THE INTERSECTION GAME

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Spot It! is a fun card game played with 55 cards, each of which carries eight pictures. What's special about this game is that any two cards have exactly one picture in common. The aim of Spot It! is to be the first player to spot this common picture. Can you spot the common picture on the two cards in figure 1?

Now let's spot the cool mathematics at the core of this game and see if we can use it to design new variants.

Spotting Lines

How would we create a 55-card Spot It! deck—a deck of cards decorated with pictures with the property that any two cards have exactly one common picture?
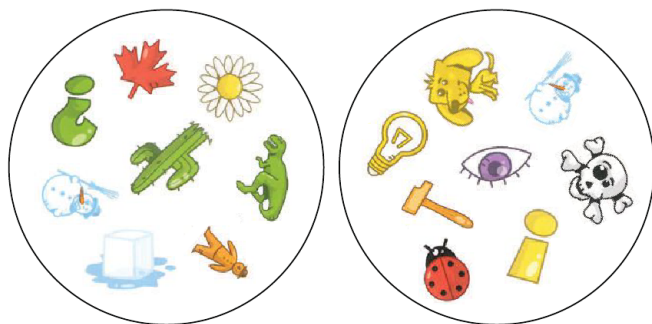

Figure 1. What picture is on both cards?

This task sounds tricky, but it's actually a piece of cake. What familiar mathematical objects have a similar property? Two nonparallel lines in a plane always intersect in a unique point.

So, to speed-design a deck of Spot It! cards, we simply draw 55 mutually nonparallel lines, label the points of intersection with different pictures, and gather all pictures on each line into its own card. For example, we can turn the four lines in figure 2 into four cards, each carrying three pictures.

Figure 2. Use intersecting lines to construct a Spot It!! deck with four cards.

Pretty neat, but also pretty disappointing. Is there really no more to spot in Spot It!? Let's have a closer look at what our brand-new Spot It! deck would look like.

Using a random assortment of lines, chances are that, as in our small example, only two lines pass through each intersection point. In other words, there will be as many points of intersection as there are ways to pair up 55 lines, amounting to $55 \cdot 54 \,/\, 2 = 1,485$ different points, and hence 1,485 pictures. Moreover, there will be 54 pictures per card. That's definitely not as compact as the commercial game—there are far too many pictures in our version to be fun.
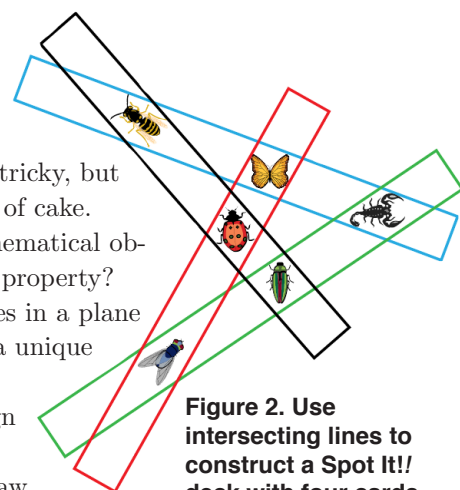
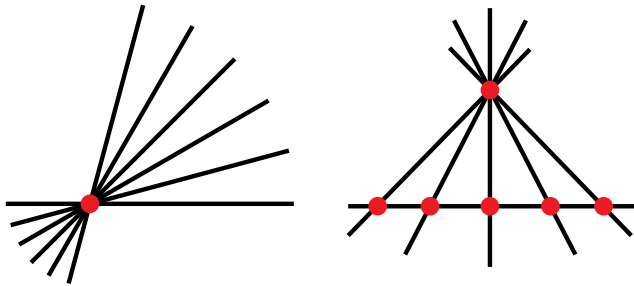To cut down the number of pictures, we have to de-

Figure 3. Sneaky configurations of points and lines.

crease the number of points of intersection. Let's give this a try with six lines. Our straightforward approach would result in 15 pictures with five pictures per card. A little experimentation leads to the two sneaky, but gamewise uninteresting, ways of keeping down the number of intersections, shown in figure 3. More interestingly, we also spot the nicely balanced approach corresponding to seven pictures with three on every card shown in figure 4. That's quite an improvement, and we cannot do better with six lines in a non-sneaky way.
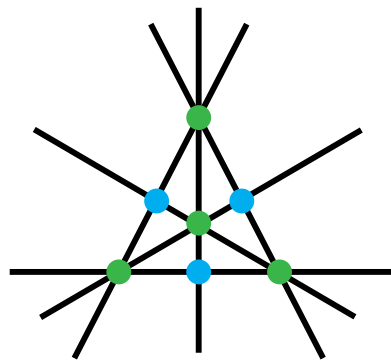

Figure 4. Every pair of lines intersects in one of seven points.

