Curriculum Burst 64: Hidden Primes

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> Barry wrote 6 different numbers, one on each side of 3 cards, and laid the cards on a table, as shown. The sums of the two numbers on each of the three cards are equal. The three numbers on the hidden sides are prime numbers. What is the average of the hidden prime numbers?
>
> | 44 | 59 | 38 |
> |---|---|---|

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 8th grade level.

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MATHEMATICAL TOPICS

Factors and primes; Averages

COMMON CORE STATE STANDARDS

6-NS.4 (Tangentially) Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 8: **SECOND GUESS THE AUTHOR**

SOURCE: This is question # 25 from the 2006 MAA AMC 8 Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the best start is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question makes me feel uneasy. I don't feel I really "get" it. (Or maybe I do and I am just contending with a feeling of panic as I don't know how to start on the problem!)

Deep breath. Let me process the question slowly.

We have three cards, with numbers on each side. Got that.

The pairs of numbers on each card add to the same value. Got that .

The numbers I don't see are prime numbers. Okay.

We are being asked for the average of the three numbers we don't see.

This question feels weird. The mention of primes seems awfully suspicious. Why does the author need the hidden numbers to be primes?

Okay, just to be clear, the primes are:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29, \ldots$$

We need

$44+$ some prime	=	something
$59+$ some prime	=	same something
$38+$ some prime	=	same something still

What is it about the primes that the author wants to make this question work?

Can we just choose some primes that work? To be easy, could we use the first prime 2? It would have to go with the largest number 59:

$$59+2=61$$

To keep the same sum 61 we need:

$$44+17=61$$
$$38+23=61$$

and 17 and 23 happen to be prime!

So the primes 2, 17 and 23 work, and they have average value $42/3=14$.

AND since the author of the question wants there to be only one answer (otherwise people taking the competition could get different answers) this must be the solution the author is looking for! Whoa!

Hmmmm…. This feels just like luck. Surely there has to be some mathematical reason to explain why 2, 13 and 17 are sure to be the hidden primes. What could that reason be? Could we add a different prime to 59? Why doesn't $59+3=62$ or $59+5=64$ or $59+19=78$ work?

All these sums here, 62, 64 and so on, are even. Oh … And this is sure to always be the case as all primes different from 2 are odd: $59+odd=even$. So … when we go back to the number 40 we'll need:

$$40+prime=even.$$

But this means the prime we use would have to, itself, be even. 2 is the only even prime, but it is not big enough.

Okay, that's it. We had to use the prime 2 with the number 59 and everything we did above is indeed the only possible scenario for this question. 14 is the answer!

Extension 1: Can you create a four-card version of this puzzle that also possesses a unique answer?

Extension 2: Suppose the numbers on the backs of the three cards are not necessarily prime. If I tell you that the sums of the two numbers on each card is the same for all three cards, and that the average of the three hidden numbers is the same as the average of the three numbers you see, do you now know the three hidden numbers?

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