so define the Lamé's curve $\mathcal{L}(\alpha, a, b)$ as

$$\mathcal{L}(\alpha, a, b) = \left\{ (x, y) \in [0, +\infty)^2 \,\middle|\, \frac{x^\alpha}{a^\alpha} + \frac{y^\alpha}{b^\alpha} = 1 \right\}.$$

In order to consider the following proof by Lamé, we introduce the envelope of a family of curves [**1**]. This is the curve that is tangent to each member of the family at some point. Thus, the envelope of a family of curves $(\mathcal{C}_\lambda)_\lambda$, each of them defined by a Cartesian equation $F(x, y, \lambda) = 0$, is obtained by solving the system of equations

$$F(x, y, \lambda) = 0, \quad \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0.$$

Theorem 3 (Lamé). *The envelope of the family of curves $\mathcal{L}(\alpha, m, n)$ when the parameter (m, n) describes the curve $\mathcal{L}(\beta, a, b)$ is the curve $\mathcal{L}(\gamma, a, b)$ with*

$$\frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta}.$$

Proof. The parameter here is a pair $\lambda = (m, n)$ constrained by the condition

$$\frac{m^\beta}{a^\beta} + \frac{n^\beta}{b^\beta} = 1$$

and so we can write the family of curves with only the only parameter m as

$$F(x, y, m) = \frac{x^\alpha}{m^\alpha} + \frac{y^\alpha}{b^\alpha \left(1 - \frac{m^\beta}{a^\beta}\right)^{\frac{\alpha}{\beta}}} - 1 = 0.$$

The equation $\frac{\partial F}{\partial m}(x, y, \lambda) = 0$ leads to the relation

$$-\frac{x^\alpha}{m^{\alpha+1}} + \frac{m^{\beta-1}}{a^\beta - m^\beta}\left(1 - \frac{x^\alpha}{m^\alpha}\right) = 0$$

from which we can obtain $x^\alpha = m^{\alpha+\beta}/a^\beta$ or $x = m(m/a)^{\beta/\alpha}$ that we prefer finally to write as

$$x = a\left(\frac{m^\beta}{a^\beta}\right)^{\frac{1}{\alpha}+\frac{1}{\beta}}.$$

It follows from this calculation and the fact that the point (x, y) belongs to the curve $\mathcal{L}(\alpha, m, n)$ that

$$y = b\left(1 - \frac{m^\beta}{a^\beta}\right)^{\frac{1}{\alpha}+\frac{1}{\beta}}.$$

If γ is the real number defined by $1/\gamma = 1/\alpha + 1/\beta$, then the previous values of x and y lead to

$$\frac{x}{a} = \left(\frac{m^\beta}{a^\beta}\right)^{\frac{1}{\gamma}} \text{ and } \frac{y}{b} = \left(1 - \frac{m^\beta}{a^\beta}\right)^{\frac{1}{\gamma}}$$

and therefore to the desired equation,

$$\frac{x^\gamma}{a^\gamma} + \frac{y^\gamma}{b^\gamma} = 1. \qquad \blacksquare$$

Lamé went on to use the this result for some geometric constructions [**4**]. Here we give examples illustrating this theorem in the cases when Lamé's curves are conics, which we know from Proposition 2 to be only those for which the exponent $r \in \{1/2, 1, 2\}$. To realize the condition $1/\gamma = 1/\alpha + 1/\beta$ using only these three values, the only possibilities are $\alpha = \beta = 2$, $\gamma = 1$ and $\alpha = \beta = 1$, $\gamma = 1/2$.

- From the first case, the envelope of the family of ellipses (given by their equations)

  $$\left(\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1\right)_{(m,n)\in\mathcal{E}},$$

  where $\mathcal{E}$ is the ellipse

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

  is the line $x/a + y/b = 1$.
- From the second case, the envelope of the family of lines (given by their equations)

  $$\left(\frac{x}{m} + \frac{y}{n} = 1\right)_{(m,n)\in\mathcal{L}},$$

  where $\mathcal{L}$ is the line $x/a + y/b = 1$, is the parabola

  $$\left|\frac{x}{a}\right|^{1/2} + \left|\frac{y}{b}\right|^{1/2} = 1,$$

  a slightly generalized form of our motivating problem.

Acknowledgments. We thank Fanny Ayache for informing us of the motivating problem for this article, our colleague Sridharan Sriraman for his proofreading, the anonymous referees for helpful advice, and the editor for ameliorating the text.

Summary. We begin with a high school exercise, determining whether a particular equation describes an arc of a circle. We give several proofs that it does not, exploring many properties of circles along the way. Yet the curve is the arc of a conic. We then explore which equations of a generalized family are arcs of conics and give some properties of the resulting Lamé curves.

References

1. M. Audin, *Geometry*. Springer, Berlin, 2002.
2. M. Berger, *Geometry I and II*. Springer, Berlin, 2009.
3. M. P. DoCarmo, *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Englewood Cliffs, NJ, 1976.
4. G. Lamé, *Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie*. Courcier, Paris, 1818.