Notice that each of the green points lies on three lines whereas each blue point is on only two lines. So, let's collect the three blue points into a new seventh card. This can't be done with a straight line, but for the purposes of designing our game, we don't care. With this seventh card added and represented by a large circle, as in figure 5, we arrive at a very compact seven-card version of Spot It! using only seven pictures.

To add to the fun of playing this game, we chose as pictures the seven letters in the words THEY WAR and distributed the
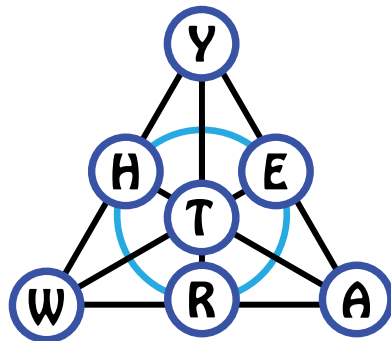

Figure 5. These six lines and the large circle generate a seven-card Spot It!deck.
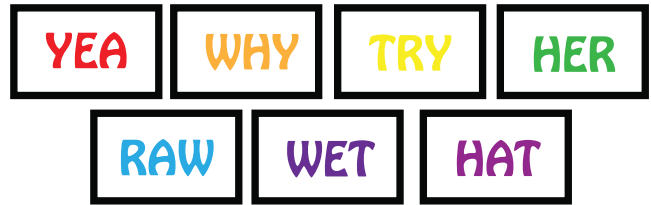

Figure 6. A seven-card Spot It!/deck.

letters as shown in the diagram. Then the seven cards consist of the words in the sentence YEA WHY TRY HER RAW WET HAT. Figure 6 shows a set of playing cards for this game.

Spotting Projective Planes

As an abstract point-line geometry, our new seven-card version of Spot It! is actually very famous. In the math biz it is known as the *Fano plane* or the *smallest projective plane*. Let's explain what we mean by this.

The *point set* of an abstract point-line geometry can be any set whatsoever, and its *line set* is a set of subsets of the point set. In this context we call the elements of the first set the POINTS and those of the second set the LINES of our geometry. Then it should be clear what it means for a POINT to be on a LINE, for two POINTS to be connected by a LINE, and so on.

We can interpret any deck of Spot It! cards as a point-line geometry simply by declaring the pictures to be the POINTS and the cards to be the LINES. In the case of our Fano plane, {T,H,E,Y,W,A,R} is the point set and {{Y,E,A}, {W,H,Y}, {T,R,Y}, {H,E,R}, {R,A,W}, {W,E,T}, {H,A,T}} is the line set.

Geometries obtained from a Spot It! deck are special point-line geometries called *dual linear spaces,* which are characterized by the fact that any two LINES intersect in a unique POINT.

Projective planes like the Fano plane are extra-special dual linear spaces that have the further property that any two POINTS are contained in exactly one LINE. In the case of the Fano plane this means that any two of the letters in THEY WAR are contained simultaneously in exactly one of the words in YEA WHY TRY HER RAW WET HAT.

One of the fundamental theorems of the theory linear spaces, discovered by de Bruijn and Erdős, implies that except for some sneaky exceptions like the ones in figure 3, Spot It! decks always have at least as many pictures as there are cards. In addition, any optimal Spot It! deck with the same number of cards and pictures is an incarnation of a finite projective plane. In this case there are n pictures on every card, every picture appears on n cards, and there are $n^2 - n + 1$ cards and pictures, for some n.

For example, in the case of the Fano plane game, there are three pictures on every card, every picture is on three cards, and the number of cards and pictures is $3^2 - 3 + 1 = 7$.

The commercial Spot It! game has eight pictures on every card, $8^2 - 8 + 1 = 57$ pictures, but only 55 cards. Interestingly, the commercial Spot It! deck is a projective plane with two lines missing. (The name of the original French game is *Dobble*, and the double 5 in 55 had market appeal and also happened to be an ideal number of cards, taking into account manufacturing considerations.) In fact, we've reconstructed the two missing cards in our deck (we've learned that not all decks are identical); they're the ones in figure 1.

