## Problem Solving Strategy Essay # 7:

# Perseverance is Key

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*Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.*

(From www.maa.org/math-competitions.)

This is the seventh Problem Solving Strategy Essay to illustrate how to do this within curriculum and daily classroom-practice demands.

Ask any scientist or mathematician what it takes to make true, significant progress with a research question, and most often the answer is tenacity, patience, perseverance, deep care and consistency of thought, the confidence to learn from false leads (which means one must follow leads, even the false ones!), and plugging on day after day, week after week, month after month. One does this until a story of some kind emerges, even if that story goes against preconceived notions!

What impression do we give students on the matter of mathematical pursuit? Answers are pre-known (they're in the back of the book); speed is important (quizzes, tests, exams are all timed); and the path of one's learning is linear and to follow a pre-described path. We give the impression that mathematics is "pre-set" enterprise with a goal of mastery of skill. Much of this structure is appropriate and necessary for classroom work – there are skill sets we want students to master – but the national STEM initiative asks for more. We must plant the seeds of thinking needed for the true pursuit of the sciences and mathematics.

In this essay, we work on one particularly scary and complex question. The big message/theme is PERSEVERANCE!

## A single MAA AMC Question for the Classroom:

One way to model tenacity and perseverance for our students is to make intentional use those occasional five- or ten-minute loose moments at the end of a class. Use them for a multi-week conversation!

Start with the MAA AMC question on the next page. Share it and start simply by asking for initial reactions, nothing more. This models the very important, first step to problem solving:

| |
|---|
| **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and reread the question. Have another emotional reaction. |

Later write the question on poster-board and pin it up on the classroom wall. Let students mull on the question themselves over the following days, and pick up class discussion on the topic only when the next task-free moment happens to occur. Really do let this one mathematical investigation extend over weeks. Model the research experience!

Record conversation thoughts and results on the poster too. That is, put the organic process of mathematic thinking on full, joyous display.) At the end, organize those thoughts into one clean, swift, elegant, pride-full presentation of ideas.

## OUR CHALLENGE TODAY

Here it is: **Question 25** from the **2011 MAA AMC 10A** competition. It is the final question, which means the competition designers themselves consider it to be mighty challenging!

> Let $R$ be a square region and $n \geq 4$ an integer. A point $X$ in the interior of $R$ is called $n$-*ray partitional* if there are $n$ rays emanating from $X$ that divide $R$ into $n$ triangles of equal area. How many points are $100$-ray partitional but not $60$-ray partitional?

WHOA!

Here is how a multi-week discussion might evolve. (Don't force this particular sequence of thoughts. Let the give-and-take of conversation with your students dictate your class's flow of ideas, false leads and all.)

## THE FIRST FIVE/TEN-MINUTE CLASSROOM MOMENT:

Let the first experience be one of acknowledging emotions and reactions as per step 1. Truly honor those emotions by writing them on the poster-board. (Their validation really is the key first step to making progress.)

My emotional reaction was "WHOA!"
Other reactions might be:

"Yick!" "Oh heavens."

"I don't have a clue what any of it means."

"Who cares?" "Scary."

"Who invents such things to torture students?"

"Will this be a question on our test?"

"I'm up for it. Bring it on! (But honestly I have no idea what to do!)"

## THE NEXT FEW FIVE/TEN-MINUTE CLASSROOM MOMENTS:

Maybe a week has gone by and nerves have calmed a little. We might be ready to attempt the question.

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STEP 2: Reread the question. Try to make
sense of some of the words in the
question. Perhaps draw a picture.
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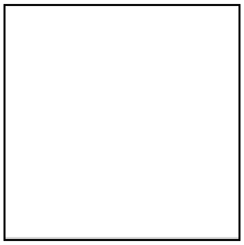
As we saw in ESSAY 4 we saw that drawing a picture is a powerful problem-solving technique. It seems the natural thing to do here.

Let's draw as we go through the question making sure we understand the words and ideas we encounter. (Or not - we are allowed to skip any parts that seem too scary. It is okay to be human!)

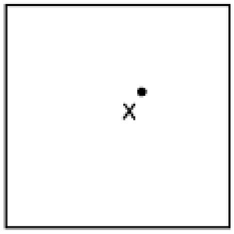
**Let $R$ be a square region and $n \geq 4$ an integer.**

Okay we have a square:



**A point $X$ in the interior of $R$ ...**

and a point inside.



