Curriculum Burst 67: Units of Big Powers

By Dr. James Tanton, MAA Mathematician in Residence

Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

QUICK STATS:

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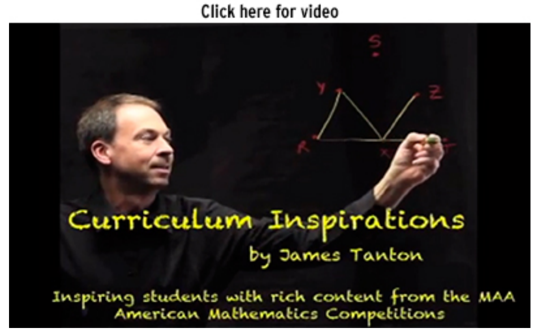

MAA AMC GRADE LEVEL

This question is appropriate for the junior high-school grade levels.

MATHEMATICAL TOPICS

Exponents

COMMON CORE STATE STANDARDS

A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: DO SOMETHING

SOURCE: This is question # 24 from the 2008 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the best start is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

So this question is asking for the units digit of

$$\left(2008^2+2^{2008}\right)^2+2^{\left(2008^2+2^{2008}\right)}.$$

Oh heavens! Obviously I am not going to work out what this number is: it's a nightmare of a number. But what can I do?

The question is only asking for its last digit. Do I know any last digits of parts of the number perhaps? Just to be able to get started and do something, let me state the obvious:

2008 ends with an 8.

Okay.

2008^2 ends with a 4.

I see this because

$$(2000+8)^2=2000^2+2\times2000\times8+64.$$

What about 2^{2008}? Hmm.

Well, let me just list some powers of two and see if anything helpful emerges:

$$1,2,4,8,16,32,64,128,256,512,1024,2048,\ldots$$

The last digits cycle $2-4-8-6$ with:

$2^1,2^5,2^9,\cdots$ ending in a 2;
$2^2,2^6,2^{10},\cdots$ ending with a 4;
$2^3,2^7,\cdots,2^{4k-1},\cdots$ ending with an 8;
$2^4,2^8,\cdots,2^{4k},\cdots$ ending with a 6.

Alright. Since 2008 is a multiple of four:

2^{2008} ends with a 6.

This is good!

So 2008^2+2^{2008} ends with a 4 plus a 6, which is the same as ending with a 0.

This means $\left(2008^2+2^{2008}\right)^2$ is a number ending with a zero squared, and so also ends with a zero.

We're halfway there. Wow!

Now for $2^{\left(2008^2+2^{2008}\right)}$.

Ooh! Isn't this exponent itself a multiple of four? Can we use the fact that the 2^{4k} s end with a 6?

Now 2008 is a multiple of four and therefore so is 2008^2, and $2^{2008}=4\times2^{2006}$. Yes! $2^{\left(2008^2+2^{2008}\right)}$ does end with a 6.

So finally…

$\left(2008^2+2^{2008}\right)^2+2^{\left(2008^2+2^{2008}\right)}$ ends with "$0+6$", which is the same as ending with a six.

Awesome!

Extension 1: What is the final digit of

$$\left(2008^3+3^{2008}\right)^3+3^{\left(2008^3+3^{2008}\right)}\ ?$$

Is the thinking needed for this question the same or a tad more delicate than work we did for this essay?

Extension 2: What is the final digit of $\left(Y^2+2^Y\right)^2+2^{Y^2+2^Y}$ where Y is the number of the current year?

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