Curriculum Burst 135: Some Algebra

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Let a, b, and c be real numbers such that $a-7b+8c=4$ and $8a+4b-c=7$.
What is $a^2-b^2+c^2$?

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Algebra: Simultaneous equations

COMMON CORE STATE STANDARDS

A-REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING!**

SOURCE: This is question # 20 from the 2002 MAA AMC 10B Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question looks scary! We have two equations in three variables:

$$a-7b+8c=4$$
$$8a+4b-c=7$$

and we're being asked something about squares that don't even appear in the equations! How am I meant to find the value of $a^2-b^2+c^2$?

I suppose I could get squares into the equations by, well, squaring the equations!

$$(a-7b+8c)^2=16$$
$$(8a+4b-c)^2=49$$

Let me be careful expanding these out.

	a	-7b	8c
a	a^2	-7ab	8ac
-7b	-7ab	$49b^2$	-56bc
8c	8ac	-56bc	$64c^2$

The first equation gives:

$$a^2+49b^2+64c^2-14ab+16ac-112bc=16.$$

In the same way, the second equation gives:

$$64a^2+16b^2+c^2+64ab-16ac-8bc=49\ .$$

Is this helpful at all?

What if I add them?

$$65a^2+65b^2+65c^2+50ab-120bc=65.$$

This doesn't seem at all helpful!

My problem is that I have an expression with $a^2+b^2+c^2$ in it, but I want $a^2-b^2+c^2$. I want the "b" part to be different.

Maybe I should try making the b parts different from the outset? Let's pull the bs away from the as and cs.

$$a+8c=4+7b$$
$$8a-c=7-4b$$

Hmm. I can't help but notice the repeats of 4 and 7 on the right, and the repeats of 8 and 1 (invisibly) on the left. I bet there is something to that.

Let's square everything again.

$$a^2+64c^2+16ac=16+49b^2+56b$$
$$64a^2+c^2-16ac=49+16b^2-56b$$

Ooh! Adding is now nice!

$$65a^2+65c^2=65+65b^2$$

So $a^2+c^2=1+b^2$ giving $a^2-b^2+c^2=1$!

Wow!

Extension: This question made use of the fact that if you square $Mx+Ny$ and $Nx-My$ and sum the squares the cross terms cancel. Can you devise you own puzzle of the type in this question that asks readers for the value of a sum of the form $a^2-b^2+c^2-d^2$ given two linear equations in a, b, c, and d to begin with?

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