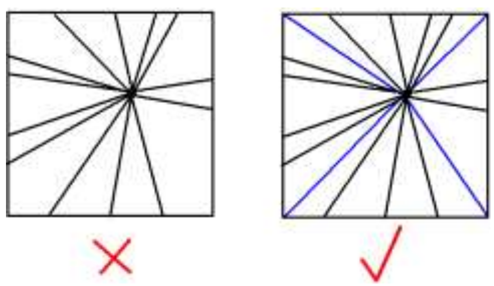
**… is called $n$*-ray partitional* if there are $n$ rays emanating from $X$ that divide $R$ into $n$ triangles of equal area …**

Oh scary! Deep breath.

Parts of this make some kind of sense. We have "rays" – I guess that just means lines – coming from $X$ making triangles.

There is some extra detail about area.

So our job, basically, is to draw lines from $X$ and make triangles.



Notice that we need the four lines that connect $X$ to each corner of the square. That feels important-ish (maybe).

**…that divide $R$ into $n$ triangles of equal area.**

Okay, we want all the triangles we make to have the same area.

**How many points are $100$-ray partitional but not $60$-ray partitional?**

That's too scary for me to think about! Let's ignore it. (Yep. I am human!)

Righteo. So we've gotten to the point that we know that "$n$-partitional" is about $n$ lines (rays) coming from the point making triangles of equal area.

I don't know why, but I feel like asking: *If there are $n$ lines, are there also $n$ triangles?*

## THE NEXT FEW FIVE/TEN MINUTE CLASSROOM MOMENTS:

Continuing to ignore the scary part of the question, let's ask:

*What feels significant about the set-up of the problem?*

We might all answer:

*That the triangles we make have the same area.*

This suggests another problem-solving strategy:

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List all that you know that could be
relevant to the issue at hand.
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What do we know about the areas of triangles? Certainly:

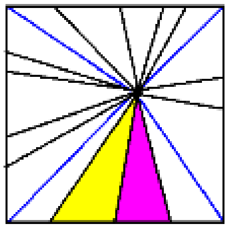
$$area = \frac{1}{2} \times base \times height \cdot$$

(We might also know some more sophisticated formulas: $A = (1/2)ab\sin\theta$ or Heron's formula, $A = \sqrt{s(s-a)(s-b)(x-c)}$, but it feels unlikely we would want to analyse angles in these triangles or their individual perimeters.)

Okay, one piece of information. Not much to play with. Oh well.

*Can we glean anything from this formula for the triangles we have?*

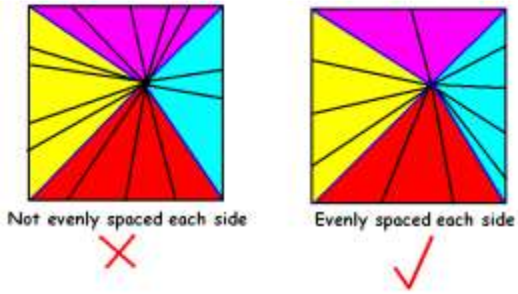
Look at two neighboring triangles. They are meant to have the same area.



<u>Epiphany</u>: *Two triangles with bases on the same side of the square have the same height. To have the same area, they must also have the same base length.*

(Well, it kind of feels like an epiphany.)

In fact, all the triangles on any one side of the square have the same height, and so all must have the same base lengths!




This definitely feels important!

> **Common Core Connection:**
> CCSS-M standard **6-G-1** asks students to find the areas of triangles and **7-G-6** has students analyze the areas of figures composed of triangles. (See the end of the essay for more on this!)

## THE NEXT SEVERAL FIVE/TEN-MINUTE CLASSROOM MOMENTS:

It feels compelling to start naming things.

