Curriculum Burst 143: Arithmetic Algebra

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> The first four terms of an arithmetic sequence are $x+y$, $x-y$, xy, and x/y. What is the [value of the] fifth term?

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Algebra: Arithmetic sequences; Quadratic equations

COMMON CORE STATE STANDARDS

A-REI.4b Solve quadratic equations by inspection (e.g., for x^2=49), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a±bi for real numbers a and b.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING!**

SOURCE: This is question # 24 from the 2003 MAA AMC 10B Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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As I read this question, my eyes skip over the algebraic expressions. But I do notice we are talking about numbers in an arithmetic sequence. And I recall that sequence of numbers is arithmetic if they change by a constant amount.

What are our numbers in the sequence?

$x+y$, $x-y$, xy, and x/y.

To go from $x+y$ to $x-y$ we subtract $2y$. So to go from $x-y$ to xy we must also subtract $2y$. (The difference must be constant.) This gives: $x-y-2y=xy$. That is, $x-3y=xy$.

To go from xy to x/y we must again subtract $2y$, so:

$xy-2y=\frac{x}{y}$. Let's rewrite this as $xy^2-2y^2=x$.

So we have the two equations:

$$x-3y=xy$$
$$xy^2-2y^2=x$$

I suppose we can try to solve to x and y. But is that what the question wants?

What is the fifth term?

Okay. We want the value the next term of the sequence: $\frac{x}{y}-2y$. If we know the values of x and y, then we know the value of this fifth term. Let's find x and y.

The two equations we have look scary. Let's work with the first one since it involves no squared terms:

$$x=xy+3y$$
$$x=y(x+3)$$

so $y=\frac{x}{x+3}$. We could substitute this into the second equation, but that would be horrid! What if I solved for x instead?

$$x-xy=3y$$
$$x(1-y)=3y$$

giving $x=\frac{3y}{1-y}$. Putting this into the second equation produces: $\frac{3y^3}{1-y}-2y^2=\frac{3y}{1-y}$. Multiplying through by $1-y$ yields: $3y^3-2y^2+2y^3=3y$. That is:

$$5y^3-2y^2-3y=0.$$

Dividing through by y gives $5y^2-2y-3=0$. (Ooh! What if $y=0$? That's a possible solution!)

Looking at $5y^2-2y-3=0$, we see that $y=1$ works. This means we can write:

$$5y^2-2y-3=0$$
$$(y-1)(\mathit{something})=0.$$
$$(y-1)(5y+3)=0$$

So $y=-\frac{3}{5}$ is another possible solution.

We have $y=0$ or $y=1$ or $y=-3/5$. Since $x=\frac{3y}{1-y}$, we can't have $y=1$. So either $y=0$ and $x=0$ or $y=-3/5$ and $x=\frac{(-9/5)}{1+3/5}=\frac{-9}{5+3}=-\frac{9}{8}$. The fifth term is $\frac{x}{y}-2y$, so $y\neq 0$. The fifth term must thus be:

$$\frac{(-9/8)}{(-3/5)}-2\left(-\frac{3}{5}\right)=\frac{45}{24}+\frac{6}{5}=\frac{369}{120}=\frac{123}{40}.$$

Extension: Are there real numbers x and y so that $x-y$, $x+y$, xy, and x/y form a geometric progression?

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