Curriculum Burst 95: A Line through Lattice Points

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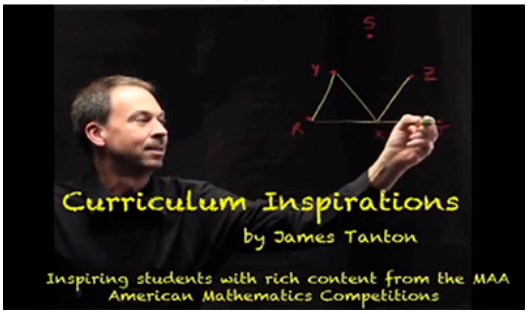
A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $\frac{1}{2} < m < a$. What is the maximum possible value of a?

Click here for video



QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grade levels.

MATHEMATICAL TOPICS

Lines. Slope.

COMMON CORE STATE STANDARDS

F-LE.B.5 Interpret the parameters in a linear or exponential function in terms of a context.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 10: **GO TO EXTREMES**

SOURCE: This is question # 24 from the 2011 MAA AMC 10B Competition.

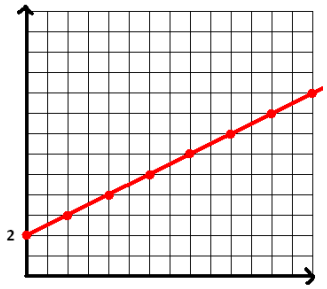
THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

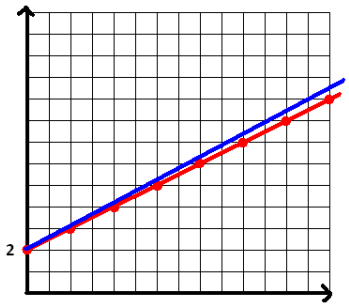
STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question is hard to understand! It's something about "lattice points," which are points in the plane with both coordinates integers. Then there is something about lines $y = mx + 2$ not passing through lattice points. And there is something about m, the slope, being between $1/2$ and some other number a. And I am meant to find a biggest value for a.

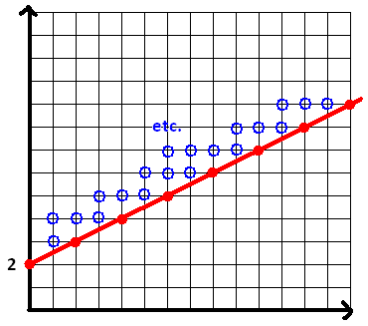
Here's my strategy: Since the only number mentioned for the slope is $\frac{1}{2}$, let me just draw the line $y = \frac{1}{2}x + 2$ and see if I can learn anything. (Well, actually, we are meant to have $\frac{1}{2} < m$. Oh well!)



Okay, this line definitely passes through lattice points. But $m = 1/2$ is an extreme position. If m is slightly larger than $1/2$ it might well miss lattice points.



Actually, we see that the slope can't be too big. We are not allowed to "hit" any of these circled lattice points.



It looks like the circled dots just to the left of each red dot are the real trouble ones. For example, the red dot at $x = 2$ has $y = \frac{1}{2} \cdot 2 + 2 = 3$, and the circled dot just to its left is at $(1,3)$. The red dot at $x = 4$ has $y = 4$ and the dot just to its left is $(3,4)$. The next is $(5,5)$. Actually, the question wants us to do this all the way up to $x = 100$. The circled dot to its left is $(99,52)$.

Okay, the lines that connect the y-intercept $(0,2)$ to each of $(1,3)$, $(3,4)$, $(5,5)$,…, $(99,52)$ are all bad lines for this question. What slope lines are they?

From $(0,2)$ to $(1,3)$, the slope is: $\frac{1}{1} = 1$.

From $(0,2)$ to $(3,4)$, the slope is: $\frac{2}{3}$.

From $(0,2)$ to $(5,5)$, the slope is: $\frac{3}{5}$.

…

From $(0,2)$ to $(99,52)$, the slope is: $\frac{50}{99}$.

The last slope is only barely larger than $1/2$. The point $(99,52)$ is "worst" circled point for this problem. So to avoid the lattice points for $0 < x \le 100$ we need a slope m satisfying $1/2 < m < 50/99$. The largest a can be is $50/99$.

Extension: What is the largest a can be if each line is allowed to pass through at most one lattice point with $0 < x \le 100$?

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