Problem Solving Strategy Essay # 4:

Draw a Picture

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Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

This is the fourth essay in a series to give credence to this claim on the MAA website, www.maa.org/math-competitions. For over six decades, AMC has been creating and sharing marvelous stand-alone mathematical tidbits. Take them out of their competition coverings and see opportunity after opportunity to engage in great conversation with your students. Everyone can revel in the true creative mathematical experience!

I personally believe that the ultimate goal of the mathematics curriculum is to teach self-reliant thinking, critical questioning and the confidence to synthesize ideas and to re-evaluate them. Content, of course, is itself important, but content linked to thinking is the key. Our complex society is demanding of the next generation not only mastery of quantitative skills, but also the confidence to ask new questions, explore, wonder, flail, innovate and succeed. Welcome to these essays!

OUR CHALLENGE TODAY

Our tidbit today is **query 22** from the **2004 MAA AMC 8** competition.

At a party, there are only single women and married men with their wives. The probability that a randomly selected woman is single is $\frac{2}{5}$. What fraction of the people in the room are married men?

I love this question as it provides an opportunity for me to share with your class a problem solving technique, not yet discussed in these essays, that represents how the minds of many mathematicians work. This approach to thinking, I think, is severely underplayed in the standard curriculum.

Let's start with the first step to analyzing any problem:

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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There is no need for speed – and there should never be when truly engaging in mathematics - so do encourage your students, as part of this class discussion, to take some time to steady their minds and relax into the problem at hand.

Having reread the question – slowly - now work to …

STEP 2: Understand the question. Understand the different components of the question.

Middle school students will likely have the greatest trouble interpreting the probability statement in the question. They will probably latch on to it right away and lose sight of "big picture" of the question. This suggests …

To help understand the question, simply ask: *What is the question about?*

Explicitly stating the obvious is a surprisingly potent first step.

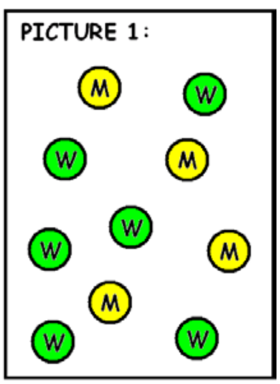
This question is about men and women who are married and not married.

Great! And now comes my favorite approach to thinking mathematically:

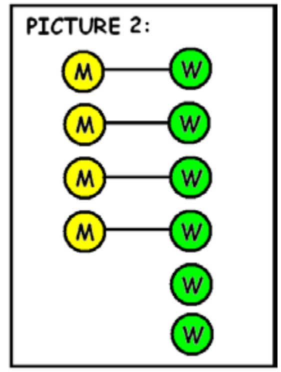
Can we draw a picture to make sense of the problem?

We certainly can! Start with a basic picture and refine it.

Here is a diagram of men and women. It's at least a start.



In <u>rereading the question</u> we recall that each man attends the party with his wife. This leads to a more refined picture:



Have we captured all elements of the question? No. We haven't yet contended with statement:

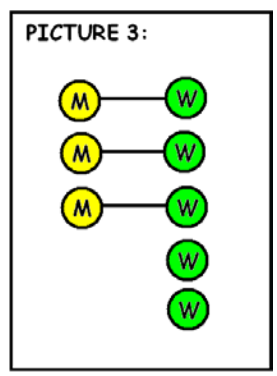
The probability that a randomly selected woman is single is $\frac{2}{5}$.

Pictures have the powerful ability to clarify and crystallize ideas. No doubt our second picture is not right but it can help us make sense of what it is we don't understand.

To understand a confusing part of the question given to you, turn it into a question about the picture you have.

Let's look at picture 2 and ask: *What is the probability that a randomly selected woman is single in picture 2?*

Even if I am not sure what "probability" really means, I can see that two women out of six are single in our picture 2. The question wants "$2/5$," which I can guess to mean two out of five. This suggests refining the picture.



Now … what was the question asking? (Hey! I am <u>rereading the question</u> again!)

What fraction of the people in the room are married men?

Picture 3 now makes it clear that of the eight people in the room, three are (married) men. The answer to the problem is $\frac{3}{8}$. GREAT!

NOTICE: Solving this problem relied on:

- Constantly rereading the question and
- Using one's wits and educated guesses to make sense of unfamiliar terms.

Many educators feel the need to explain all terms and define all concepts before letting students examine challenges and play with ideas. Unfortunately, problems and challenges in science, business, and life don't usually come with the safety of pre-established preamble! Let your students flail with jargon, make good guesses as to what unfamiliar terms might mean, and rely on their abilities and wits to "nut their way" through challenges.