Spot It Cubed!

What about the natural question: Is it possible to construct a deck in which any *three* cards have exactly one picture in common? The answer is "Yes."

Any three mutually nonparallel planes in space intersect in a unique point or in a line. So, we just need to take a collection of mutually nonparallel planes, no three of which pass through a common line (which is easy). We label the resulting points of intersection with different pictures and gather all pictures on the different planes into cards and voilà: instant Spot It Cubed!

It is just as easy to create cards for Spot It^4!, Spot It^5!, and so on, using intersecting hyperplanes. However, as in the case of regular Spot It!, when designing these new decks, the real trick is to keep the number of pictures small. Luckily, there are mathematical objects that can be translated into efficient decks.

Here is a nice simple one. It's a point-circle geometry. Again, this is an abstraction of ordinary geometry—CIRCLES are sets of POINTS in which every triple of POINTS lie on exactly one CIRCLE. (Here the analogue is points and circles on a *sphere* rather than points and circles in the plane.)

In our geometry, the POINTS are the eight vertices of a cube and the CIRCLES are 14 sets of four vertices each, represented by the six faces of the cube (which we'll color green), the six (orange) diagonal rectangle cuts, and the two (blue) tetrahedra inscribed in the cube.

Figure 7 shows one representative each of the three

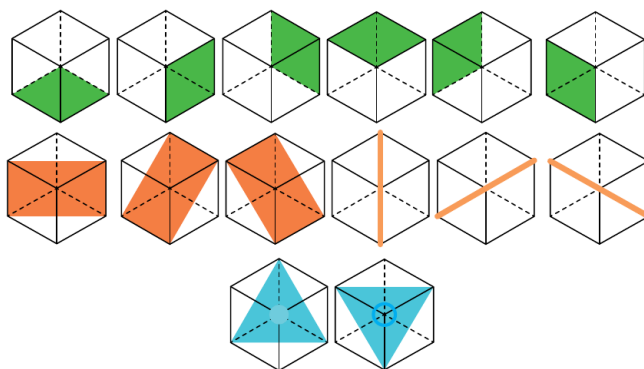Figure 7. The three types of CIRCLES.

Figure 8. The 14 CIRCLES.

types of CIRCLES. Convince yourself that any three POINTS in our minigeometry are contained in exactly one CIRCLE.

To make this geometry work for us, we must *dualize* it: We make the CIRCLES the pictures and the POINTS the cards. The card corresponding to a POINT contains all the CIRCLES through the POINT.

Confused? Let's build the deck. Using a regular hexagon picture of the cube we can draw the 14 CIRCLES as in figure 8.

Figure 9 shows the card corresponding to the purple POINT/vertex; it contains the pictures of all the CIRCLES that contain this vertex.

Figure 10 shows the complete set of cards. But of course we can also use pictures that don't encode any of the geometry, as in figure 11.
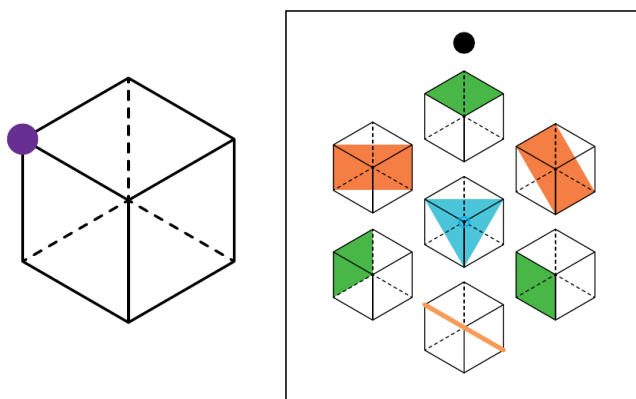

Figure 9. The card corresponding to the purple vertex.

Spot a Biplane!

If we want to be innovative Spot It! game designers, the mathematical objects to check out are the so-called t-(v, k, λ) *(block) designs*. A t-(v, k, λ) design has a point set consisting of v POINTS and BLOCKS consisting of k POINTS each, such that any t POINTS are contained in exactly λ BLOCKS.

