Problem Solving Strategy Essay # 9:

Avoid Hard Work

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Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

There is an aphorism in the mathematics community:

> If walking through a department office you see a mathematician busy at his desk typing furiously at a computer keyboard he is not really working on mathematics, just engaged in busy work. But, if in the next office, you see a mathematician with her feet up on the table and staring at the ceiling, then you know that she is working very hard!

Of course, the keyboard-tapping scholar could very well be engaged in deep, extensive thinking while the upward-staring academic is having a mental snooze. But the goal of this story is to point out that a significant component of mathematical work is tough mental effort through sustained, deep reflection, mulling and pondering. Mathematicians will work and think very hard to find a good and elegant approach that avoids hard work! This is the feature problem-solving strategy for today:

AVOID HARD WORK!

Laborious, tedious computation should be left as a last resort.

The materials in the MAA AMC archives provide repeated opportunity to foster this effective and potent habit of mind. And, we educators can find good place for this work in our classrooms. Read on to see how!

OUR CHALLENGE TODAY

Our query this time is **question 15** from the **2011 MAA AMC 12b** exam, the twelfth-grade exam:

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How many positive two-digit integers are factors of 2^24 − 1?
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Offer this question as a topic for class discussion. (Group conversation provides emotional safety for all to explore ideas, follow leads, and possibly get things beautifully wrong! Your role in such an experience is to simply facilitate the conversation and be the scribe of ideas on the board.)

Students might offer one strategy: Just work out the number $2^{24}-1$, factor the answer, and count the two-digit factors.

Let's do just that!

The Brute-Force Approach:

Let's do go through the brute-force approach. It will be terribly labored, but be jovial about it. The point is to push the tedium so as to learn, for later, it is something we wish to avoid!

BY THE WAY*: If a student happens to see an efficient way to factor* $2^{24}-1$ *right off the bat, give her a knowing smile. Promise to let her show her approach later on.*

Let's compute $2^{24}-1$. We have:

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ \ldots$$

and we can keep doubling the numbers pretty easily:

16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, …

Ooh! We have $2^{12} = 4096$, so:

$$2^{24} = 2^{12} \times 2^{12} = 4096 \times 4096$$

We can save some doubling work. (Clever!)

COMMENT: *Students will tire of doubling and will invariably invent some short-cut of this kind for themselves. Don't force this short-cut. Let them own their cleverness.*

Doing the long multiplication on scratch paper we get $2^{24} = 16,777,216$, and so:

$$2^{24} - 1 = 16,777,215.$$

That wasn't too bad.

So all we need do now is find the factors of this number. It is clearly divisible by 5.

COMMENT: *Most students are aware that if a number ends in a five, it is divisible by five.* (How does one prove this?)

Long division on the side gives:

$$16,777,215 = 5 \times 3,355,443.$$

What can we do with $3,355,443$?

`Engage in wishful thinking.`

It would be lovely if this number possessed a small factor we could recognize.

COMMENT: *This is the sort of "nudge" I make when leading this group discussion. But I do this only if the class seems truly stuck on what to do next. (Don't be afraid of minutes of silence!)*

Is it divisible by three?

Yes! $3,355,443 = 3 \times 1,118,481$

Can we divide by three again?

Yes! $1,118,481 = 3 \times 372,827$

COMMENT: *It is fine if the group does not think to check for multiple factors. You can "nudge" the class back to wishful thinking if later they seem stuck.*

COMMENT: *Some students might know a divisibility rule for the number three. But please be leery of memorized rules that might or might not come with understanding. (See* ESSAY 8 *for a discussion of divisibility rules. See also the videos* www.jamestanton.com/?p=1287 *.)*

Can we divide by three yet again?

Alas, no. Long division shows that $372,827$ leaves a remainder.

Okay .. So what can we do with $372,827$?

Is it divisible by 7, the next smallest possible factor? (*Why did we skip 4 and 6 as possible factors?*)

Long division gives: $372,827 = 7 \times 53,261$
We're lucking out!

Another factor of 7? Long division shows not. Bother!

