Curriculum Burst 96: Successive Triangles

By Dr. James Tanton, MAA Mathematician in Residence

> Let T_1 be a triangle with sides 2011, 2012, and 2013. For $n \geq 1$, if $T_n = \Delta ABC$ and D, E, and F are the points of tangency of the incircle of ΔABC to the sides AB, BC, and AC, respectively, then T_{n+1} is a triangle with side lengths AD, BE, and CF, if it exists. What is the perimeter of the last triangle in the sequence $\left(T_n\right)$?

QUICK STATS:

Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grade levels.

MATHEMATICAL TOPICS

Recursive sequences, Triangle Inequality, Systems of Equations, Circle Theorems

COMMON CORE STATE STANDARDS

G-C.A Understand and apply theorems about circles
A-SSE.A1b Interpret complicated expressions by viewing one or more of their parts as a single entity.
A-REI.C Solve systems of equations.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING**

SOURCE: This is question # 25 from the 2011 MAA AMC 10B Competition.

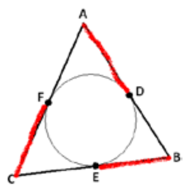
THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

```
STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
```

This question looks positively scary! T_1 is a triangle with three big sides-lengths: 2011, 2012, and 2013. That's okay. But it is the middle commentary about triangles T_n and T_{n+1} that is truly frightening.

Let me draw a picture of T_n. I can do that at least.

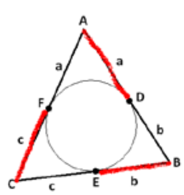


Deep breath. The next triangle T_{n+1} is made of the three segments highlighted. So I guess we keep doing this: start with our first triangle T_1, draw its incircle and find the three lengths as indicated, and use those lengths as the sides of a new triangle T_2, and do this again, and again, and again.

The final line of the question "*What is the perimeter of the last triangle in the sequence* $\left(T_n\right)$*?*" suggests that this process stops. We must eventually get three side-lengths that don't make a triangle. This happens if two lengths don't sum to more than the third.

I suppose we just have to start with the numbers 2011, 2012, and 2013, and see what the sides of the next triangle will be, and then the next, and so on.

When I look at the picture, I can't help but notice common tangent lengths:



We have:

$$\begin{aligned} a+b&=2011 \\ b+c&=2012 \\ c+a&=2013 \end{aligned}$$

I suppose we can solve for each of a, b, and c.

Adding all three equations gives:

$$2a+2b+2c=2011+2012+2013=3\times 2012.$$

So $2a+2\times 2012=3\times 2012$, and $a=2012/2$.

Also $2b=3\times 2012-2\times 2013$, giving $b=\frac{2012}{2}-1$, and $2c=3\times 2012-2\times 2011$, giving $c=\frac{2012}{2}+1$.

Oh look! We started with three side-lengths $x-1$, x, $x+1$ and the new triangle has side-lengths $\frac{x}{2}-1, \frac{x}{2}, \frac{x}{2}+1$. Whoa! So the next triangle has sides $\frac{x}{4}-1, \frac{x}{4}, \frac{x}{4}+1$, and T_n has side-lengths $\frac{x}{2^{n-1}}-1, \frac{x}{2^{n-1}}, \frac{x}{2^{n-1}}+1$.
Here $x=2012$.

Okay, which n fails to be a triangle? We need $\frac{x}{2^{n-1}}+\frac{x}{2^{n-1}}-1\le\frac{x}{2^{n-1}}+1$. This gives $x\le 2^n$. That is, $2012\le 2^n$. The first n that fails is thus $n=11$ (since $2^{10}=1024$). So T_{10} is the final triangle that exists and its perimeter is:

$$\frac{x}{2^9}-1+\frac{x}{2^9}+\frac{x}{2^9}+1=3\times\frac{x}{2^9}=3\times\frac{2012}{2^9}=3\times\frac{503}{2^7}=\frac{1509}{128}$$

Wow!

Extension: Is there a scalene triangle with three side-lengths such that the sequence of triangles $\left(T_n\right)$ constructed this way from it never stops?

MAA acknowledges with gratitude the generous contributions of
the following donors to the Curriculum Inspirations Project:

The TBL and Akamai Foundations
for providing continuing support

The Mary P. Dolciani Halloran Foundation for providing seed
funding by supporting the Dolciani Visiting
Mathematician Program during fall 2012

MathWorks for its support at the Winner's Circle Level