Curriculum Burst 75: Oneless Numbers

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How many whole numbers between 1 and 1000 do not contain the digit 1?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the middle-school grade levels.

MATHEMATICAL TOPICS

Counting methods

COMMON CORE STATE STANDARDS

S.CP.9 (Tangentially) Use permutations and combinations to compute probabilities of compound events and solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 9: **AVOID HARD WORK**

SOURCE: This is question # 22 from the 2009 MAA AMC 8 Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the best start is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

On approach to this question is to list all the one-digit numbers without a 1:

$$2\ 3\ 4\ 5\ 6\ 7\ 8\ 9$$

(there are eight of these), all the two-digit numbers without 1:

$$20\ 22\ 23\ ..30\ 32\ ..\ 40\ 42...98\ 99$$

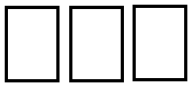
(there is some number of these, which I might be able to count), and all the three-digit numbers without a 1:

$$200\ 202\ ...\ 998\ \ 999$$

(and these seem a challenge to count).

How can I do these individual counts without hard work?

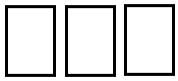
A three-digit number is made by selecting digit for each of three slots:



There are nine choices for each slot – each of the digits $0, 2, 3, 4, 5, 6, 7, 8, 9$ - except for the first digit, which can't be zero.

Actually, why can't the first digit be zero?

038, for example, corresponds to the two-digit number 38, and 007 the one-digit number 7. We want those so let's include them! So we actually do have nine-choices for each slot!



$9 \times 9 \times 9 = 81 \times 9 = 729$ choices.

We thus have 729 one-, two-, and three-digit numbers that lack a 1.

OOPS! Let me revise that. We could get "000,"a value we don't want to include in our considerations.

There are 728 one-less numbers at most three digits long. Final answer!

Question: a) Quentin said that the answer to this question is given by $8 + 8 \times 9 + 8 \times 9 \times 9 = 728$. What line of reasoning did he follow?

b) Explain why $1 + 9 + 9^2 + 9^3 + \cdots + 9^{n-1} = \dfrac{9^n - 1}{8}$.

Extension 1: How many one-, two-, and three-digit numbers lack the digit 1 and lack the digit 2?

Extension 2: How many one-, two-, and three-numbers have at most one 1 among their digits? How many have exactly one 1?

Extension 3:
Let A be the total count of one-, two-, and three-digit numbers with no 1s.
Let B be the total count of all one-, two-, and three-digit numbers possessing exactly one 7.
Let C be the total count of all one-, two-, and three-digit numbers possessing exactly two 5s.

Quickly: What is the value of the sum $A + B + C$?

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