Curriculum Burst 119: Circles in Squares

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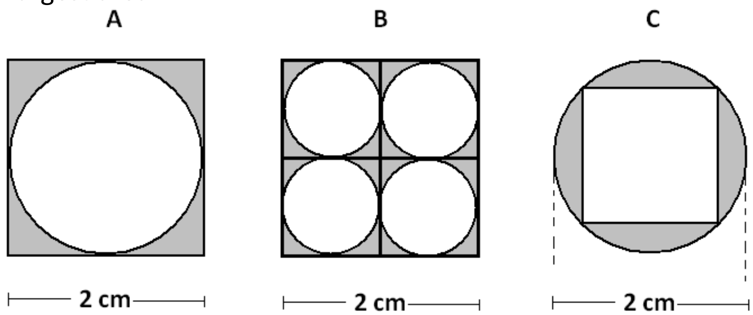
The following figures are composed of squares and circles. Which figure has a shaded region with largest area?



QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the middle-school grade levels.

MATHEMATICAL TOPICS

Geometry: Circle areas, Pythagorean theorem.

COMMON CORE STATE STANDARDS

7.G.4 Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 7: **PERSEVERENCE IS KEY**

SOURCE: This is question # 22 from the 2003 MAA AMC 8 Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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Hmmm. This question feels straightforward. Can I just work out each of the three areas and see which is the largest in value? Why not?

In figure A we have a circle of radius 1 sitting inside a square of side-length 2. The area of the shaded region is thus $2\times 2-\pi\cdot 1^2=4-\pi$ square centimeters. (I'll leave the number like that for now. If I need actual value later, I'll work it out then!)

In figure B we have four copies of the same figure: a circle of radius $\frac{1}{2}$ sitting in side a square of side-length 1. The shaded area is thus:

$$4\times\left(1\times 1-\pi\cdot\left(\frac{1}{2}\right)^2\right)=4\left(1-\frac{\pi}{4}\right)=4-\pi$$

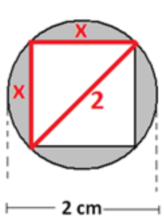
square centimeters. Ooh! It's the same. Weird!

The question is asking for the figure with the largest shaded area. Since the first two choices have the same area, the answer to this question must be C !!

But to check, let's work out the shaded area in figure C. We have a square inside a circle of radius 1. What is the side-length of that square?

I do see that the diagonal of the square is the same as diameter of the circle.

C



diameter = 2

And I see a right triangle! Let's call the side-length of the square x. Then, by the Pythagorean Theorem, we have:

$$x^2+x^2=1^2$$
$$2x^2=1$$
$$x^2=\frac{1}{2}$$
$$x=\sqrt{\frac{1}{2}}$$

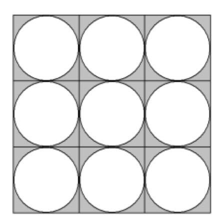
That's a nasty number! But it is the area of the square we want, which is $x^2=\frac{1}{2}$, and that is not nasty. The shaded part of figure C is thus:

$$\pi\cdot 1^2-\frac{1}{2}=\pi-\frac{1}{2}\ .$$

Hmm. Is $\pi-\frac{1}{2}$ larger than $4-\pi$?

Now π is a little larger than three, so $4-\pi$ is smaller than one and $\pi-\frac{1}{2}$ is certainly larger than one. Yep $\pi-\frac{1}{2}$ is bigger than $4-\pi$.

Extension: That the areas of the shaded parts of figures A and B are the same is curious! If the figure below is also 2 cm wide, is the area of the shaded region again $4-\pi$ sq. cm?



What about a figure like this composed of 16 small circles in 16 small squares? 25 ? 36 ? 256 ?

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