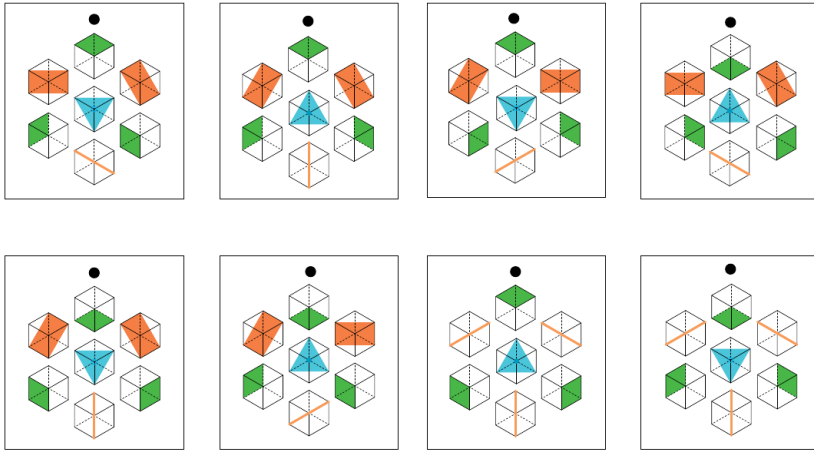For example, projective planes are the

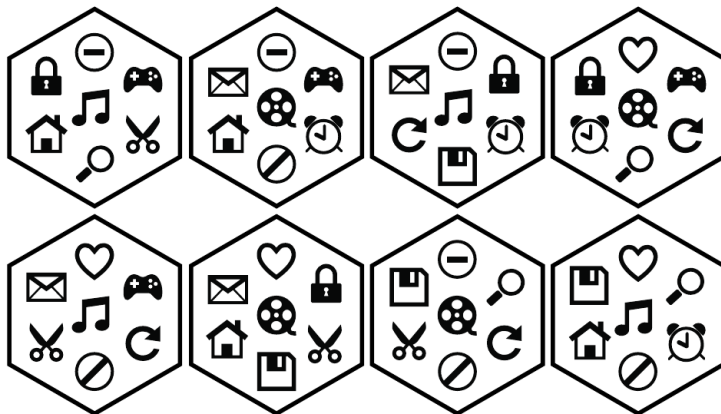**Figure 10. Cards corresponding to the eight POINTS.**

Figure 11. Any three cards in this deck have one picture in common.

Figure 12. A Spot a Biplane! deck.

$2\text{-}(n^2 - n + 1, n, 1)$ designs and our geometry on the cube is a 3-(8,4,1) design. Dualizing any $t\text{-}$(whatever,whatever,1) design using the method we just demonstrated gives a deck of cards in which any t cards have exactly one picture in common. The problem with larger t is that no matter how hard we try, we'll always end up with a lot of pictures on our cards.

For example, starting with the amazingly compact 5-(12,6,1) Mathieu design, we'd get a deck of 12 cards with 66 pictures on each card out of a total of 132 pictures. Pretty mindboggling—but the game is almost certainly not much fun to play.

Instead of increasing t, why not play with the other parameters in the definition of block designs? In particular, some of the 2-(whatever,whatever,2) designs translate into interesting decks of cards with small numbers of pictures.

These very rare and exotic designs are called *biplanes.* Similar to projective planes any two POINTS of a biplane are contained in exactly two BLOCKS and (thrown in for free!) any two BLOCKS intersect in exactly two POINTS. This intersection property means that we don't have to worry about dualizing a biplane. Just make the POINTS into the pictures and the BLOCKS into the cards and we've got a Spot a Biplane! deck.

For example, the blocks of the unique 2-(7,4,2) biplane happen to be the complements of the LINES of the Fano plane within its point set. Figure 12 shows the corresponding Spot a Biplane! deck.

Ready to play? Then spot the two common letters on the red and green cards, and spot the two cards that contains the letters AT.

Further Reading

"The Mathematics of 'Spot It!'" by Rebekah Coggin and Anthony Meyer, ΠΜΕ *Journal* 13(8), 459–467, has more details on projective planes, as do a number of articles online.

Check out Spot It! (or Dobble) for mobile devices. It incorporates (complete!) decks corresponding to projective planes with 31, 57, and 91 cards.

If you are interested in making your own serious Spot a Biplane! deck, search the Web for "Gordon Royle, biplane." ∎

Burkard Polster specializes in fun mathematics. Readers may be familiar with some of his books, such as The Mathematics of Juggling *or* Math Goes to the Movies *(with Marty Ross). Burkard teaches math at Monash University in Melbourne, Australia.*
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