We are meant to have $n$ triangles. Suppose we have $a$ yellow triangles to the left, $b$ red triangles in the bottom section, $c$ blue triangles to the right, and $d$ pink triangles on top. Then:

$$a+b+c+d=n$$

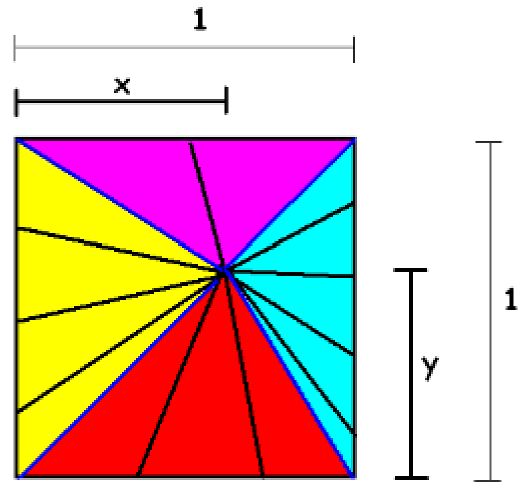
I'd hate to introduce any more variables into the problem (we've already got five!) I suppose we could call the side-length of the square $s$, but would it hurt to assume that we're dealing with a square of side-length $1$ unit?

> **Common Core Connection:**
> Suppose someone was thinking that the side-length was actually 8 inches.
>
> Fine. But let's now declare a new unit of length called a "flooble," with one flooble just happening to be a length of eight inches. In this setting, our square is now indeed 1 unit wide! (And if later we ever want to convert lengths back into inches, all we need do is multiply our answers in floobles by the scale factor of $8$. For example, a length of $3.5$ floobles is a length of $28$ inches.)
>
> There is thus no problem then in assuming that the side length of the square is 1 unit.
>
> This is a sophisticated way of thinking about ratio and proportion: "a change of unit is equivalent to a change of scale." CCSS-M grade six and seven standards **6-RP** and **7-RP** address the topic of scale, ratio and proportions directly, but the idea mentioned here offers a sophisticated way for high-school students to revisit this topic.

The $a$ yellow triangles each have the same base length. Since the square is 1 unit wide, this base length is $\frac{1}{a}$.

These yellow triangles also all have the same height. It seems another variable is inevitable.

Let's call the height of the yellow triangles $x$ (though, in our picture, this height is horizontal!). We might as well give the height of each red triangle a name as well, say $y$.




Thankfully we don't need any more names! The height of each blue triangle is $1-x$ and each pink triangle $1-y$.

*I've forgotten. Why are we naming things?*

Well, we never had a reason other than it felt like the thing to do. We're still just trying to get a feel for things, seeing if anything leads to what seems like a meaningful path.
We can link things back to our epiphany, which was all about areas.

Let's write:

$$area_{yellow\,triangle} = \frac{1}{2}\cdot\frac{1}{a}\cdot x$$

$$area_{red\ triangle} = \frac{1}{2}\cdot\frac{1}{b}\cdot y$$

$$area_{blue\,triangle} = \frac{1}{2}\cdot\frac{1}{c}\cdot(1-x)$$

$$area_{pink\ triangle} = \frac{1}{2}\cdot\frac{1}{d}\cdot(1-y)$$

All these areas are meant to be equal in value. This gives us lots of algebra. (Is that good or bad?)

For example, the first and second areas being equal give $\frac{x}{a}=\frac{y}{b}$ which says:

$$x=\frac{a}{b}y.$$

The second and fourth area formulas being equal will give an equation just for $y$, in terms of $a$, $b$, $c$ and $d$. And from $x=\frac{a}{b}y$ we could then find a formula for $x$. And so on.

Before launching into this, let's pause and ask:

*Do we want to get stuck in a morass of algebra? Would having formulas for $x$ and $y$ in terms of the numbers $a$, $b$, $c$ and $d$ be helpful?*

## THE NEXT SEVERAL FIVE/TEN-MINUTE CLASSROOM MOMENTS:

I am nervous wading through lots of equations with many symbols ($a$, $b$, $c$, $d$, $x$, $y$ and $n$!). But I don't see what else to do.

**When stuck, reread the question.**

I am reminded that we ignored the actual question part of the question!

**How many points are $100$-ray partitional but not $60$-ray partitional?**

This still feels scary.

**When stuck, do something!**

Okay, let's just take one part of this, the "$100$-ray partitional" piece. What would $100$ rays look like? (I am just doing something!)

The obvious things to draw would be $25$ rays in each section of the square, $a=25$, $b=25$, $c=25$, $d=25$, and have a completely symmetrical picture: 25 triangles of each colour.

