Curriculum Burst 92: Point of Contact Triangle

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> Circles with radii 1, 2, and 3 are mutually externally tangent. What is the area of the triangle determined by the points of tangency?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the upper high-school grade levels.

MATHEMATICAL TOPICS

Trigonometry; Geometry: Area formulas

COMMON CORE STATE STANDARDS

G-SRT.D Apply trigonometry to general triangles

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1: **ENGAGE IN SUCCESSFUL FLAILING**

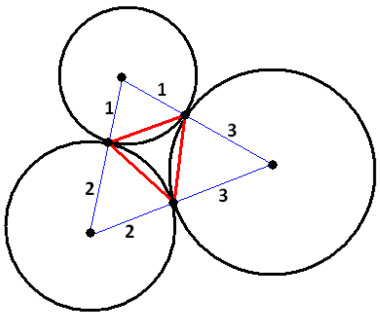
SOURCE: This is question # 17 from the 2011 MAA AMC 12A Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

> **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I need to draw a picture in order to properly take in the question.



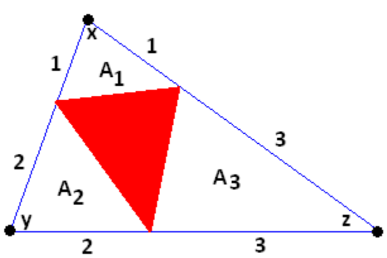
We want the area of the inner red triangle. (It seemed compelling to draw in all the radii.) Hmm. Do I know how to work out areas of triangles? I know two formulas. If a, b, and c are the sides lengths of the triangle, then its area A is:

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

$$A=\frac{1}{2}ab\sin\theta$$

Here $s=(a+b+c)/2$ and θ is the angle between the sides of lengths a and b. Helpful?

I feel like it might be productive to work out the area A of the big blue triangle and subtract from it the areas of the three "outer" triangles. Call those areas A_1, A_2, and A_3.



The area of the big triangle is:

$$A=\sqrt{s(s-3)(s-4)(s-5)}.$$

Ooh! Stop! It's a $3-4-5$ triangle. The angle labeled as x is a right angle, and so $A=\frac{1}{2}\cdot 3\cdot 4=6$.

We also have:

$$A_1=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}.$$

Okay, we also have

$$A_2=\frac{1}{2}\cdot 2\cdot 2\cdot\sin(y)$$

$$A_3=\frac{1}{2}\cdot 3\cdot 3\cdot\sin(z)$$

Do we know $\sin(y)$ and $\sin(z)$? Well, angles y and z are each part of a right $3-4-5$ triangle, and so $\sin(y)=4/5$ and $\sin(z)=3/5$. Thus:

$$A_2=\frac{1}{2}\cdot 2\cdot 2\cdot\frac{4}{5}=\frac{8}{5}$$

$$A_3=\frac{1}{2}\cdot 3\cdot 3\cdot\frac{3}{5}=\frac{27}{10}$$

That's all we need!

$$A=6-\frac{1}{2}-\frac{8}{5}-\frac{27}{10}=\frac{60-5-16-27}{10}=\frac{12}{10}=\frac{6}{5}.$$

> **Extension:** This question was "nice" in that it gave us a right triangle to work with. What if the radii of the three mutually tangent circles were instead 2, 3, and 4? Is it still possible to work out the area of the triangle determined by the points of contact? (That is a YES/NO question! If the answer is YES, would you want to determine the area of that triangle? Is there a pleasant way to find it?)

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