Curriculum Burst 80: A Tricky Length

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> Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E, $AC = 14$, and ΔAED and ΔBEC have equal areas. What is AE?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grade levels.

MATHEMATICAL TOPICS

Geometry

COMMON CORE STATE STANDARDS

G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: DO SOMETHING!

SOURCE: This is question # 23 from the 2009 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the best start is …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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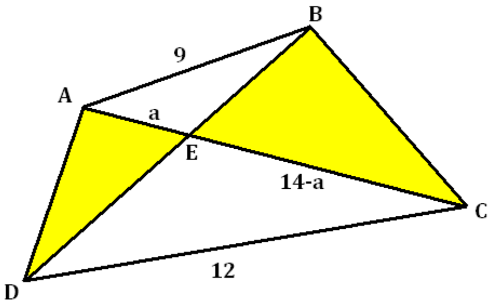
My reaction is that I need a picture!

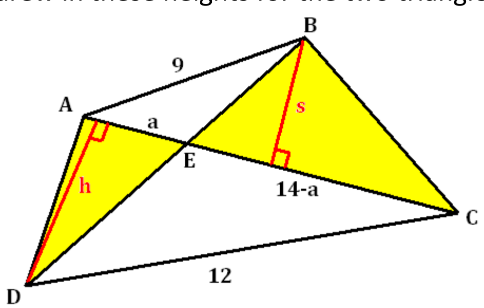


I split the length $AC = 14$ into two parts, a and $14 - a$. The question wants the length a. The only thing we know to help us out are the lengths indicated in the diagram and the fact that the two shaded triangles have the same area.

Are those two triangles congruent? They have vertical angles, but nothing else. I can't tell if they are congruent.

I know that the formula $\frac{1}{2} base \times height$ evaluates to the same value for each triangle. Can I choose a base in common or a height in common for each triangle? Hmm. Doesn't look obvious.

What if I drew in these heights for the two triangles?



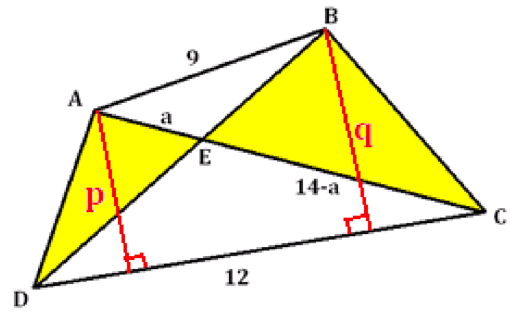
We have: $\frac{1}{2}ah = \frac{1}{2}(14 - a)s$ giving $ah + as = 14s$. Is that helpful? Hmm.

Actually the altitude of length s is also an altitude of ΔAEC. We have $Area\,\Delta AEC = \frac{1}{2} \cdot 14 \cdot s = 7s$.

In the same way, $Area\,\Delta ADC = \frac{1}{2} \cdot 14 \cdot h = 7h$. The area of the whole quadrilateral is $7h + 7s$. Helpful?

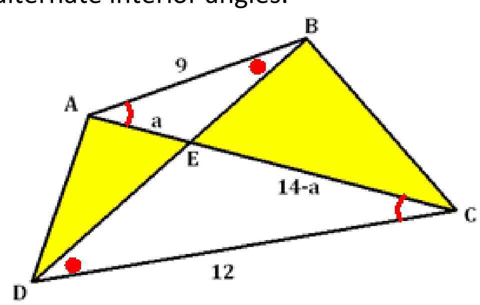
I haven't used the sides of lengths 9 or 12 in any way. (I really am groping for ideas. I have no idea where any of this is going!)

The side 12 can be viewed as a base of ΔAEC and – whoa – it is a base of ΔDBC. And these two triangles have the same area! (Do you see why? ΔDEC is in common.)



We have: $\frac{1}{2} \cdot 12 \cdot p = \frac{1}{2} \cdot 12 \cdot q$ giving $p = q$. Is this helpful? So the quadrilateral with the sides of lengths p and q and $\overline{AC}$ and the portion of $\overline{CD}$ has one pair of congruent parallel sides. It is a parallelogram. So $\overline{AB}$ and $\overline{CD}$ are parallel. Is that helpful?

We have alternate interior angles.



So triangles ABE and CDE are similar.

Okay! This means $\frac{a}{14 - a} = \frac{9}{12}$ giving $4a = 42 - 3a$, and so $a = 6$! Wow!

Extension: What is the ratio of the area of $ABCD$ to the area of ΔCED?

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