From

$$area_{yellow\,triangle} = area_{blue\,triangle}$$

we get:

$$\frac{1}{2}\cdot\frac{1}{25}x=\frac{1}{2}\cdot\frac{1}{25}(1-x)$$

or $x=1-x$, and so $x=\frac{1}{2}$.

Also, the red and pink triangles having equal areas gives $y=1-y$, and so $y=\frac{1}{2}$.So for $a=b=c=d=25$, the point $X$ must be half way to the right and half way up. That is, $X$ must be the center point of the square. Hmm.

What if we tried some asymmetrical numbers: $a=10$, $b=20$, $c=30$, $d=40$, perhaps?

The equation:

$$\frac{1}{2}\cdot\frac{1}{10}x=\frac{1}{2}\cdot\frac{1}{30}(1-x)$$

gives $x=\frac{1}{4}$ and the equation:

$$\frac{1}{2}\cdot\frac{1}{20}y=\frac{1}{2}\cdot\frac{1}{40}(1-y)$$

gives $y=\frac{1}{3}$. (Check these!) The point $X$ is one-third over and one-quarter up.

It seems that knowing the numbers $a,b,c,d$ will pin down where the point $X$ has to be.

**How many points are $100$-ray partitional but not $60$-ray partitional?**

There has to be something about the places for $X$ that work for "$100$" but don't work for "$60$."
I don't want to keep trying random examples of numbers that add to $100$ and/or $60$. I think we do need actual formulas for $x$ and $y$.

Let's do the algebra!

## THE NEXT SEVERAL/MANY FIVE/TEN-MINUTE CLASSROOM MOMENTS:

Okay, we have the equations

$$area_{yellow} = \frac{1}{2}\cdot\frac{1}{a}\cdot x \qquad area_{red} = \frac{1}{2}\cdot\frac{1}{b}\cdot y$$

$$area_{blue} = \frac{1}{2}\cdot\frac{1}{c}\cdot(1-x) \qquad area_{pink} = \frac{1}{2}\cdot\frac{1}{d}\cdot(1-y)$$

All four area formulas have the same value.

To get a formula for $x$ we can use $\frac{1}{2}\cdot\frac{1}{a}x = \frac{1}{2}\cdot\frac{1}{c}(1-x)$.

This reads $cx = a - ax$ and so: $x = \dfrac{a}{a+c}$.

From $\frac{1}{2}\cdot\frac{1}{b}y = \frac{1}{2}\cdot\frac{1}{d}(1-y)$ we get: $y = \dfrac{b}{b+d}$.

While we are at it, from $\frac{1}{2}\cdot\frac{1}{a}x = \frac{1}{2}\cdot\frac{1}{b}y$ we get $\frac{x}{y} = \frac{a}{b}$.

Hang on! This is weird!

We just worked out $x$ and $y$ so this reads:

$$\frac{\frac{a}{a+c}}{\frac{b}{b+d}} = \frac{a}{b}$$

and simplifying gives:

$a + c = b + d$. (Check this!)

Let's keep setting two equations equal to each other and see what more we can learn. (This seems a neat thing to try!)

**Exercise:** There are ten pairs of formulas to set equal to each other. (Why ten?). Check them all and verify that they yield the same information we already have. The remaining equations turn out to be redundant. (One doesn't know this until one actually checks!)

In summary we have:

$$x = \frac{a}{a+c}$$

$$y = \frac{b}{b+d}$$

$$a + c = b + d$$

We also shouldn't forget:

$$a + b + c + d = n.$$

[Hang on! Didn't everything work out in our example: $a = 10$, $b = 20$, $c = 30$, $d = 40$? Here $a+c$ fails to equal $b+d$. Do we not have triangles of equal area in this case after all? ]

**Common Core Connection:**
It is clear we are connecting with the CCSS-M high-school standards **A-CED**, creating equations, and **A-REI**, reasoning with equations and inequalities.

## THE NEXT SEVERAL/MANY FIVE/TEN-MINUTE CLASSROOM MOMENTS:

So what?

**`When stuck, reread the question.`**

Okay …

**How many points are $100$-ray partitional but not $60$-ray partitional?**

If $X$ is $100$-partitional, what do we know?

