Curriculum Burst 82: Gothic Windows

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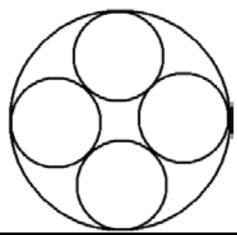
Many Gothic cathedrals have windows with portions containing a ring of congruent circles that are circumscribed by a larger circle. In the figure shown, the number of smaller circles is four. What is the ratio of the sum of the areas of the four smaller circles to the area of the larger circle?



QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grade levels.

MATHEMATICAL TOPICS

Geometry

COMMON CORE STATE STANDARDS

G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1: **ENGAGE IN SUCCESSFUL FLAILING**

SOURCE: This is question # 21 from the 2009 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the best start is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

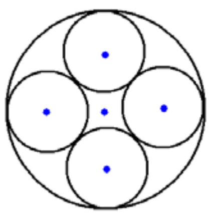
I understand the question and have a sense of what I need to do. But I don't really know how I am going to do it.

It seems I need the radius r of each small circle and the radius R of the big circle. Then the ratio the question seeks is $\frac{4\pi r^2}{\pi R^2} = \frac{4r^2}{R^2}$.

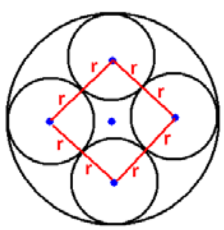
But there are no numbers in this question! How am I meant to figure out the values of r and R?

Hmm. Let's just play with the question and see if anything comes to me.

It feels compelling to draw the centers of the circles.

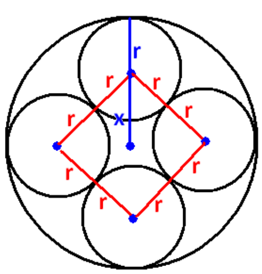


The centers of the four small circles make a square.



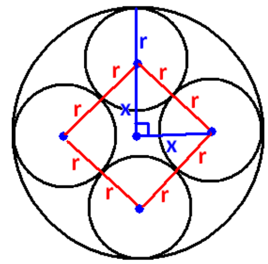
And the sides of that square are two small radii.

Do I know anything about the bigger radius R? Let me draw in one big radius.



I see it is composed of two parts: a length I've marked x and another radius r. So $R = x + r$. I don't know if that is helpful.

Can I say anything about the x part?



If I draw another section of length x I see I have a right triangle. By the Pythagorean Theorem, $x^2 + x^2 = (2r)^2$. That is, $2x^2 = 4r^2$. This gives $x = \sqrt{2}r$. Hmm. Is that helpful? Well, $R = x + r = \sqrt{2}r + r = r\left(1+\sqrt{2}\right)$, but that's ugly. And I still don't have any actual numbers! All I have is $R = \left(1+\sqrt{2}\right)r$.

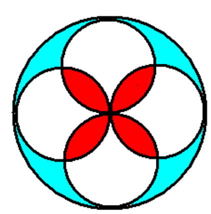
What are we looking for? The value of $\frac{4\pi r^2}{\pi R^2} = \frac{4r^2}{R^2}$. Well, I can at least write:

$$\frac{4r^2}{R^2} = \frac{4r^2}{\left(1+\sqrt{2}\right)^2 r^2} = \frac{4}{\left(1+\sqrt{2}\right)^2} = \frac{4}{1+2+2\sqrt{2}} = \frac{4}{3+\sqrt{8}}.$$

Ooh! That's an actual number and is the answer to the question!

Extension 1: We said that the centers of the small circles form a square with sides of length $2r$. Is that true? Does each side of that square pass through the point of contact of two circles?

Extension 2: Discover something interesting about the red and blue areas:



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