Curriculum Burst 141: Multi-Base Three-Digit Numbers

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A base-10 three-digit number n is selected at random. Which of the following is closest to the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals?

(A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6 (E) 0.7

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Polynomials: Parallels to base arithmetic.

COMMON CORE STATE STANDARDS

A-APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING!**

SOURCE: This is question # 20 from the 2003 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I definitely have a feeling of alarm with this question!

A base-10 three-digit number n is selected at random.

Okay. I can handle that part. We're choosing one of the numbers $100, 101, \ldots, 999$.

… the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals.

Gulp!

Alright. Deep breath. What do I know about base nine?

Well, it's like base ten except that it uses powers of nine rather than powers of ten. So, the "abc" in base nine means the number:

$$a \times 9^2 + b \times 9 + c.$$

Okay. So the smallest base-nine three-digit number is "100," which is really $1 \times 81 + 0 \times 9 + 0 = 81$. The largest base-nine three-digit number is "999." Ooh. No! You don't use the digit 9 in base nine (they get confused with the powers of nine). You only use the digits $0, 1, 2, \ldots, 8$. The largest base-nine three-digit number is "888," which is really $8 \times 81 + 8 \times 9 + 8 = 728$.

Working in base eleven is similar. It is all about the powers of eleven.

The smallest base-eleven three-digit number is "100," which is really $1 \times 11^2 + 0 \times 11 + 0 = 121$, and the largest is …? Hmm. What digits do we use in base eleven?

In any base you always use the digits 0, 1, 2, up to one less than the base number. So in base eleven we use the digits $0, 1, 2, \ldots, 9$ and one more thing! I'll call that thing "e" for eleven! The largest base-eleven three-digit number is "eee," which is really:

$$e \times 11^2 + e \times 11 + e$$

Oh dear! What is e? Hmm.

Well it's the digit after 9, which in ordinary arithmetic corresponds to the number 10. So when working with ordinary arithmetic we should write 10 for e.

Okay, the largest three-digit number in base eleven eee is really the number:

$$10 \times 11^2 + 10 \times 11 + 10 = 1330.$$

Now I've forgotten the question!

A base-10 three-digit number n is selected at random. Which of the following is closest to the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals?

Okay, we have one of the numbers from:

$$100, 101, \ldots, 999.$$

To have three digits in base nine, it must be between 81 and 728. To have three digits in base eleven it must be between 121 and 1330. So we need the number to be in the range:

$$121, 122, \ldots, 728.$$

The probability of this being the case is:

$$\frac{728 - 120}{900} = \frac{608}{900} \approx \frac{2}{3} \approx 0.7.$$

Extension: Is the largest three-digit number in base k always $k^3 - 1$?

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