Curriculum Inspirations

Inspiring students with rich content from the MAA American Mathematics Competitions

MAA

Curriculum Burst 137: Changing Mean

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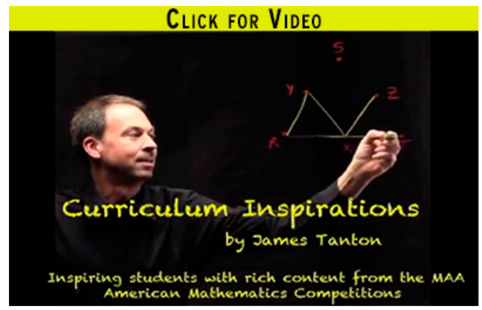
> When 15 is appended to a list of integers, the mean in increased by 2. When 1 is appended to the enlarged list, the mean of the enlarged list is decreased by 1. How many integers were in the original list?



QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Algebra: Systems of equations

COMMON CORE STATE STANDARDS

A-REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1: **ENGAGE IN SUCCESSFUL FLAILING!**

SOURCE: This is question # 25 from the 2002 MAA AMC 10B Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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Okay. I need to make sure I understand this question.

We have some list of integers: a_1, a_2, …, a_k, say k of them.

They have some mean.

We add 15 to the list and the mean goes up by 2.

Alright, let's call the original mean M. As equations in mathematics we have:

$$\frac{a_1+a_2+\cdots+a_k}{k}=M$$
$$\frac{a_1+a_2+\cdots+a_k+15}{k+1}=M+2$$

Next we add 1 to the list and the mean goes down by one:

$$\frac{a_1+a_2+\cdots+a_k+15+1}{k+2}=M+1.$$

And we want the number of integers in the original list. That's my k.

Right now these equations look scary. But let's multiply each through by the denominator that appears. That should make them look friendlier:

$$a_1+a_2+\cdots+a_k=Mk$$
$$a_1+a_2+\cdots+a_k+15=(M+2)(k+1)$$
$$a_1+a_2+\cdots+a_k+16=(M+1)(k+2)$$

The first equation says that the sum equals Mk. That makes the second two equations look more manageable:

$$Mk+15=(M+2)(k+1)$$
$$Mk+16=(M+1)(k+2)$$

It feels compelling to expand each right side:

$$Mk+15=Mk+M+2k+2$$
$$Mk+16=Mk+2M+k+2$$

Rewriting makes these friendlier still

$$M+2k=13$$
$$2M+k=14$$

Since I just want k, let's double the first equation and subtract the second from it:

$$2M+4k=26$$
$$2M+k=14$$

$$\Rightarrow 0+3k=12$$

So $k=4$. Done! Cool!

Extension: Consider a list of 100 integers. Let M_{99} be the mean of the first 99 numbers in the list, M_{100} the mean of all 100 numbers, and let a_{100} be the 100th number.

If two of the three numbers M_{99}, M_{100}, and a_{100} are equal, must the third number equal that common value as well?

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