Curriculum Burst 128: Arranging Mean, Median, and Mode

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> When the mean, median, and mode of the list
>
> $$10, 2, 5, 2, 4, 2, x$$
>
> are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grade levels.

MATHEMATICAL TOPICS

Statistics: Central measures. Algebra: Arithmetic sequences

COMMON CORE STATE STANDARDS

S-ID.A Summarize, represent, and interpret data on a single count or measurement variable.

F-BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING!**

SOURCE: This is question #23 from the 2000 MAA AMC 10 Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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I feel like my first job in this question is to recall what mean, median, and mode each mean.

mean = the average of all the numbers.

In this question, the average is $\frac{25+x}{7}$.

mode = most frequent data value.

We have a mode of 2.

Median = middle value (after the data values have been placed in numerical order).

We don't know the value of x, but we do have the list:

$2, 2, 2, 4, 5, 10$

with x still to be included.

If $x \le 2$, then the median is 2.
If $2 < x < 4$, then the median is x.
If $4 \le x$, then the median is 4.

Okay. All that feels helpful. What was the question?

When the mean, median, and mode … are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x*?*

We have the numbers $\frac{25+x}{7}$, 2, and either 2, x, or 4.
These three numbers are meant to make a non-constant arithmetic progression. What does that mean? Hmm.
I guess we want the three numbers to be of the form: $a, a+d, a+2d$.

The three numbers are meant to be different, and so we can't have a median of 2. This means that the case $x \le 2$ is out. So either $2 < x < 4$ or $4 \le x$.

<u>Case</u> $2 < x < 4$: We have the numbers $\frac{25+x}{7}$, 2, and x.

Since x is larger than 2, $(25+x)/7$ is too.

The arithmetic progression is either: $2, x, \frac{25+x}{7}$ or $2, \frac{25+x}{7}, x$. Hmm.

In the first situation we should have $x = 2+d$ and $\frac{25+x}{7} = 2+2d$. These give $\frac{25+2+d}{7} = 2+2d$, which yields $d = 1$, giving $x = 3$.

In the second situation we should have $\frac{25+x}{7} = 2+d$ and $x = 2+2d$. These give $d = \frac{13}{5}$ and a value for x out of the range we are considering.

<u>Case</u> $4 \le x$: We have the numbers $\frac{25+x}{7}$, 2, and 4.

Since $x \ge 4$, $(25+x)/7 > 4$, and we must have the numerical order $2, 4, \frac{25+x}{7}$. In which case, $\frac{25+x}{7} = 6$ and $x = 17$.

There are two possible values of x, 3 and 17, and their sum is 20. Done!

Extension: Find six data values that have a mean of 10, a mode of 10, and a median of 1000. How about six data values with a mean of 10, a mode of 1000, and a median of 10? Or six with a mean of 1000, a mode of 10, and a median of 10?

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