How to Make a Torus

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Ask a topologist how to make a model of a torus and your instructions will be to tape two edges of a piece of paper together to form a tube and then to tape the ends of the tube together. This process is mathematically correct, but the result is either a flat torus or a crinkled mess.

A more deformable material such as cloth gives slightly better results, but sewing a torus exposes an unexpected property of tori. Start by sewing the torus inside out so that the stitching will be hidden when the torus is reversed. Leave a hole in the side for this purpose. Wait a minute! Can you even turn a punctured torus inside out?

Surprisingly, the answer is yes. However, doing so yields a baffling result. A torus that started with a pleasing doughnut shape transforms into a one with different, almost unrecognizable dimensions when turned right side out. (See figure 1.)

So, if the standard model of a torus doesn't work well, we'll need to take another tack. A torus is generated by sweeping a circle around an axis in the same plane as the circle. This means that any plane containing the axis intersects the torus in two circles.

We can use this property to make a different model. Cut a stack of sticky notes that have adhesive on alternate sides into a circle and fan it to form a torus, as in figure 2. This model is attractive, if a bit flimsy.

Figure 1. A cloth torus sewed inside out, turned right side out.

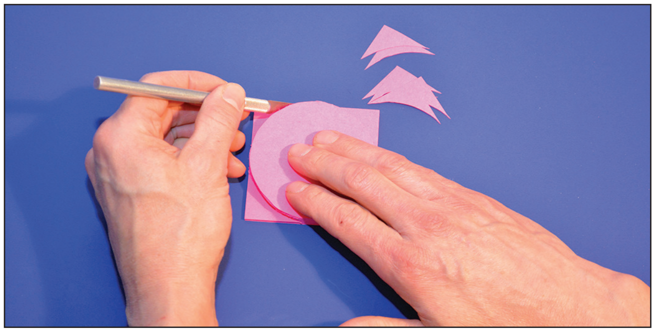


Figure 2. A torus made from sticky notes.

The sticky notes illustrate one of three types of circular cross sections. We find the second circular cross section by cutting a torus with a plane perpendicular to the axis of revolution (both such cross sections are shown on the green torus in figure 3).

Figure 3. Circular cross sections.

But there is third, less obvious way to create a circular cross section. If we cut the torus with a plane slanted at just the right angle—tangent to the torus as shown in figure 4—the cross section will be two intersecting circles called Villarceau circles (shown on the orange torus in figure 3).

Villarceau circles are the foundation of the most visually arresting models. One model, shown in figure 5, was created independently by Yoshinobu Miyamoto (*http://bit.ly/1G7beYd*) and the team of María García Monera and Juan Monterde ("Building a Torus with Villarceau Sections," *Journal for Geometry and Graphics* 15, no. 1 [2011] 93–99). Its intersecting crescent-shaped pieces are defined by Villarceau circles. Instructions and patterns to make this model can be found at *maa.org/math-horizons-supplements.*

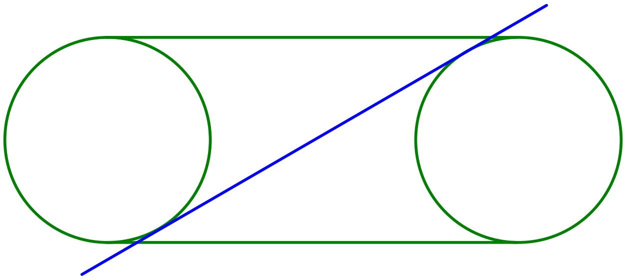
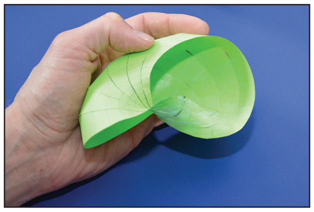


Figure 4. The blue plane slices the green torus in a pair of Villarceau circles.


(a)


(b)


(c)


(d)


(e)

Figure 7(a)–(e). Instructions for building the torus.

The torus in figure 6 has seams that lie along Villarceau circles. To make this model, cut out two copies of the pattern found at *maa.org/math-horizons-supplements* and follow these instructions, which are shown in figure 7 (the website also contains a link to video instructions).

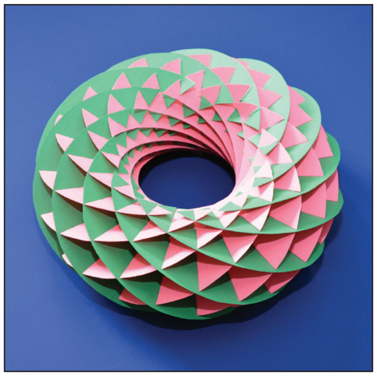


Figure 5. The Miyaoto-Moneroa-Monterde model.

(a) Tape the edges together, placing the tape at the back of the pattern and lining up the dashed lines as you go. Repeat for the second pattern.

(b) Hold the two ends of one half as shown.

(c) Give each end a 180-degree twist and tape them together.

(d) At this point, you can see the outline of a Villarceau circle around the opening of the piece.

(e) Tape the two halves together along the circles to form a torus. ■

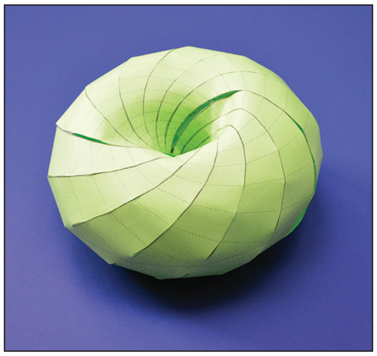


Figure 6. A torus taped along Villarceau circles.

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