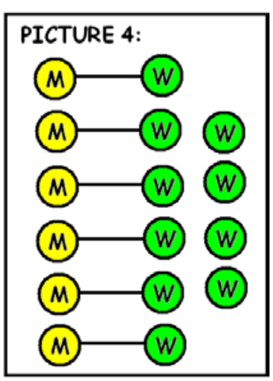
OOH! I AM WORRIED!

Most students would be satisfied with the answer given here and would stop at this point. But, is it the correct answer?

I bet you are wondering about an issue I've let slide.

Many educators also feel the need to correct unspoken assumptions before they even come into question. The desire to direct a path to an answer that is unerringly straight and logically solid right off the bat is remarkably strong. But, alas, the human mind does not work in unerringly straight ways! In life, one often reaches an answer, and then doubts the answer! The same could (should?) be true in mathematics as well.

Is there reason to doubt our answer of $3/8$? As the teacher you might want to suggest a reason for doubt by asking: *Is picture 3 the only picture that fits the question?* Depending on how the class reacts, you might wish to offer picture number 4!



Here four out of ten women are single.

Some questions to ask your students might include …

Is the probability that a randomly selected woman is single again $2/5$ *in this new picture? What do you think "probability' means?*

Is the fraction of married men in picture 4 again $3/8$*?*

Are pictures 3 and 4 the only pictures that fit the question? Draw another picture that works. Is the final answer for the fraction of married men always $3/8$*?*

You and your students are now engaged in a wonderful discussion on proportion, fraction and probability!

Question: In picture one I happened to draw more women than men. How might your students respond if they first draw a picture with more men than women?

COMMON CORE STANDARDS and PRACTICES:

This problem is connected to standards:

6.RP: Ratios and Proportions
7.RP: Ratios and Proportional Relationships
7.SP: Investigate Chance Processes and Develop, Use and Evaluate Probability Models.

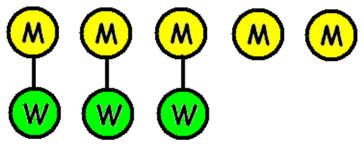
We are also right on the mark with the following practice standards:

MP1: Make sense of problems and persevere in solving them.
MP2: Reason abstractly and quantitatively.
MP3: Construct viable arguments and critique the reasoning of others.
MP4: Model with Mathematics
MP7: Look for and make use of structure.

Comment: A picture is a model! That fits standard 4!

SOME EMPOWERING FUN FOR YOUR STUDENTS!

Have students draw diagrams of married relations and devise their own interesting questions for their classmates. For example, this picture:



makes me think of the questions …

At a party, three-quarters of the attendees are married and two people were not. If each married person came with his or her spouse, what is the smallest possible number of people at the party?

OR

At a party, 3/5 of the men attending were married. Each came with his spouse and no other women attended. What proportion of folk at the party were women?

OR …

Better yet … Have a student draw a single diagram like this on the board and see how many different interesting questions the class can devise from it.

How empowering for your students to see questions galore generated from the same answer! Now we're owning and creating mathematics!

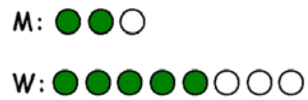
SOME FUN FOR YOU!

This MAA AMC problem reminds me of one of my favourite puzzlers. I'll let you be the judge whether or not to offer it to your students. But do take to the time to nut on it yourself.

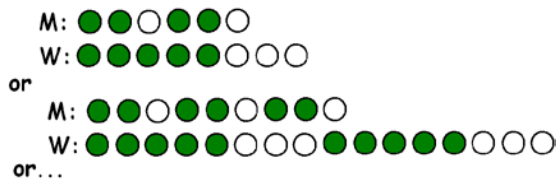
In a particular town, two-thirds of the adult men are married to five-eighths of the adult women. What fraction of the entire adult population is married? (Assume each woman is married to one man, and vice versa!)

A visual answer:

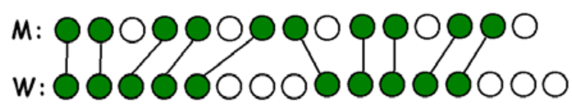
Represent the fraction of married men and married women each pictorially:



Actually any multiple of these diagrams depicts the same proportion of married men and women.



But since each woman is matched with one man, and vice versa, a diagram that shows this matching must have an equal number of green dots on each line. This suggests we should think "$10/15$ of men and $10/16$ of women."



It is now clear, for the entire adult population, 20 out of every 31 people are married. (How often does a fraction question call for common numerators?!)

EXTRA EXTRA: MORE FOR YOU!

Can you see that the following two problems are actually the same problem in two different guises? (The answer to each is 45.) The power of a picture again, even in algebra!

Problem: Guise 1: Here are ten dots in a row. We wish to colour two of them red. In how many ways can this be done?

Problem: Guise 2: How many solutions are there to the equation $10 = x + y + z$ with each of x, y and z a whole number greater than or equal to zero?

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.