Curriculum Burst 112: Remainders of Two

By Dr. James Tanton, MAA Mathematician in Residence

> A whole number larger than 2 leaves a remainder of 2 when divided by each of the numbers 3, 4, 5 and 6. The smallest such number lies between which two numbers?
>
> (A) 40 and 49 (B) 60 and 79 (C) 100 and 129 (D) 210 and 249 (E) 320 and 369

QUICK STATS:

MAA AMC GRADE LEVEL
This question is appropriate for the middle-school grade levels.

MATHEMATICAL TOPICS
Number Sense: Least Common Factor

COMMON CORE STATE STANDARDS
6.NS.B4 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor.

MATHEMATICAL PRACTICE STANDARDS
MP1   Make sense of problems and persevere in solving them.
MP2   Reason abstractly and quantitatively.
MP3   Construct viable arguments and critique the reasoning of others.
MP7   Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 9:   **AVOID HARD WORK**

SOURCE: This is question # 19 from the 2004 MAA AMC 8 Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

> **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Okay. This question feels doable. All we have to do is find the first number (bigger than 2) that is two more than a multiple of three, and two more than a multiple of four, and of five, and of six.

Here are all the numbers two more than a multiple of three: $5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, \dots$.

Here are all the numbers two more than a multiple of four: $6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, \dots$.

Here are all two more than a multiple of five: $7, 12, 17, 22, 27, 32, 37, 42, 47, 52, \dots$.

And here all two more than a multiple of six: $8, 14, 20, 26, 32, 38, 44, 50, \dots$.

Oh dear. I don't see any number in common to all the lists. I've changed my mind: this question is harder than I first thought!

Hmm. Algebra?

We need to solve a system of equations that looks like:

$$3n + 2 = 4m + 2 = 5k + 2 = 6r + 2.$$

Ooh. That's hard too!

What else can I do? I could keep extending the four lists I started with. But that doesn't seem fun.

[At this point I should stop doing this problem and go for a walk. I need an epiphany!]

The number we want keeps leaving a remainder of two.

Imagine a big pile of pennies. If we divide those pennies into groups of three, two sit left over. If we divide it into fours, those two sit left over again. With groups of five and with six those two same pennies sit left over.

Hmm.

Let's take those two pennies away! The rest of them make a group evenly divisible by three, four, five, and six.

Oh! And what's the smallest number divisible by $3, 4, 5$, and 6? It's 60.

That's it. The smallest number that keeps leaving a remainder of two just as we want is 62, and so the answer to the problem is (B).

Extension 1: What is the smallest number that leaves a remainder of 2 upon division by three, a remainder of 3 upon division by four, a remainder of 4 upon division by five, and a remainder of 5 upon division by six?

Extension 2: Pick any four values a, b, c, and d. Is there sure to be a number that leaves the same remainder a does upon division by three, as b does upon division by four, as c does upon division by five, and as d does upon division by six?

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