Well …

$$a + b + c + d = 100$$

and $X$ lies at the position with:

$$x = \frac{a}{a+c},\; y = \frac{b}{b+d}.$$

We also know $a + c = b + d$.

If $X$ is $60$-partitional, what do we know?

Well …

$$a+b+c+d=60$$

and $X$ lies at the position with:

$$x=\frac{a}{a+c},\ y=\frac{b}{b+d}$$

We also know $a+c=b+d$.

The question wants us to count how many $X$ s work for $a+b+c+d=100$ but not for $a+b+c+d=60$. Yeesh!

I am stuck.

## THE NEXT SEVERAL/MANY FIVE/TEN-MINUTE CLASSROOM MOMENTS:

The location of each possible point $X$ is given by fractions:

$$x=\frac{a}{a+c},\ y=\frac{b}{b+d}$$

There infinitely many fractions. So "infinity" many points work for $100$, and "infinity" many points work for $60$, so the answer is "infinity minus infinity"? Yeesh!

I am truly stuck.

## THE NEXT NUMBER OF FIVE/TEN MINUTE CLASSROOM MOMENTS:

*Have we used everything we know?*

We've got:

$$x=\frac{a}{a+c},\ y=\frac{b}{b+d}$$

We have: $a+b+c+d=100$ (or $60$ if we are looking at the other case).

Have we used $a+c=b+d$?

Oooh!

$$a+b+c+d=100$$
$$(a+c)+(b+d)=100$$
$$(a+c)+(a+c)=100$$

We see $a+c=b+d=50$! (and this is $30$ in the $60$-ray case). Whoa! This gives:

If $X$ is $100$-ray partitional, then:

$$x=\frac{a}{50},\ y=\frac{c}{50}.$$

If $X$ is $60$-ray partitional, then:

$$x=\frac{a}{30},\ y=\frac{c}{30}.$$

Only certain fractions work! We're on to something!

*How many points work for the $100$-ray case?*

**"A point $X$ in the interior of $R$ is called …"**

The point $X$ must be <u>inside</u> the square. So the fraction $x=\frac{a}{50}$ must be strictly between $0$ and $1$. This means $a$ can have any of the values $1,2,...,49$. Similarly, for $y=\frac{b}{50}$, the number $b$ can be any of the numbers $1,2,..,49$.

There are $49^2=2401$ possible locations for the point $X$ in the $100$-ray case.

**Aside:** To be logically solid we should establish that each of these 2401 possible locations is actually a valid option! For example, if we were told that $X$ lay at the position $x=\frac{22}{50},\ y=\frac{5}{50}$, can we actually draw $100$ triangles of equal area? The answer is yes, because from these fractions we see $a=22, b=28, c=5, d=45$ and we can verify the arithmetic to show all triangles are equal in area. (And one should generalise this argument as an abstract piece of algebra.)

*How many points work for the $60$-ray case?*

Here $x=\frac{a}{30}$ and $y=\frac{c}{30}$. There are $29^2=841$ possible locations for $X$ in the $60$-ray case.

So are we essentially done?

**How many points are $100$-ray partitional but not $60$-ray partitional?**

Oh dear. This is another twist! We want to know which of the $2401$ points that work in the $100$-ray case don't also work for the $60$-point case. How do we wrap our minds around that?

## THE NEXT NUMBER OF FIVE/TEN-MINUTE CLASSROOM MOMENTS:

Which points in the square satisfying

$x = \frac{a}{50}, y = \frac{c}{50}$ $(1 \le a, c \le 49)$ are not also

satisfying

$x = \frac{a}{30}, y = \frac{c}{30}$ $(1 \le a, c \le 29)$?

I am worried about using the same symbols $a$ and $c$ in two different contexts. Let's rewrite this:

How many times do we have $x = \frac{a}{50}, y = \frac{c}{50}$ with $1 \le a, c \le 49$, but not $x = \frac{e}{30}, y = \frac{f}{30}$ for $1 \le e, f \le 29$?

I am also worried about the bad wording of this question! I just can't wrap my mind around it!

**When stuck (and you are beyond rereading the question) take a step back: *What is the general issue at hand?***

We have to count how many times one thing is happening while a second thing is not happening. What's the best way to do that?

Answer: Count the number of times the first thing is happening and subtract from that the number of times both happen!

