Carl B. Allendoerfer Awards

William Dunham

"The Early (and Peculiar) History of the Möbius Function," *Mathematics Magazine*, 91:2, 83–91, 10.1080/0025570X.2017.1413921.

Upon first encountering the Möbius function in a number theory course, a student might well find it "neither useful nor obvious," as the author writes in this lively story of the early appearances of the function. Seeking to perform a kind of inversion on infinite series, August Ferdinand Möbius was led in 1832 to introduce a certain function $\mu(x)$ on the positive integers—although as the author notes, the now-ubiquitous notation $\mu(x)$ only took hold after being introduced some 40 years later by Franz Mertens. In order to achieve his inversion, Möbius found that $\mu(x)$ must satisfy:

1. $\mu(1) = 1$
2. $\mu(n) = 0$ if n is divisible by the square of some prime
3. $\mu(n) = (-1)r$ if n is a product of r distinct primes

While it seems non-intuitive at first, Möbius' function gets at something fundamental about the integers – both the prime number theorem and the Riemann hypothesis can be recast as statements about $\mu(x)$. The author leads us on a tour of Möbius' construction from his 1832 paper, and illustrates the power of Möbius' method by showing how the complicated-looking sum

$$\sum_{k=1}^{\infty} \frac{\mu(k)x^k}{1-x^k}$$

equals, remarkably, x.

But the story doesn't end there, and those familiar with the author's previous work might have an inkling where it goes. Some 80 years earlier, Euler had already met the same function. In his typical fashion, Euler derived eye-popping formulas for sums such as $\sum_{k=1}^{\infty} \mu(k)/k$ and $\sum_{k=1}^{\infty} \mu(k)/k^2$ without ever explicitly defining $\mu(x)$. He considered the interplay between infinite products and infinite sums, and starting with the product

$$\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\cdots$$

and its reciprocal. Niftily using the unique factorization of integers into products of prime powers, he showed that this product is given by $\sum_{k=1}^{\infty} \mu(k)/k$ and its reciprocal by the harmonic series. And presto: $\sum_{k=1}^{\infty} \mu(k)/k = 0$. With similar verve, Euler derived $\sum_{k=1}^{\infty} \mu(k)/k^2 = 6/\pi^2$. As the author writes, "These wonderful results are examples of Euler being Euler, manipulating symbols with a gusto that can take one's breath away. In so doing, he not only anticipated the Möbius function, but generated formulas more sophisticated than anything its namesake would discover eight decades later. Euler was, once again, far ahead of his time."

The article is written with its own gusto, guiding the reader with eloquence through the peculiar history of this foundational number-theoretic function. In a most entertaining way, "this tale reminds us—if we need reminding—that the history of mathematics can provide a host of unexpected rewards."

Response

It is a thrill to receive the Allendoerfer Award for my article on the origins of the Möbius function. Many thanks to the MAA and to those committee members who directed this honor my way.

Let me share a little story. When Penny and I retired from Muhlenberg College in 2014, we moved to Bryn Mawr on Philadelphia's Main Line. This put us near Bryn Mawr College, and we were pleased when their mathematics department gave us an affiliation that let us enjoy a new academic home a few blocks from our real one.

As a historian, I especially appreciated the College's excellent Science Library. There, one might find an old book that once belonged to Charlotte Angas Scott, Bryn Mawr's first math professor, or a volume bearing the signature of Emmy Noether, the illustrious mathematician who was welcomed by Bryn Mawr after fleeing Nazi Germany in 1933.

One day my browsing led me to the collected works of August Ferdinand Möbius. I figured I'd thumb through it to find the famous Möbius function from the theory of numbers. But nothing inside smacked of number theory. It took some time before I spotted a version of the function buried within an 1832 paper on analysis. This suggested that there was more to the topic than meets the eye, a thought reinforced when I found that Leonhard Euler had stumbled upon the same function in his famous *Introductio in analysin infinitorum* from 1748. My attempts to unravel the history of this idea became an article for *Mathematics Magazine*. And here we are.

The take-away from my little tale: grazing through a great library can have unexpected rewards.

Biographical Sketch

William Dunham is a historian of mathematics who has written/edited six books on the subject, including *Euler: The Master of Us All* (MAA, 1999) and *The Calculus Gallery* (Princeton, 2005). Since retiring from Muhlenberg College in 2014, he has held visiting positions at Princeton, Penn, Cornell, Harvard, and at Bryn Mawr College, where he is currently a Research Associate in Mathematics.