What next? 11?
Long division shows that $53,261$ is not divisible by 11.

How about 13?
Yes! We get: $53,261 = 13 \times 4097$.

We can check to see if a second factor of 13 lies in 4097. It doesn't.

Is 17 next?
On the side we see $4097 = 17 \times 241$ and there is not a second factor of 17 present.

So this leaves the number 241.

This number is not divisible by $2, 3, 7, 11, 13$ or 17. (We have already "pulled out" as many of these factors as we can.)

COMMENT: *The class might suspect that* 241 *is prime. Push them to understand as to how we can be certain of this.* [*We could try dividing by* 19, *and by* 23, *and by* 29, *and by* 31, *and by … Heavens!*]

If 241 factors, $241 = a \times b$, then one of the factors must be smaller than $\sqrt{241}$. (*Why? What goes wrong if both* a *and* b *are larger than this?)*

Actually, as every number factors into primes, if 241 factors, it will have a prime factor smaller than $\sqrt{241}$.

Also $17^2 = 289$ shows $\sqrt{241} < 17$. (Actually it is less than 16.)

So we only need to check if 241 factors with one of the primes $2, 3, 7, 11, 13$.

But we have done that already - and it doesn't! This means 241 is prime! Phew!

COMMENT: *This was a complex line of thought.*

Okay, putting it all together we have:

$$2^{24} - 1 = 3 \times 3 \times 5 \times 7 \times 13 \times 17 \times 241$$

We have completely factored our number.

But I've lost sight of the question! What are we meant to be doing?

`How many positive two-digit integers are factors of` $2^{24} - 1$`?`

SOME MORE BRUTE FORCE:

Okay, we see that 13 and 17 are two two-digit factors. But there are more. For example, $3 \times 5 = 15$ is another two-digit factor of $2^{24} - 1$.

COMMENT: *Students might not think of this. I would then simply ask: "So there are two two-digit factors,* 13 *and* 17, *and we are done?"*

Let's find all the two-digit products we can make using the numbers $3, 3, 5, 7, 13$ and 17. These will be all the two-digit products of $2^{24} - 1$. (*We can ignore making use of the prime* 241. *Are you clear why?*)

COMMENT: *Most young scholars would probably list products in a haphazard manner. Let them. And let your board-work consequently be haphazard. At some point you should ask:*

"Have we got them all?"

The need and desire for a systematic search will follow.

FOLLOW-UP COMMENT: *It is tempting as an adult educator, with experience and practice under one's belt, to advise students to avoid difficulties before those difficulties arise. Unfortunately, from the student perspective, this has little context or meaning. One must first personally experience awkwardness in order to know to want to avoid it! We have an opportunity for students to experience and appreciate the advantages of systematic work.*

Let's compute the products we need, ordering them by the largest prime we use.

COMMENT: *Of course, this is not the only systematic method.*

Using seventeen as largest prime:

17

$3 \times 17 = 51$

$5 \times 17 = 85$

$7 \times 17 = \textit{too big}$

Using thirteen as largest prime:

13

$3 \times 13 = 39$

$5 \times 13 = 65$

$7 \times 13 = 91$

$3 \times 3 \times 13 = \textit{too big}$

Using seven as largest prime:

$3 \times 7 = 21$

$5 \times 7 = 35$

$3 \times 3 \times 7 = 63$

$3 \times 5 \times 7 = \textit{too big}$

Using five as largest prime:

$3 \times 5 = 15$

$3 \times 3 \times 5 = 45$

Using three as largest prime:

$$3 \times 3 = \textit{not big enough}$$

We see that there are 12 two-digit products and so $2^{24}-1$ has 12 two-digit integer factors.

We're done!

WHY THIS MATHEMATICALLY LABORED DISCUSSION?

The previous discussion should take place in an **ALGEBRA II** class once one has practiced factoring basic polynomials.

The discussion, as labored and awkward as it is, does have pedagogical advantages. The pain has gain!

Why pursue factoring $2^{24}-1$ by hand?

