Curriculum Burst 71: Counting Two Sets

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> Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the junior high-school grade levels.

MATHEMATICAL TOPICS

Permutations and Combinations

COMMON CORE STATE STANDARDS

S.CP.9 Use permutations and combinations to compute probabilities of compound events and solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 4: **DRAW A PICTURE**

SOURCE: This is question # 23 from the 2008 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

As always, one starts the problem-solving process with …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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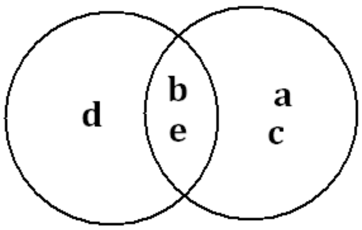
I read this question and I am a little bit confused by it. We have two choose two subsets of a five-element set S so that their union is S and their intersection contains exactly two elements. Huh?

What if I said the word "make" instead of "choose." Does that help? We have to make two subsets of S that have two elements in their intersection, and whose union is all of S. Can I even make one? Sure. Something like this works.

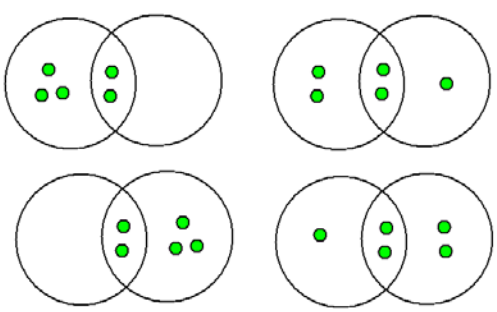


Okay. I've got the gist of the question. We have to sprinkle the five elements a, b, c, d, and e in Venn diagram like the one above so that there are two elements in the intersection and all five elements appear in the sets. But there is this part of the question:

In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

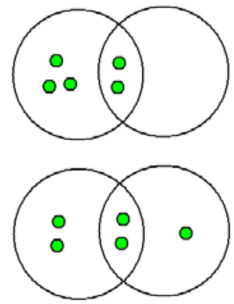
I am not sure yet what that means. Well, let's ignore it for now!

It seems there are four basic ways to have the five elements in two sets with two in the intersection:



"…assuming that the order in which the subsets are chosen does not matter…"

Hmm. These pictures definitely have a "left set" and a "right set" and so my four pictures are relying on an order. If I couldn't tell left from right (say, I could flip the diagrams over), which is what I think the question wants, then there are only TWO distinct configurations to consider:

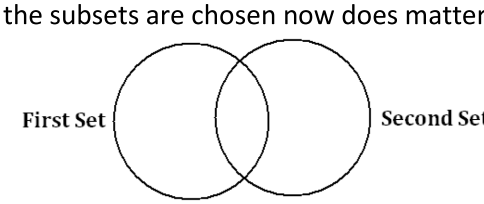


I think it is these I need to count.

In the first diagram three of a, b, c, d, e are to be labeled "part of a set outside of the intersection" and two are to be labeled "in the intersection." There are $\frac{5!}{3!2!} = \frac{5\times4}{2} = 10$ ways this can be so. In the second diagram, two of the elements a, b, c, d, e are to be labeled "part of a pair outside of the intersection," two as "in the intersection," and one as "on its own outside of the intersection." There are $\frac{5!}{2!2!1!} = \frac{5\times4\times3}{2} = 30$ ways this can be done. This gives a total of $10+30=40$ ways to make subsets meeting the conditions of the question.

Comment: See the "Permutations and Combinations" course at http://gdaymath.com/courses/ for more on the "labeling" technique I followed here.

Extension: What is the answer to this question if the order in which the subsets are chosen now does matter?



Does the answer surprise you?

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