Curriculum Burst 130: Counting Multiples

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> How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

QUICK STATS:

MAA AMC GRADE LEVEL
This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS
Algebra: linear inequalities. Number sense.

COMMON CORE STATE STANDARDS
A-REI.3   Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

MATHEMATICAL PRACTICE STANDARDS
MP1   Make sense of problems and persevere in solving them.
MP2   Reason abstractly and quantitatively.
MP3   Construct viable arguments and critique the reasoning of others.
MP7   Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1:   **ENGAGE IN SUCCESSFUL FLAILING.**

SOURCE: This is question # 25 from the 2001 MAA AMC 10 Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

> **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I have the feeling that this question is deceptively hard!

Just to get a feel for it, let me just note that every third integer is a multiple of 3, so about a third of the integers from 1 to 2001 are multiples of 3. About a quarter are a multiple of 4. Does that mean then that about $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ of them are a multiple of three or four? Over half of them? Is that right? Hmm.

Let's be more precise:

Since 2001 is a multiple of 3 (it is 667×3), the multiples of 3 up to 2001 are:
$$3 = 3 \times 1, \ 6 = 3 \times 2, \ …, 2001 = 3 \times 667.$$
There are 667 of them.

Since $2000 = 4 \times 500$ is a multiple of 4, there are 500 multiples of four up to 2001.

Since $2000 = 5 \times 400$, there are 400 multiples of 5 - but do I need to know that? I only want to take away the multiples of 5 that lie among the multiples of 3 or 4.

Ooh. This means I want to take away the multiples of 15 (those multiples of 3 divisible by 5) and take away the multiples of 20 (those multiples of 4 divisible by 5).

I am starting to get lost!

It looks like I need to think about the multiples of 3, 4, 15, and 20. Anything else?

I am going to spell it all out:

Multiples of 3: $3, 6, 9, 12, 15, 18, 21, 24, …, 2001$; 667 of them.

Multiples of 4: $4, 8, 12, 16, 20, 24, …, 2000$; 500 of them.

Multiples of 3 or 4:
$3, 4, 6, 8, 9, 12, 15, 16, 18, 20, 21, 24, …, 2001$

Aah! This is less than $667 + 500$ numbers because numbers like 12 and 24 appeared on both lists. Okay, we need to count the multiples of 12 since every multiple of 12 appears on both lists and we need to make sure we don't count them twice.

Multiples of 12: $12, 24, 36, …, 166 \times 12 = 1992$. There are 166 of them.

So there are $667 + 500 - 166 = 1001$ numbers that are a multiple of 3 or 4.

Now, among these, we don't want the numbers that are a multiple of 15 or a multiple of 20.

Multiples of 15: $15, 30, 45, …, 1995 = 133 \times 15$. There are 133 of them.

Multiples of 20: $20, 40, …, 2000 = 100 \times 20$. There are 100 of them.

Now I am worried about double counting. Are there any multiples of 3 or 4 that are both a multiple of 15 and 20?

Any number that is a multiple of 60 is counted twice!

Multiples of 60: $60, 120, …, 33 \times 60 = 1980$. There are 33 of them.

So there are $133 + 100 - 33 = 200$ multiples of 3 or 4 we need to discard. This means that there are
$$1001 - 200 = 801$$
numbers of the type we seek. Whoa!

> **Extension:** Find a general formula for the number of multiples of an integer k among all the integers from 1 up to N.

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