Students will indeed bore of doubling numbers over and over again, and their realization that $2^{24}=2^{12}\times 2^{12}$ (or any equivalent shortcut they devise) offers a moment to revisit some exponent properties.

Moreover, the relief offered by the shortcut gives a purpose to those exponent properties: they save work!

The real pedagogical advantage:

Struggling through clunky, hard work provides good motivation and context for wanting to avoid it! We can now ask the class, with legitimate clout …

If we put our feet up on the table, stared at the ceiling and mulled on this problem for a while, could we have avoided hard work?

Mulling readies the mind for epiphanies.

AN ENLIGHTENED ALGEBRA II APPROACH

Timing key MAA AMC problem-discussions with pieces of the curriculum can be powerful.

We want to factor $2^{24}-1$.

In algebra II we learn various factored formulas.

$$x^2-1=(x-1)(x+1)$$
$$x^3-1=(x-1)(x^2+x+1)$$
$$x^3+1=(x+1)(x^2-x+1)$$

etc.

Can we connect the ideas?

Epiphany: In algebra class, x can actually be a number!

People seem to forget this!

PRACTICE: Work out 97×103 in your head. (While you are at it, work out 103×103 and 97×97 too!)

Epiphany Continued: $2^{24}-1=\left(2^{12}\right)^2-1$. So x can be 2^{12}.

We can use the difference of two squares formula $x^2-1=(x-1)(x+1)$ to start the factoring without ever actually computing $2^{24}-1$. Wow!

$$\begin{aligned}2^{24}-1&=\left(2^{12}\right)^2-1\\&=\left(2^{12}-1\right)\left(2^{12}+1\right)\end{aligned}$$

Can we keep factoring? YES! Apply the same cleverness to $2^{12}-1$:

$$2^{24}-1=\left(2^6-1\right)\left(2^6+1\right)\left(2^{12}+1\right)$$

And again to 2^6-1!

$$2^{24}-1=\left(2^3-1\right)\left(2^3+1\right)\left(2^6+1\right)\left(2^{12}+1\right)$$

Since $2^3=8$ this reads:

$$2^{24}-1=7\times 9\times\left(2^6+1\right)\left(2^{12}+1\right)$$

Can we keep factoring? YOU BET!

Using

$$x^3+1=(x+1)(x^2+x+1)$$

we have

$$\begin{aligned} 2^6+1 &= \left(2^2\right)^3+1 = 4^3+1 \\ &= (4+1)(16-4+1) \\ &= 5\times 13 \end{aligned}$$

$$\begin{aligned} 2^{12}+1 &= \left(2^4\right)^3+1 = 16^3+1 \\ &= (16+1)(16^2-16+1) \\ &= 17\times(256-16+1) \\ &= 17\times 241 \end{aligned}$$

and so

$$2^{24}-1 = 7\times 9\times(5\times 13)(17\times 241)$$

This swiftly and beautifully gives the result we had before:

$$2^{24}-1 = 3\times 3\times 5\times 7\times 13\times 17\times 241$$

We still need to check if 241 factors.

Question: Is there an efficient way to check if 241 is prime? Or is the approach we took before the best approach?

And we still need to look for the two-digit factors of $2^{24}-1$.

Question: *Is there a way to make our previous systematic approach more efficient? (The answer could be no!)*

COMMENT: *Sometimes elements of brute-force checking are just unavoidable.*

COMMON CORE STATE STANDARDS and MATHEMATICAL PRACTICES:

Clearly we have hit the mark with:

A-APR-4 : Prove polynomial identities and use them to describe numerical relationships.

We have also modeled the practice standards:

MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP3. Construct viable arguments and critique the reasoning of others.
MP7. Look for and make use of structure.

ASIDE: The fifth practice standard states:

MP5. Use appropriate tools strategically.

The most efficient way to solve this problem would be to go to some sophisticated online mathematics program (*Wolfram Alpha*, for example), type in "factor 2^24 – 1" and simply count the number of two-digit factors! What do you think?

THINK BEFORE YOU LEAP QUESTIONS!