Thing one: There are 2401 $100$-ray points.
Thing two: There are 841 $60$-ray points.

How many are both?

## THE NEXT NUMBER OF FIVE/TEN-MINUTE CLASSROOM MOMENTS:

We want the number of points $X$ with an $x$-value that can be written both as $\frac{a}{50}$ for some $1 \le a \le 49$, and as $\frac{e}{30}$ for some $1 \le e \le 29$. (And ditto for the $y$-coordinates.) That is, we need $\frac{a}{50} = \frac{e}{30}$ or:

$$a = \frac{5}{3}e.$$

Of the values $1, 2, 3, \ldots, 29$ for $e$ only $3, 6, 9, \ldots, 27$ are multiples of three. And for each of these $\frac{5}{3}e$ is still less than or equal to $49$. So $\frac{a}{50} = \frac{e}{30}$ happens nine times. (With $e = 3, 6, 9, \ldots, 27$.)

For the $y$-values we want $\frac{c}{50} = \frac{f}{30}$, and this happens nine times as well.

Thus there are $9 \times 9 = 81$ points $X$ for which $x$ and $y$ each correspond to the coordinates of a $100$-ray point and a $60$-ray point.

We can finally, after all these months, answer the question!

**There are $2401 - 81 = 2320$ points inside a square that are $100$-ray partitional, but not $60$-ray partitional!**

### Common Core Practice Standards:

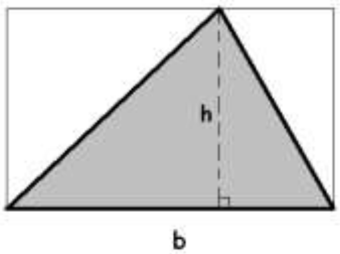
Oh boy! We have certainly hit the mark on a good number of practice standards!

***MP1:*** *Make sense of problems and persevere in solving them.*
***MP2:*** *Reason abstractly and quantitatively.*
***MP3:*** *Construct viable arguments and critique the reasoning of others.*
***MP4:*** *Model with mathematics.*
***MP5:*** *Use appropriate tools strategically.*
***MP7:*** *Look for and make use of structure.*

# ON THE AREA OF A TRIANGLE:

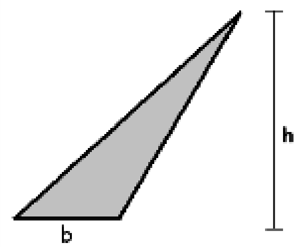
The formula $A = \frac{1}{2} base \times height$ for the area of a triangle is discussed in the middle-school curriculum and is often dubbed as "known" and "obvious" by high-school students. (After all, they have been using it for years!)

The formula is easily seen as true if the altitude of triangle (line of height) lies in the interior of the triangle:




Area = half the area of the rectangle that encloses the triangle

The trouble is that it is not at all obvious the formula holds for <u>all</u> types of triangles.




*Consider the obtuse triangle shown. If I insist on regarding the side labeled "b" as the base of the triangle, does the formula $area = \frac{1}{2} bh$ hold?*

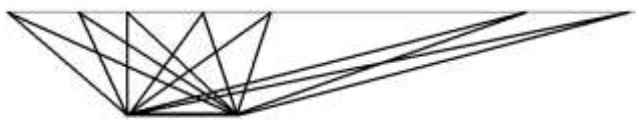
A common student response to this is:

"Just don't use the side labeled $b$ as the base! Use the longest side of the triangle instead and you'll be okay. (The height of the triangle is then inside the triangle.)"

Fair enough!

But of course "$area = \frac{1}{2} bh$" is valid in all contexts. (See the video http://www.jamestanton.com/?p=1271 if you are interested in the proof.) But it is hard to believe!

*On an intuitive level, do you personally feel that each and every triangle in the picture below has the same area? Truly? Even one that goes out forty-thousand miles to the right and the triangle is nothing more than the merest of slivers? (Each triangle has the same base and the same height, and so the formula says the area is constant.)*



**A QUESTION:** Abigail and Beatrice are standing 20 feet apart. Draw a diagram to show <u>all</u> the possible places Charlene could stand so that the triangle formed by the three girls has area 50 square feet. (Make sure you don't have just half the answers!)

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