The Common Core State Standards in Mathematics place a strong emphasis on moving students away from rote doing and procedure for the sake of getting answers. There is significant emphasis on promoting conceptual understanding and deeper reflection.

Nonetheless, assessment tools often, by their structure, foster "memorization" and quick-response. They may focus on "what" questions with numerical answers.

Is it possible to promote reflection and mulling even within an enlightened assessment structure?

I was recently asked to share my thoughts on this very matter – where is the "wiggle room" within classical assessment? Have a look at the essay ***A Few Thoughts about Assessment*** available at

www.jamestanton.com/?p=968.

In it I offer a number of personal ideas, coupled with concrete examples, for taking opportunity to promote independent, confident thinking even within the structure of homework assignments, quiz and test questions, and exams. I call one class of questions the "Think Before You Leap" category. (I have eight categories of questions in all.)

Here are some examples of such meta-questions from the essay. Be inspired too by the MAA AMC materials to create more questions of this type for your students.

We should always look for opportunity to open up that door of creative thinking for our young scholars, even if only a smidgeon at a time. Let's actively work to foster a next generation of confident, independent and creative thinkers for the world!

SOME "THINK BEFORE YOU LEAP" CURRICULUM QUESTIONS:

Question: Here are four quadratic equations:

(A) $y = 4(x-3)(x-7)$

(B) $y = 3(x-2)^2 + 6$

(C) $y = 2x^2 - 4x + 8$

(D) $y = x^2 + x(x-3)$

i) For which equation would it be easiest to answer the question: *What is the vertex of the quadratic?*
ii) For which equation would it be easiest to answer the question: *Where does the quadratic cross the x-axis?*
iii) For which equation would it be easiest to answer the question: *What is the smallest value the quadratic adopts?*
iv) For which equation would it be easiest to answer the question: *What is the line of symmetry of the quadratic?*
v) For which equation would it be easiest to answer the question: *What is the y-intercept of the quadratic?*

Question: Which of the following problems is <u>not</u> easy to work out in your head?

$$23 \times 37 - 13 \times 37$$
$$27 \cdot 153 + 73 \cdot 153$$
$$3(7) + 87(7)$$
$$105(105) - 95(105)$$
$$17 \times 13 + 13 \times 3$$
$$34 \times 7 + 34 \times 6$$

Question: Compute:
$(7-18)(8-18)(9-18)(10-18)\cdots\cdots(25-18)(26-18)$

Question: A parabola passes through the points $(2,5)$, $(3,-6)$ and $(10,5)$. What is the x-coordinate of its vertex?

Question: Which of the following statements seem they could be true? Which are definitely wrong? (Answer this question without actually computing the products. Not one of them is actually correct! This is an exercise in estimation only.)

$$999 \times 31 = 30999$$
$$12 \times 198 = 1996$$
$$106 \times 213 = 206,816$$
$$9458 \times 9786 = 192837261748$$
$$19990 \times 4 = 76987$$

Question: Quickly … solve:

a) $(x-2)(x-14)(x-22) = 0$

b) $(x+1)^2 = 25$

Question: Find the area of a triangle with side lengths 9 inches, 8 inches and 19 inches.

Question: For each of the following describe an easy way to compute the answer without a calculator. Either describe your method in words or write a line of arithmetic that illustrates your way of proceeding.

a) 82×5

b) $35 \times 35 \times 40$

c) 7×16

d) 198×32

e) $87 \cdot 903 + 13 \cdot 903 + 17$

f) $196 - 37$

g) $817 - 69$

h) 621 divided by 5

i) 15% of 62

j) $\frac{13}{66} \cdot \frac{33}{28} \cdot \frac{7}{13}$

k) $603 \div 97$

l) $813 \div 198$

TWO FINAL PRACTICE PIECES

To practice avoiding hard work, try mulling on these two puzzlers:

What is the last digit of 777^{777} *?*

What are the last two digits of 777^{777} *?*

Think about ways to avoid hard work!

COMMENT: There is always *Wolfram Alpha!*

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.